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Gaussian Fluctuation for the Number of Particles in Airy, Bessel, Sine, and Other Determinantal Random Point Fields

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We prove the Central Limit Theorem (CLT) for the number of eigenvalues near the spectrum edge for certain Hermitian ensembles of random matrices. To derive our results, we use a general theorem, essentially due to Costin and Lebowitz, concerning the Gaussian fluctuation of the number of particles in random point fields with determinantal correlation functions. As another corollary of the Costin–Lebowitz Theorem we prove the CLT for the empirical distribution function of the eigenvalues of random matrices from classical compact groups.

KEY WORDS: Determinantal random point fields; central limit theorem; random matrices; Airy and Bessel kernels; classical compact groups.

1. INTRODUCTION AND FORMULATION OF RESULTS

Random Hermitian matrices were introduced in mathematical physics by Wigner in the fifties [Wig1, Wig2]. The main motivation of pioneers in this field was to obtain a better understanding of the statistical behavior of energy levels of heavy nuclei. An archetypical example of random matrices is the Gaussian Unitary Ensemble (G.U.E.) which can be defined by the probability distribution on a space of n-dimensional Hermitian matrices as

$$P(dA) = \mathrm{const}_n \cdot e^{-2n \,\mathrm{Trace}\, A^2}\, dA \tag{1.1}$$

Here dA is the Lebesgue measure on n^2-parameters set

$$\{\mathrm{Re}\, a_{ij},\ 1 \leqslant i < j \leqslant n;\ \mathrm{Im}\, a_{ij},\ 1 \leqslant i < j \leqslant n,\ a_{ii},\ 1 \leqslant i \leqslant n\} \tag{1.2}$$

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and $\text{const}_n = (\pi n)^{-n^2/2} \cdot 2^{n(2n-1)/2}$ is a normalization constant. (1.1) implies that matrix entries (1.2) area independent Gaussian random variables $N(0, (1+\delta_{ij})/8n)$. It is well known that the G.U.E. is the only ensemble of Hermitian random matrices (up to a trivial rescaling) that satisfies both of the following properties:

(1) probability distribution $P(dA)$ is invariant under unitary transformation

$$A \to U^{-1} A U, \qquad U \in U(n)$$

(2) matrix entries up from the diagonal are independent random variables (see [Me, Chap. 2]).

The n eigenvalues, all real, of a Hermitian matrix A will be denoted by $\lambda_1, \lambda_2, ..., \lambda_n$. For the formulas for their joint distribution density $p_n(\lambda_1, ..., \lambda_n)$ and k-point correlation functions $\rho_{n,k}(\lambda_1, ..., \lambda_k)$ we refer to [Me]. One has

$$p_n(\lambda_1, ..., \lambda_n) = \text{const}'_n \cdot \prod_{1 \leqslant i < j \leqslant n} |\lambda_i - \lambda_j|^2 \cdot \exp\left(-2n \cdot \sum_{i=1}^{n} \lambda_i^2\right) \tag{1.3}$$

$$\begin{aligned}\rho_{n,k}(\lambda_1, ..., \lambda_k) &:= \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(\lambda_1, ..., \lambda_n)\, d\lambda_{k+1} \cdots d\lambda_n \\ &= \det(K_n(\lambda_i, \lambda_j))_{i,j=1}^{k} \end{aligned} \tag{1.4}$$

where $K_n(x, y)$ is a projection kernel,

$$K_n(x, y) = \sqrt{2n} \cdot \sum_{\ell=0}^{n-1} \psi_\ell(\sqrt{2n}\, x) \cdot \psi_\ell(\sqrt{2n} \cdot y) \tag{1.5}$$

and

$$\psi_\ell(x) = \frac{(-1)^\ell}{\pi^{1/4} \cdot (2^\ell \cdot \ell!)^{1/2}} \cdot \exp\left(\frac{x^2}{2}\right) \cdot \frac{d^\ell}{dx^\ell} \exp(-x^2) \tag{1.6}$$

$\ell = 0, 1, ...,$ are Weber–Hermite functions. The global behavior of eigenvalues is governed by the celebrated semicircle law, which states that the empirical distribution function of the eigenvalues weakly converges to a nonrandom (Wigner) distribution:

$$\mathscr{F}_n(\lambda) = \frac{1}{n} \#\{\lambda_i \leqslant \lambda\} \xrightarrow[n \to \infty]{w} \mathscr{F}(\lambda) = \int_{-\infty}^{\lambda} \rho(x)\, dx \tag{1.7}$$

with probability one ([Wig1, Wig2]) where the spectral density ρ is given by

$$\rho(t) = \begin{cases} \frac{2}{\pi}\sqrt{1-t^2}, & |t| \leqslant 1 \\ 0, & |t| > 1 \end{cases} \tag{1.8}$$

To study the local behavior of eigenvalues near an arbitrary point in the spectrum $x \in [-1, 1]$, one has to consider rescaling

$$\lambda_i = x + \frac{y_i}{\rho_{n,1}(x)}, \qquad i = 1,..., k \tag{1.9}$$

and study the rescaled k-point correlation functions

$$R_{n.k}(y_1,..., y_k) := (\rho_{n,1}(x))^{-k} \cdot \rho_{n,k}(\lambda_1,..., \lambda_k) \tag{1.10}$$

The biggest interest is paid to the asymptotics of rescaled correlation functions when n goes to infinity. For G.U.E. the answer can be obtained from the Plancherel–Rotach asymptotic formulas for Hermite polynomials [PR]:

$$\lim_{n \to \infty} R_{n,k}(y_1,..., y_k) = \rho_k(y_1,..., y_k) = \det(K(y_i, y_j))_{i,j=1}^{k} \tag{1.11}$$

The kernel K actually also depends on x but in a very simple way. It can be represented as

$$K(y, z) = \frac{\mathscr{A}(y) \cdot \mathscr{A}'(z) - \mathscr{A}(z)\, \mathscr{A}'(y)}{y - z} \tag{1.12}$$

where for all $|x| < 1$ the function $\mathscr{A}$ is just $\sin(\pi y)/\pi$, and for $x = \pm 1$ it is

$$\mathscr{A}i(\pm y) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 \pm yt\right) dt \tag{1.13}$$

The function defined by (1.13) is known as the Airy function and the kernel (1.12)–(1.13) is known as the Airy kernel (see [Me, TW1, F]). The limiting correlation functions (1.11)–(1.13) determine a random point field on the real line, i.e., probability measure on the Borel σ-algebra of the space of locally finite configurations,

$$\Omega = \{\omega = (y_i)_{i \in \mathbb{Z}} : \forall T > 0 \, \# \{y_i : |y_i| < T\} < \infty\} \tag{1.14}$$

The distribution of random point field is uniquely defined by the generating function

$$\psi(z_1,..., z_k; I_1,..., I_k) = E \prod_{j=1}^{k} z_j^{\nu_j}$$

where I_j, $j = 1,..., k$, are disjoint intervals on the real line, $\nu_j = \#\{y_i \in I_j\} = \#(I_j)$, the number of particles in I_j, and $k \in \mathbb{Z}^1_+$. It follows from the general theory of existence and uniqueness for random point fields [L1, L2], that if $K(y, z)$ is locally bounded than determinantal correlation functions uniquelly determine random point field assuming that such random point field exists. The generating function $\psi(z_1,..., z_k)$ is given by Fredholm determinant of the intergal operator in $L^2(\mathbb{R}^1)$:

$$\psi(z_1,..., z_k) = \det(\delta(x-y) + \sum_{j=1}^{k} (z_j - 1) \cdot K(x, y) \cdot \chi_{I_j}(y)) \tag{1.15}$$

where χ_{I_j} is an indicator of I_j.

In particular these results are applicable to the Airy kernel (1.12)–(1.13). We shall call the corresponding random point field the Airy random point field. For one-level density formulas (1.11)–(1.13) produce

$$\rho_1(y) = -y \cdot (\mathscr{A}i)^2\,(y) + (\mathscr{A}i'(y))^2 \tag{1.16}$$

The asymptotic expansion of the Airy function is well known (see [Ol]). One can deduce from it

$$\rho_1(y) \sim \begin{cases} \dfrac{|y|^{1/2}}{\pi} - \dfrac{\cos(4 \cdot |y|^{3/2}/3)}{4\pi \cdot |y|} + \underline{0}(|y|^{-5/2}) & \text{as} \quad y \to -\infty \\ \dfrac{17}{96 \cdot \pi y^{1/2}} \cdot \exp(-4y^{3/2}/3) & \text{as} \quad y \to +\infty. \end{cases} \tag{1.17}$$

ρ_1 satisfies the third order differential equation

$$\rho_1'''(y) = -2\rho_1(y) + 4y \cdot \rho_1'(y) \tag{1.18}$$

One can think about the one-point correlation function as a level density, since for any interval $I \subset \mathbb{R}^1$ we have $E\#(I) = \int_I \rho_1(y)\, dy$. It follows from (1.17) that $E\#((-T, +\infty))$ is finite for any T and $E\#((-T, +\infty)) \sim (2T^{3/2}/3\pi) + \underline{0}(1)$ when T goes to $+\infty$. The last formula means that $E\#((-T, +\infty)) - (2T^{3/2}/3\pi)$ stays bounded for large positive T. Let us denote $\nu_1(T) := \#\{y_i > -T\} = \#((-T, +\infty))$, $\nu_k(T) := \#((-kT, -(k-1)\,T]))$, $k = 2, 3,...$. Theorem 1 establishes the Central Limit Theorem for $\nu_k(T)$, $k \geqslant 1$.

Theorem 1. The variance of $v_k(T)$ grows logarithmically

$$\mathrm{Var}\, v_k(T) \sim \frac{11}{12\pi^2} \cdot \log T + \underline{0}(1)$$

and the sequence of normalized random variables $(v_k(T) - Ev_k(T))/\sqrt{\mathrm{Var}\, v_k(T)}$ converges in distribution to the centalized gaussian random sequence $\{\xi_k\}$ with the covariance function $E\xi_k \xi_l = \delta_{k,l} - \frac{1}{2}\delta_{k,l+1} - \frac{1}{2}\delta_{k,l-1}$.

Remark 1. The first result about Gaussian fluctuation of the number of particles in random matrix model was established by Costin and Lebowitz [CL] for the kernel $(\sin \pi(x-y))/\pi(x-y)$. See Section 2 for a more detailed discussion.

Remark 2. Basor and Widom [BaW] recently proved the Central Limit Theorem for a large class of smooth linear statistics $\sum_{i=-\infty}^{+\infty} f(y_i/T)$ where f satisfies some decay and differentiality conditions. Similar results for smooth linear statistics in other random matrix ensembles were proven in [Sp], [DS], [Jo1], [SiSo1], [KKP], [SiSo2], [Ba], [BF], [BdMK], [Br]; see also [So1] for the results about global distribution of spacings.

Another class of random Hermitian matrices, called Laguerre ensemble, was introduced by Bronk in [Br]. This one is the ensemble of positive $n \times n$ Hermitian matrices. Any positive Hermitian matrix H can be represented as $H = AA^*$, where A is some complex valued $n \times n$ matrix and A^* is its conjugate. The distribution on such matrices is defined as

$$P(dH) = \mathrm{const}''_n \cdot \exp(-n \cdot \mathrm{Trace}\, A \cdot A^*) \cdot [\det(AA^*)]^\alpha \, dA \tag{1.19}$$

where $\alpha > -1$ and dA is Lebesgue measure on $2n^2$-dimensional space of complex matrices. The joint distribution of n (positive) eigenvalues of H is given by

$$p_n(\lambda_1,..., \lambda_n) = \mathrm{const}'''_n \cdot \exp\left(-n \cdot \sum_{i=1}^{n} \lambda_i\right) \cdot \prod_{i=1}^{n} \lambda_i^\alpha \cdot \prod_{1 \leqslant i < j \leqslant n} (\lambda_i - \lambda_j)^2 \tag{1.20}$$

The Vandermonde factor in (1.20) implies that correlation functions still have the determinantal form (1.4) with the kernel

$$K_n(x, y) = n \cdot \sum_{\ell=0}^{n-1} \phi_\ell(nx) \cdot \phi_\ell(ny) \tag{1.21}$$

where the sequence $\{\phi_\ell(x)\}$ is obtained by orthonormalizing the sequence $\{x^k \cdot x^{\alpha/2} \cdot e^{-x/2}\}$ on $(0, +\infty)$. The limiting level density is supported on $[0, 1]$ and given by the formula

$$\rho_1(x) = \frac{1}{2\pi} x^{-1/2} \cdot (1-x)^{1/2} \tag{1.22}$$

(It is not surprising to see that (1.22) is the density of a square of the Wigner random variable!) Plancherel–Rotach type asymptotics for Laguerre polynomials [E], [PR] imply that by scaling the kernel $K_n(x, y)$ in the bulk of the spectrum, we obtain the sine kernel $(\sin \pi(x-y))/\pi(x-y)$, and by scaling at $x = 1$ ("soft edge"), we obtain the Airy kernel. As we already know, the same kernels appear after rescaling in G.U.E. This feature, called universality of local correlations, has been established recently for a variety of ensembles of Hermitian random matrices (see [PS], [BI], [DKMVZ], [Jo2], [So2], [BZ]). To scale the kernel at the "hard edge" $x = 0$, we need an asymptotic formula of Hilb's type (see [E]), which leads to

$$\lim_{n \to \infty} \frac{1}{4n} K_n\left(\frac{x}{4n}, \frac{y}{4n}\right) = \frac{J_\alpha(\sqrt{x}) \cdot \sqrt{y} \cdot J'_\alpha(\sqrt{y}) - \sqrt{x}\, J'_\alpha(\sqrt{x}) \cdot J_\alpha(\sqrt{y})}{2(x-y)} \tag{1.23}$$

where J_α is the Bessel function of order α [F, TW2, Ba]. The kernel (1.23) is also known to appear at hard edges in the Jacobi ensemble [NW]. For a quick reference, we note that in the Jacobi case, a sequence $\{\phi_\ell(x)\}$ from (1.21) is obtained by orthonormalizing $\{x^k(1-x)^{\alpha/2}\,(1+x)^{\beta/2}\}$. The random point field on $[0, +\infty]$ with the determinantal correlation functions defined by (1.23) will be referred to as the Bessel random point field. There is a general belief among people working in random matrix theory that in the same way as the sine kernel appears to be a universal limit in the bulk of the spectrum for random Hermitian matrices, Airy and Bessel kernels are universal limits at the soft and hard edge of the spectrum. The next theorem establishes the CLT for $\nu_k(T) = \#(((k-1)\,T, kT]),\ k = 1, 2,\ldots$.

Theorem 2. Let $\nu(T)$ be the number of particles in $(0, T)$ for the Bessel random point field. Then

$$E\nu_k(T) \sim \frac{1}{\pi} T^{1/2}(k^{1/2} - (k-1)^{1/2}) + \underline{0}(1)$$

$$\operatorname{Var} \nu_k(T) \sim \frac{1}{4\pi^2} \log T + \underline{0}(1)$$

and the sequence of the the normalized random variable $(\nu_k(T) - E\nu_k(T))/\sqrt{\operatorname{Var} \nu_k(T)}$ converges in distribution to the gaussian random sequence from the Theorem 1.

Theorems 1 and 2, as well as similar results for the random fields arising from the classical compact groups (see Section 4) are the corollaries of the general result about determinantal random point fields, which is essentially due to Costin and Lebowitz. Recently a number of discrete determinantal random point fields appeared in two-dimensional growth models [Jo3, Jo5], asymptotics of Plancherel measures on symmetric groups and the representation theory of the infinite symmetric group [BO1, BO2, BOO, Jo4, Ok1, Ok2]. If one can show the infinite growth of the variance of the number of particles in these models (the goal which is probably attainable since the asymptotics of the discrete orthogonal polynomials arising in some of these problems are known) the Costin–Lebowitz theorem should work there as well.

The rest of the paper is organized as follows. We discuss the general (Costin–Lebowitz) theorem in Section 2. Theorems 1 and 2 will be proven in Sections 3 and 4. In the Bessel case, we will see that the kernel $(\sin \pi(x-y)/\pi(x-y)) \pm (\sin \pi(x+y)/\pi(x+y))$ naturally appears in our considerations. We recall in Section 4 that the sine kernel also appears in the limiting distribution of eigenvalues in unitary group and the even and odd sine kernels appear in the distribution of eigenvalues in orthogonal and symplectic groups and then prove the Gaussian fluctuation for the number of eigenvalues in these models in Theorem 3–6.

2. THE CENTRAL LIMIT THEOREM FOR DETERMINANTAL RANDOM POINT FIELDS

Let $\{\mathscr{P}_t\}_{t \in \mathbb{R}^1_+}$ be a family of random point fields on $\mathbb{R}^d$ such that their correlation functions have determinantal form at the r.h.s. of (1.11) with Hermitian kernels $K_t(y, z)$, and $\{I_t\}_{t \in \mathbb{R}^1_+}$ a collection of Borel subsets $\mathbb{R}^d$. We denote by A_t an integral operator on I_t with the kernel $K_t(y, z)$, $A_t \colon L^2(I_t) \to L^2(I_t)$, by ν_t the number of particles in I_t, $\nu_t = \#(I_t)$, and by E_t, Var_t the mathematical expectation and variance with respect to the probability distribution of the random field $\mathscr{P}_t$. In many applications the random point field $\mathscr{P}_t$, and therefore the kernel K_t will be the same for all t. In such situations I_t will be expanding.

Theorem (O. Costin, J. Lebowitz). Let $A_t = K_t \cdot \chi_{I_t}$ be a family of trace class Hermitian operators associated with determinantal random point fields $\{\mathscr{P}_t\}$ such that $\mathrm{Var}_t\, \nu_t = \mathrm{Trace}(A_t - A_t^2)$ goes to infinity as $t \to +\infty$. Then the distribution of the normalized random variable $(\nu_t - E_t \nu_t)/\sqrt{\mathrm{Var}_t\, \nu_t}$ with respect to the random point field $\mathscr{P}_t$ weakly converges to the normal law $N(0, 1)$.

Remark 3. The result has been proven by Costin and Lebowitz when $d = 1$, $K_t(x, y) = (\sin \pi(x-y))/\pi(x-y)$ for any t and $|I_t| \xrightarrow{t \to \infty} \infty$ (see [CL]). The original paper contains a remark, due to Widom, that the result holds for more general kernels.

Remark 4. There is a general result that a (locally) trace class Hermitian operator K defines a determinantal random point field iff $0 \leqslant K \leqslant 1$ (see [So4] or, for a slightly weaker version, [Ma]).

The idea of the proof is very clear and consists of two parts. Let us denote the ℓth cumulant of ν_t by $C_\ell(\nu_t)$. We remind that by definition

$$\sum_{\ell=1}^{\infty} C_\ell (iz)^\ell / \ell\,! = \log(E_t \exp(iz\nu_t))$$

Lemma 1. The following recursive relation holds for any $\ell \geqslant 2$:

$$C_\ell(\nu_t) = (-1)^\ell \cdot (\ell - 1)!\ \mathrm{Trace}(A_t - A_t^\ell) + \sum_{s=2}^{\ell-1} \alpha_{s\ell}\, C_s(\nu_t) \tag{2.1}$$

where $\alpha_{s\ell}$, $2 \leqslant s \leqslant \ell - 1$, are some combinatorial coefficients (irrelevant for our purposes).

The proof can be found in [CL] or [So1, Section 2]; (of course one has to replace everywhere $(\sin \pi(x-y))/\pi(x-y)$ by $K_t(x, y)$). For the convinience of the reader we sketch the main ideas here. We start by introducing the Ursell (cluster) functions:

$$r_1(x_1) = \rho(x_1), \qquad r_2(x_1, x_2) = \rho_2(x_1, x_2) - \rho_1(x_1)\, \rho(x_2)$$

and, in general,

$$r_k(x_1,..., x_k) = \sum_{m=1}^{k} \sum_{G} (-1)^{m-1}\, (m-1)! \prod_{j=1}^{m} r_{G_j}(\bar{x}(G_j)) \tag{2.2}$$

where G is a partition of indices $\{1, 2,..., k\}$ into m subgroups $G_1,..., G_m$, and $\bar{x}(G_j)$ stands for the collection of x_i with indices in G_j.

It appears that the integral of k-point Ursell function $r_k(x_1,..., x_k)$ over dk-dimensional cube $I_t \times \cdots I_t$ is equal to the linear combination of $C_j(\nu_t)$, $j=1,..., k$. Namely, let us denote

$$T_k(\nu_t) = \int_{I_t} \cdots \int_{I_t} r_k(x_1,..., x_k)\, dx_1 \cdots dx_k$$

Then

$$\sum_k^{\infty} C_k(iz)^k/k! = \sum_{k=1}^{\infty} (\exp(z)-1)^k\, T_k(\nu_t)/k! \tag{2.3}$$

Taking into account that for the determinantal random point fields

$$T_k(\nu_t) = (-1)^k \cdot (k-1)!\ \mathrm{Trace}(A_t)^k$$

the last two equations imply (2.1). The next lemma allows us to estimate $\mathrm{Trace}(A_t - A_t^{\ell})$.

Lemma 2. $0 \leqslant \mathrm{Trace}(A_t - A_t^{\ell}) \leqslant (\ell-1) \cdot \mathrm{Trace}(A_t - A_t^2)$.

The proof is elementary: $0 \leqslant \mathrm{Trace}(A_t - A_t^{\ell}) = \sum_{j=1}^{\ell-1} \mathrm{Trace}(A_t^j - A_t^{j+1}) \leqslant \sum_{j=1}^{\ell-1} \|A_t^{j-1}\| \cdot \mathrm{Trace}(A_t - A_t^2) \leqslant (\ell-1) \cdot \mathrm{Trace}(A_t - A_t^2)$. ∎

As a corollary of the lemmas we have $C_\ell(\nu_t) = \underline{0}(C_2(\nu_t))$ for any $\ell \geqslant 2$. Since $C_2(\nu_t) = \mathrm{Trace}(A_t - A_t^2) \xrightarrow{t\to\infty} +\infty$, we conclude that for $\ell > 2$, $C_\ell((\nu_t - E\nu_t)/\sqrt{\mathrm{Var}_t\, \nu_t}) = C_\ell(\nu_t)/((C_2(\nu_t))^{\ell}/2) \xrightarrow{t\to\infty} 0$.

At the same time the first two cumulants of the normalized random variable are 0 and 1, respectively. The convergence of cumulants implies the convergence of moments to the moments of $N(0,1)$. The theorem is proven. ∎

To generalize the Costin–Lebowitz theorem to the case of several intervals we consider $I_t^{(m)}$, $m=1,..., s$, and define $\nu_t^{(m)} = \#(I_t^{(m)})$. The equation

$$\sum C_{k_1,..., k_s}(iz_1)^{k_1}/k_1! \cdots (iz_s)^{k_s}/k_s! = \log(E_t \exp(i(z_1 \nu_t^{(1)} + \cdots + z_s \nu_t^{(s)}))) \tag{2.4}$$

defines the joint cumulants of $\nu_t^{(m)}$'s.

Proposition 1. $C_{k_1,..., k_s}(\nu_t^{(1)},..., \nu_t(s))$ is equal to a linear combination of the traces

$$\mathrm{Trace}\, K_t \cdot \chi_{I_t}^{(\cdots)} \cdot K_t \cdot \chi_{I_t}^{(\cdots)} \cdots K_t \cdot \chi_{I_t}^{(\cdots)}$$

with some combinatorial coefficients (irrelevant for our purposes), such that for any nonzero k_j at least one indicator in each term of the linear combination is the indicator of $I_t^{(j)}$.

The proof immediately follows from the analogue of (2.4) for the case of several intervals.

In the next section we will apply these results to prove Theorem 1.

3. PROOF OF THEOREM 1

For the most part of the section we will study the case of one interval $(-T, +\infty)$. We start by recalling the asymptotic expansion of Airy function for large positive and negative y (see [Ol]).

$$\mathscr{A}_i(|y|) \sim \frac{e^{-\pi z}}{2\pi^{1/2} \cdot |y|^{1/4}} \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{u_s}{z^s} \tag{3.1}$$

$$\mathscr{A}'_i(|y|) \sim \frac{|y|^{1/4} \cdot e^{-\pi z}}{2\pi^{1/2}} \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{v_s}{z^s} \tag{3.2}$$

$$\mathscr{A}_i(-|y|) \sim \frac{1}{\pi^{1/2} \cdot |y|^{1/4}} \cdot \left\{ \cos\left(\pi z + \frac{\pi}{4}\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{u_{2s}}{z^{2s}} + \sin\left(\pi z + \frac{\pi}{4}\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{u_{2s+1}}{z^{2s+1}} \right\} \tag{3.3}$$

$$\mathscr{A}'_i(-|y|) \sim \frac{|y|^{1/4}}{\pi^{1/2}} \cdot \left\{ \sin\left(\pi z + \frac{\pi}{4}\right) \cdot \sum_{s=0}^{\infty} (-1)^{s+1} \cdot \frac{v_{2s}}{z^{2s}} - \cos\left(\pi z + \frac{\pi}{4}\right) \cdot \sum_{s=0}^{\infty} (-1)^{s+1} \cdot \frac{v_{2s+1}}{z^{2s+1}} \right\} \tag{3.4}$$

where $z = (2/3\pi)\, y \cdot |y|^{1/2}$; $u_0 = v_0 = 1$, and $u_s = ((2s+1) \cdot (2s+3) \cdot \cdots \cdot (6s-1))/(216 \cdot \pi)^2 \cdot s!$, $v_s = -((6s+1)/(6s-1))\, u_s$, $s \geqslant 1$. In particular, as a consequence of (3.1)–(3.4) one has (1.17). It follows from (3.1)–(3.4) together with the boundedness of $\mathscr{A}_i(y)$, $\mathscr{A}'_i(y)$ on any compact set that for any fixed $a \in \mathbb{R}^1$ all moments of $\#((a, +\infty))$ are finite. Therefore it is enough to establish the CLT for $\#((-T, a))$. We choose $a = -(3\pi/2)^{2/3}$ ($y = -(3\pi/2)^{2/3}$ corresponds to $z = -1$). We are going to show that the conditions of the theorem from Section 2 are satisfied by $\chi_{(-T,a)} K \cdot \chi_{(-T,a)}$, where, as above, this notation is reserved for the integral operator with the kernel $\chi_{(-T,a)}(x) K(x, y) \cdot \chi_{(-T,a)}(y)$.

Lemma 3. $0 \leqslant \chi_{(-T,a)} K \cdot \chi_{(-T,a)} \leqslant 1$ and $\chi_{(-T,a)} K \cdot \chi_{(-T,a)}$ is trace class. ∎

The kernel $K(y_1, y_2) = (\mathscr{A}_i(y_1) \cdot \mathscr{A}_i'(y_2) - \mathscr{A}_i'(y_1) \cdot \mathscr{A}_i(y_2))/(y_1 - y_2)$ was obtained from $K_n(x_1, x_2) = \sqrt{2n} \cdot \sum_{\ell=0}^{n-1} \psi_\ell(\sqrt{2n}\, x_1) \cdot \psi_\ell(\sqrt{2n} \cdot x_2)$ after rescaling $x_i = 1 + (y_i/2n^{2/3})$, $i = 1, 2$, and taking the limit $n \to \infty$. The convergence is uniform on compact sets. As a projection operation K_n satisfies $0 \leqslant K_n \leqslant 1$. We immediately conclude that $\chi_{(-T, a)} K \cdot \chi_{(-T, a)}$ satisfies the same inequalities and since the kernel is continuous and non-negative definite the operator is trace class (see, e.g., [GK] or [RS, Section XI.4]). Now the main step of the proof consists of

Proposition 2.

$$\operatorname{Var}\left(\#\left(-T, -\left(\frac{3\pi}{2}\right)^{2/3}\right)\right) \sim \frac{11}{12\pi^2} \log T + \underline{0}(1)$$

Proof. We introduce the chanA5 of variables

$$z_i = \frac{2}{3\pi}\, y_i \cdot |y_i|^{1/2} \tag{3.5}$$

and agree to use the notations $Q(z_1, z_2)$, $q_k(z_1,..., z_k)$, $k = 1, 2,...$, for the kernel and k-point correlation function of the new random point field obtained after the change of coordinates. It follows from (1.17) that $q_1(z) \sim 1 + (\cos 2\pi z/6\pi z) + \underline{0}(z^{-2})$ for $z \to -\infty$, so we see that the configuration (z_i) is equally spaced at $-\infty$.

The kernel $Q(z_1, z_2)$ is defined by

$$\begin{aligned} Q(z_1, z_2) &= \frac{\pi}{|y_1|^{1/4} \cdot |y_2|^{1/4}} \cdot K(y_1, y_2) \\ &= \frac{\pi}{|y_1|^{1/4} \cdot |y_2|^{1/4}} \cdot \frac{\mathscr{A}_i(y_1) \cdot \mathscr{A}_i'(y_2) - \mathscr{A}_i'(y_1) \cdot \mathscr{A}_i(y_2)}{y_1 - y_2} \end{aligned}$$

Formulas (3.3)–(3.4) allow us to represent Q as the sum of six kernels $Q^{(i)}$, $i = 1,..., 6$, with the known asymptotic expansions: $Q(z_1, z_2) = \sum_{i=1}^{6} Q^{(i)}(z_1, z_2)$, where

$$\begin{aligned} Q^{(1)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{2/3} - z_2^{2/3}} \cdot \sin \pi(z_1 - z_2) \\ \cdot \left\{ \sum_{m, n=0}^{\infty} (-1)^{m+n} \cdot u_{2m} \cdot v_{2n} \right. \\ \left. \cdot (z_1^{-2m-1/3} \cdot z_2^{-2n} + z_1^{-2n} \cdot z_2^{-2m-1/3}) \right\} \end{aligned} \tag{3.6}$$

$$Q^{(2)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{2/3} - z_2^{2/3}} \cdot \cos \pi(z_1 + z_2)$$
$$\cdot \left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \cdot u_{2m} \cdot v_{2n} \right.$$
$$\left. \cdot \left(z_1^{-2n} \cdot z_2^{-2m-1/3} - z_1^{-2m-1/3} \cdot z_2^{-2n}\right) \right\} \tag{3.7}$$

$$Q^{(3)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{2/3} - z_2^{2/3}} \cdot 2 \cos\left(\pi z_1 + \frac{\pi}{4}\right) \cdot \cos\left(\pi z_2 + \frac{\pi}{4}\right)$$
$$\cdot \left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n+1} \cdot u_{2m} \cdot v_{2n+1} \right.$$
$$\left. \cdot \left(z_1^{-2m-1/3} \cdot z_2^{-2n-1} - z_1^{-2n-1} \cdot z_2^{-2m-1/3}\right) \right\} \tag{3.8}$$

$$Q^{(4)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{2/3} - z_2^{2/3}} \cdot 2 \sin\left(\pi z_1 + \frac{\pi}{4}\right) \cdot \sin\left(\pi z_2 + \frac{\pi}{4}\right)$$
$$\cdot \left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \cdot u_{2m+1} \cdot v_{2n} \right.$$
$$\left. \cdot \left(z_1^{-2m-4/3} \cdot z_2^{-2n} - z_1^{-2n} \cdot z_2^{-2m-4/3}\right) \right\} \tag{3.9}$$

$$Q^{(5)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{2/3} - z_2^{2/3}} \cdot \sin \pi(z_1 - z_2)$$
$$\cdot \left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n+1} \cdot u_{2m+1} \cdot v_{2n+1} \right.$$
$$\left. \cdot \left(z_1^{-2m-4/3} \cdot z_2^{-2n-1} + z_1^{-2n-1} \cdot z_2^{-2m-4/3}\right) \right\} \tag{3.10}$$

$$Q^{(6)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{2/3} - z_2^{2/3}} \cdot \cos \pi(z_1 + z_2)$$
$$\cdot \left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \cdot u_{2m+1} \cdot v_{2n+1} \right.$$
$$\left. \cdot \left(z_1^{-2n-1} \cdot z_2^{-2m-4/3} - z_1^{-2m-4/3} \cdot z_2^{-2n-1}\right) \right\} \tag{3.11}$$

We denote by $Q^{(i)}_{m,n}(z_1, z_2)$ the (m, n)th term in the asymptotic expansion of $Q^{(i)}(z_1, z_2)$. Then

$$Q^{(1)}_{0,0}(z_1, z_2) = \frac{\sin \pi(z_1 - z_2)}{z_1^{2/3} - z_2^{2/3}} \cdot \frac{1}{3\pi} \cdot (z_1^{-1/3} + z_2^{-1/3})$$
$$= \frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} \cdot \frac{z^{2/3}{}_1 + z_1^{1/3} \cdot z_2^{1/3} + z_2^{2/3}}{3 \cdot z_1^{1/3} \cdot z_2^{1/3}}$$

We note that near the diagonal $Q^{(1)}_{0,0}$ is essentially the sine kernel. We also will need

$$Q^{(2)}_{0,0}(z_1, z_2) = \frac{\cos \pi(z_1 + z_2)}{\pi(z_1 + z_2)} \cdot \frac{z_1^{2/3} - z_1^{1/3} \cdot z_2^{1/3} + z_2^{2/3}}{3 \cdot z_1^{1/3} \cdot z_2^{1/3}}$$

Let us define $S(z_1, z_2) = Q^{(1)}_{0,0}(z_1, z_2) + Q^{(2)}_{0,0}(z_1, z_2)$, $U(z_1, z_2) = Q(z_1, z_2) - S(z_1, z_2)$.

Lemma 4.

$$\int_{-L}^{-1} \int_{-L}^{-1} (Q^{(1)}_{0,0}(z_1, z_2))^2 \, dz_1 \, dz_2 = L - \frac{2}{3\pi^2} \log L + \underline{0}(1) \tag{3.12}$$

Proof. The integral can be written as

$$\frac{1}{9} \int_1^L \int_1^L \left(\frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} \right)^2 \cdot \left(\frac{z_1^{2/3} + z_1^{1/3} \cdot z_2^{1/3} + z_2^{2/3}}{z_1^{1/3} \cdot z_2^{1/3}} \right)^2 dz_1 \, dz_2$$
$$= \frac{2}{9} \cdot \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot \int_1^{L-u} \left[\left(\frac{z+u}{z} \right)^{1/3} + 1 + \left(\frac{z}{z+u} \right)^{1/3} \right]^2 dz \, du$$

We represent the inner integral as

$$\int_1^{L-u} \left[\left(\frac{z+u}{z} \right)^{2/3} + 2 \cdot \left(\frac{z+u}{z} \right)^{1/3} + \left(\frac{z}{z+u} \right)^{2/3} + 2 \cdot \left(\frac{z}{z+u} \right)^{1/3} + 3 \right] dz$$
$$= I_1(u) + I_2(u) + 3 \cdot (L - u - 1) \tag{3.13}$$

with

$$I_1(u) = \int_1^{L-u} \left(\frac{z+u}{z} \right)^{2/3} + 2 \cdot \left(\frac{z+u}{z} \right)^{1/3} dz$$

$$I_2(u) = \int_1^{L-u} \left(\frac{z}{z+u} \right)^{2/3} + 2 \cdot \left(\frac{z}{z+u} \right)^{1/3} dz$$

To calculate $I_1(u)$ we introduce the change of variables $t=((z+u)/z)^{1/3}$. Then

$$I_1(u)=\int_{(L/(L-u))^{1/3}}^{(1+u)^{1/3}}(t^2+2t)\cdot\left(\frac{u}{1-t^3}\right)'\,dt$$
$$=((1+u)^{2/3}+2\cdot(1+u)^{1/3})\cdot(-1)$$
$$-\left(\left(\frac{L}{L-u}\right)^{2/3}+2\cdot\left(\frac{L}{L-u}\right)^{1/3}\right)\cdot(-L+u)$$
$$+u\int_{(L/(L-u))^{1/3}}^{(1+u)^{1/3}}(2t+2)\cdot\frac{1}{t^3-1}\,dt \tag{3.14}$$

We have

$$u\cdot\int_{(L/(L-u))^{1/3}}^{(1+u)^{1/3}}(2t+2)\cdot\frac{1}{t^3-1}\,dt$$
$$=u\cdot\int_{(L/(L-u))^{1/3}}^{(1+u)^{1/3}}\frac{4}{3}\cdot\left(\frac{1}{t-1}-\frac{t+1/2}{t^2+t+1}\right)dt$$
$$=\frac{4u}{3}\cdot\log[(1+u)^{1/3}-1]-\frac{4u}{3}\log\left[\left(\frac{L}{L-u}\right)^{1/3}-1\right]$$
$$-\frac{2u}{3}\log[(1+u)^{2/3}+(1+u)^{1/3}+1]$$
$$+\frac{2u}{3}\log\left[\left(\frac{L}{L-u}\right)^{2/3}+\left(\frac{L}{L-u}\right)^{1/3}+1\right] \tag{3.15}$$

and

$$I_1(u)=-(1+u)^{2/3}-2(1+u)^{1/3}+L\cdot\left(1-\frac{u}{L}\right)^{1/3}+2L\cdot\left(1-\frac{u}{L}\right)^{2/3}$$
$$+\frac{4u}{3}\log[(1+u)^{1/3}-1]-\frac{4u}{3}\log\left[\left(\frac{L}{L-u}\right)^{1/3}-1\right]$$
$$+\frac{2u}{3}\log\left[\left(\frac{L}{L-u}\right)^{2/3}+\left(\frac{L}{L-u}\right)^{1/3}+1\right]$$
$$-\frac{2u}{3}\log[(1+u)^{2/3}+(1+u)^{1/3}+1] \tag{3.16}$$

In a similar way in order to calculate

$$I_2(u) = \int_{1+u}^{L} \left(\frac{z-u}{z}\right)^{2/3} + 2 \cdot \left(\frac{z-u}{z}\right)^{1/3} dz$$

we consider the change of variables $t = ((z-u)/z)^{1/3}$. Then

$$\begin{aligned}
I_2(u) &= \int_{(1+u)^{-1/3}}^{(L/(L-u))^{-1/3}} (t^2+2t) \cdot \left(\frac{u}{1-t^3}\right)' dt \\
&= \left(\left(\frac{L-u}{L}\right)^{2/3} + 2 \cdot \left(\frac{L-u}{L}\right)^{1/3}\right) \cdot L \\
&\quad - ((1+u)^{-2/3} + 2 \cdot (1+u)^{-1/3}) \cdot (1+u) \\
&\quad + u \cdot \int_{(1+u)^{-1/3}}^{(L/(L-u))^{-1/3}} (2t+2) \cdot \frac{1}{t^3-1}\, dt \\
&= -(1+u)^{1/3} - 2(1+u)^{2/3} + L \cdot \left(1 - \frac{u}{L}\right)^{2/3} + 2L \cdot \left(1 - \frac{u}{L}\right)^{1/3} \\
&\quad + \frac{4}{3} u \cdot \log\left[1 - \left(\frac{L-u}{L}\right)^{1/3}\right] - \frac{4}{3} u \cdot \log[1-(1+u)^{-1/3}] \\
&\quad - \frac{2}{3} u \cdot \log\left[\left(\frac{L-u}{L}\right)^{2/3} + \left(\frac{L-u}{L}\right)^{1/3} + 1\right] \\
&\quad + \frac{2}{3} u \cdot \log[(1+u)^{-2/3} + (1+u)^{-1/3} + 1].
\end{aligned} \tag{3.17}$$

It follows from (3.13) that

$$\begin{aligned}
&\int_{-L}^{-1} \int_{-L}^{-1} (Q_{0,0}^{(1)}(z_1, z_2))^2\, dz_1\, dz_2 \\
&\quad = \frac{2}{9} \cdot \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot (I_1(u) + I_2(u) + 3L - 3u - 3)\, du.
\end{aligned} \tag{3.18}$$

We note that

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot L\, du = \frac{L}{2} + \underline{0}(1), \tag{3.19}$$

$$\int_0^{L-1} \frac{(\sin \pi u)^2}{\pi u}\, du = \frac{1}{2\pi^2} \log L + \underline{0}(1), \tag{3.20}$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot L \cdot \left(1-\frac{u}{L}\right)^{1/3} du$$

$$= L \cdot \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot \left(1-\frac{1}{3}\frac{u}{L}+\underline{0}\left(\frac{u^2}{L^2}\right)\right) du$$

$$= \frac{L}{2} - \frac{1}{6\pi^2} \log L + \underline{0}(1), \tag{3.21}$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot L \cdot \left(1-\frac{u}{L}\right)^{2/3} du$$

$$= \frac{L}{2} - \frac{1}{3\pi^2} \log L + \underline{0}(1). \tag{3.22}$$

The combined contribution of all other terms to (3.18) is $\underline{0}(1)$. Indeed,

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot (1+u)^{2/3} = \underline{0}(1),$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot (1+u)^{1/3} = \underline{0}(1),$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot \frac{4u}{3} \log[(1+u)^{1/3}-1]\, du - \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2$$

$$\cdot \frac{2u}{3} \cdot \log[(1+u)^{2/3}+(1+u)^{1/3}+1]\, du$$

$$= \frac{2}{3\pi^2} \int_0^{L-1} \frac{\sin^2 \pi u}{\pi^2 u} \cdot \log\left[\frac{((1+u)^{1/3}-1)^2}{(1+u)^{2/3}+(1+u)^{1/3}+1}\right] du$$

$$= \frac{2u}{3\pi^2} \int_0^{L-1} \frac{\sin^2 \pi u}{u} \cdot \underline{0}(u^{-1/3})\, du = \underline{0}(1),$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot \frac{4u}{3} \log[1-(1+u)^{-1/3}]\, du = \underline{0}(1),$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot \frac{2u}{3} \log[(1+u)^{-2/3}+(1+u)^{-1/3}+1]\, du = \underline{0}(1).$$

The last expression to consider is

$$
\begin{aligned}
&-\int_0^{L-1}\left(\frac{\sin \pi u}{\pi u}\right)^2\cdot\frac{4}{3}\,u\cdot\log\left[\left(\frac{L}{L-u}\right)^{1/3}-1\right]du+\int_0^{L-1}\left(\frac{\sin \pi u}{\pi u}\right)^2\\
&\qquad\cdot\frac{2}{3}\,u\cdot\log\left[\left(\frac{L}{L-u}\right)^{2/3}+\left(\frac{L}{L-u}\right)^{1/3}+1\right]du+\int_0^{L-1}\left(\frac{\sin \pi u}{\pi u}\right)^2\cdot\frac{4}{3}\,u\\
&\qquad\cdot\log\left[1-\left(\frac{L-u}{L}\right)^{1/3}\right]du-\int_0^{L-1}\left(\frac{\sin \pi u}{\pi u}\right)^2\cdot\frac{2}{3}\,u\\
&\qquad\cdot\log\left[\left(\frac{L-u}{L}\right)^{2/3}+\left(\frac{L-u}{L}\right)^{1/3}+1\right]du\\
&\quad=\frac{2}{3\pi^2}\int_0^{L-1}\frac{\sin^2\pi u}{u}\cdot\log\left[\frac{(1-((L-u)/L)^{1/3})}{((L/(L-u))^{1/3}-1)}\right.\\
&\qquad\times\left.\frac{((L/(L-u))^{2/3}+(L/(L-u))^{1/3}+1)}{((L-u)/L)^{2/3}+((L-u)/L)^{1/3}+1)}\right]du\\
&\quad=\frac{2}{3\pi^2}\int_0^{L-1}\frac{\sin^2\pi u}{u}\cdot\log\left(\frac{L}{L-u}\right)du\\
&\quad=\frac{1}{3\pi^2}\cdot\int_1^{L-1}\frac{1}{u}\cdot\left(-\log\left(1-\frac{u}{L}\right)\right)du+\underline{0}(1)\\
&\quad=\sqrt{\frac{1}{3\pi^2}}\cdot\int_1^{L-1}\frac{1}{u}\cdot\sum_{k=1}^{\infty}\frac{1}{k}\cdot\left(\frac{u}{L}\right)^k du+\underline{0}(1)\\
&\quad=\frac{1}{3\pi^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}+\underline{0}(1)=\underline{0}(1).
\end{aligned}
$$

Combining all the above integrals and looking specifically for the contributions from (3.19)–(3.22) we obtain

$$
\begin{aligned}
&\int_{-L}^{-1}\int_{-L}^{-1}(Q_{0,0}^{(1)}(z_1,z_2))^2\,dz_1\,dz_2\\
&\qquad=\frac{2}{9}\left(3\cdot\frac{L}{2}-3\cdot\frac{1}{2\pi^2}\log L+\frac{L}{2}-\frac{1}{6\pi^2}\cdot\log L+2\cdot\frac{L}{2}-2\cdot\frac{1}{3\pi^2}\cdot\log L\right.\\
&\qquad\qquad\left.+\frac{L}{2}-\frac{1}{3\pi^2}\log L+2\cdot\frac{L}{2}-2\cdot\frac{1}{6\pi^2}\log L+\underline{0}(1)\right)\\
&\qquad=L-\frac{2}{3\pi^2}\log L+\underline{0}(1).
\end{aligned}
$$

In the next lemma we evaluate the integrals involving $Q^{(2)}_{0,0}$.

Lemma 5.

(a) $\int_{-L}^{-1}\int_{-L}^{-1}(Q^{(2)}_{0,0}(z_1, z_2))^2\,dz_1\,dz_2 = (1/18\pi^2)\log L + \underline{0}(1)$.

(b) $\int_{-L}^{-1}\int_{-L}^{-1} Q^{(1)}_{0,0}(z_1, z_2)\cdot Q^{(2)}_{0,0}(z_1, z_2)\,dz_1\,dz_2 = \underline{0}(1)$.

Proof. The integral in part (a) can be written as

$$\frac{1}{9}\int_1^L\int_1^L\left(\frac{\cos\pi(z_1+z_2)}{\pi(z_1+z_2)}\right)^2\cdot\left(\frac{z_1^{2/3}-z_1^{1/3}\cdot z_2^{1/3}+z_2^{2/3}}{z_1^{1/3}\cdot z_2^{1/3}}\right)^2 dz_1\,dz_2$$

$$=\frac{1}{9}\int_2^{L+1}\left(\frac{\cos\pi u}{\pi u}\right)^2\cdot\int_1^u\left[\left(\frac{z}{u-z}\right)^{1/3}-1+\left(\frac{u-z}{z}\right)^{1/3}\right]^2 dz\,du+\underline{0}(1).$$

We denote the inner integral by

$$I_3(u):=\int_1^u\left[\left(\frac{z}{u-z}\right)^{2/3}-2\left(\frac{z}{u-z}\right)^{1/3}+\left(\frac{u-z}{z}\right)^{2/3}-2\left(\frac{u-z}{z}\right)^{1/3}+1\right]dz$$

$$=2\cdot\int_1^u\left(\frac{u-z}{z}\right)^{2/3}-2\left(\frac{u-z}{z}\right)^{1/3}dz+(u-1)$$

$$=2\cdot\int_{(u-1)^{1/3}}^0(t^2-2t)\cdot\left(\frac{u}{t^3+1}\right)' dt+(u-1)$$

$$=6u\cdot\int_0^{(u-1)^{1/3}}\frac{(t^2-2t)\cdot t^2}{(t^3+1)^2}dt+(u-1)$$

$$=u\left(1+6\int_0^\infty\frac{t^4-2t^3}{(t^3+1)^2}dt\right)+\underline{0}(u^{2/3}).$$

Taking into account $\int_0^\infty((t^4-2t^3)/(t^3+1)^2)\,dt=0$, we obtain $I_3(u)=u+\underline{0}(u^{2/3})$, and

$$\frac{1}{9}\int_2^{L+1}\left(\frac{\cos\pi u}{\pi u}\right)^2\cdot(u+\underline{0}(u^{2/3}))\,du=\frac{1}{18\pi^2}\log L+\underline{0}(1)$$

Now let us consider the integral in part (b):

$$\frac{1}{9}\int_1^L\int_1^L \frac{\cos\pi(z_1+z_2)}{\pi(z_1+z_2)}\cdot\frac{\sin\pi(z_1-z_2)}{\pi\cdot(z_1-z_2)}$$
$$\times\left(\frac{z_1^{2/3}+z_1^{1/3}\cdot z_2^{1/3}+z_2^{2/3}}{z_1^{1/3}\cdot z_2^{1/3}}\right)\cdot\left(\frac{z_1^{2/3}-z_1^{1/3}\cdot z_2^{1/3}+z_2^{2/3}}{z_1^{1/3}\cdot z_2^{1/3}}\right)dz_1\,dz_2$$
$$=\frac{1}{9}\cdot\int_2^{2L}\frac{\cos\pi u}{\pi u}\cdot\int_0^{u-2}\frac{\sin\pi v}{\pi v}\cdot\left[\left(\frac{u+v}{u-v}\right)^{2/3}+\left(\frac{u-v}{u+v}\right)^{2/3}+1\right]dv\,du$$

It is not difficult to see that because of the oscillations of trigonometric functions this integral is of order of constant. To show this we write the inner integral as $I_4(u)+I_5(u)+I_6(u)$, where

$$I_4(u)=\int_0^{u-2}\frac{\sin\pi v}{\pi v}\cdot\left(\frac{u+v}{u-v}\right)^{2/3}dv$$

$$I_5(u)=\int_0^{u-2}\frac{\sin\pi v}{\pi v}\cdot\left(\frac{u+v}{u-v}\right)^{1/3}dv$$

$$I_6(u)=\int_0^{u-2}\frac{\sin\pi v}{\pi v}\,dv$$

Integration by parts gives $I_6(u)=\frac{1}{2}+\underline{0}(u^{-1})$. Consider now

$$I_4(u)=\int_0^{u-2u^{1/4}}\left(\frac{\sin\pi v}{\pi v}\right)\cdot\left(\frac{u+v}{u-v}\right)^{2/3}dv$$
$$+\int_{u-2u^{1/4}}^{u-2}\left(\frac{\sin\pi v}{\pi v}\right)\cdot\left(\frac{u+v}{u-v}\right)^{2/3}dv \qquad (3.23)$$

The second integral in (3.23) is $\underline{0}(u^{1/4}\cdot u^{-1}\cdot u^{2/3})=\underline{0}(u^{-1/12})$. As for the first one we introduce $\chi([v]\text{ even})$, an indicator of the set where the integer part of v is even, and write the integral in the following form

$$\int_0^{u-2u^{1/4}}\frac{\sin\pi v}{\pi}\cdot\chi([v]\text{ even})$$
$$\cdot\left(\frac{1}{v}\cdot\left(\frac{u+v}{u-v}\right)^{2/3}-\frac{1}{v+1}\left(\frac{u+v+1}{u-v-1}\right)^{2/3}\right)dv+\underline{0}(u^{-1/2})$$

The absolute value of the last expression is estimated from above by

$$\int_0^{u-2u^{1/4}} \frac{|\sin \pi v|}{\pi} \cdot \left(\frac{v\cdot(u-v)^{2/3}-(u-v-1)^{2/3}\cdot(v+1)}{v\cdot(v+1)\cdot(u-v)^{2/3}\cdot(u-v-1)^{2/3}}\right) dv$$

$$= \int_0^{u-2u^{1/4}} \frac{|\sin \pi v|}{\pi} \cdot \left(\frac{(u-v)^{2/3}-(u-v-1)^{2/3}}{(v+1)\cdot(u-v)^{2/3}\cdot(u-v-1)^{2/3}}\right) dv$$

$$+ \int_0^{u-2u^{1/4}} \frac{|\sin \pi v|}{\pi} \cdot \left(\frac{1}{v\cdot(v+1)\cdot(u-v)^{2/3}}\right) dv$$

$$= \underline{O}(u^{-1/6})$$

Therefore $I_4(u) = \underline{O}(u^{-1/6})$, and similarly $I_5(u) = \underline{O}(u^{-1/12})$. As a result

$$\frac{1}{9}\int_2^{2L} \frac{\sin \pi u}{\pi u}\,(I_4(u)+I_5(u)+I_6(u))\,du$$

$$= \frac{1}{18}\int_2^{2L} \frac{\sin \pi u}{\pi u}\,du + \frac{1}{9}\int_2^{2L} \frac{\sin \pi u}{\pi u} \cdot \underline{O}(u^{-1/12})\,du = \underline{O}(1)$$

Lemma 5 is proven. ∎

As a result of the last two lemmas we have

$$\int_{-L}^{-1}\int_{-}^{-1} S^2(z_1, z_2)\,dz_1\,dz_2 = L - \frac{2}{3\pi^2}\log L + \frac{1}{18\pi^2}\log L + \underline{O}(1)$$

$$= L - \frac{11}{18\pi^2}\log L + \underline{O}(1)$$

Remember that S was defined as $S(z_1, z_2) = Q^{(1)}_{0,0}(z_1, z_2) + Q^{(2)}_{0,0}(z_1, z_2)$. To finish the proof of the CLT for $\#(-T, +\infty)$ we just need to show that the remainder term $U(z_1, z_2) = Q(z_1, z_2) - S(z_1, z_2)$ is negligible in the following sense:

Lemma 6. (a) $\int_{-L}^{-1}\int_{-L}^{-1} U^2(z_1, z_2)\,dz_1\,dz_2 = \underline{O}(1)$
(b) $\int_{-L}^{-1}\int_{-L}^{-1} U(z_1, z_2)\cdot S(z_1, z_2)\,dz_1\,dz_2 = \underline{O}(1)$
(c) $\int_{-L}^{-1} U(z, z)\,dz = \underline{O}(1)$

Proof. We shall establish part (a). The estimates (b) and (c) can be obtained in a similar manner. Repeating the calculations of the last two lemmas, it is easy to see that for any fixed indices (i, m, n) such that $(i-1)(i-2)+m+n>0$, we have

$$\int_{-L}^{-1}\int_{-L}^{-1}(Q_{m,n}^{(i)}(z_1, z_2))^2\,dz_1\,dz_2=\underline{0}(1)$$

Let us now choose N to be sufficiently large and write $U(z_1, z_2)= U_N(z_1, z_2)+V_N(z_1, z_2)$, where

$$U_N(z_1, z_2)=\sum_{i=1}^{2}\sum_{0<m+n\leqslant N}Q_{m,n}^{(i)}(z_1, z_2)+\sum_{i=3}^{6}\sum_{0\leqslant m+\leqslant N}Q_{m,n}^{(i)}(z_1, z_2)$$

We see that $\int_{-L}^{-1}\int_{-L}^{-1}(U_N(z_1, z_2))^2\,dz_1\,dz_2=\underline{0}(1)$. Asymptotic formulas (3.6)–(3.11) imply that

$$\begin{aligned}|V_N(z_1, z_2)|&\leqslant\frac{\text{const}_N}{|z_1^{2/3}-z_2^{2/3}|}\cdot(z_1^{-2N}+z_n^{-2N})\\&\leqslant\frac{\text{const}_N}{|z_1-z_2|}\cdot\frac{3}{2}\cdot(z_1^{1/3}+z_2^{1/3})\cdot(z_1^{-2N}+z_2^{-2N})\end{aligned}\tag{3.24}$$

It follows from (3.24) that if we choose $N\geqslant 2$ then

$$\iint_{|z_1-z_2|\geqslant(1/z_2^2)}(V_n(z_1, z_2))^2\,dz_1\,dz_2=\underline{0}(1)\tag{3.25}$$

(The integration in (3.25) is over the subset of $[1, L]\times[1, L]$). Indeed, to estimate the integral over z_2 we write

$$\begin{aligned}\int_{|z_1-z_2|\geqslant(1/z_2^2)}\left(\frac{z_1}{(z_1-z_2)}\cdot\frac{1}{z_2^{2N}}\right)^2dz_2&\leqslant\int_1^L\frac{z_2^4}{(z_1-z_2)^2+1}\cdot z_1^{2/3}\cdot\frac{1}{z_2^{4N}}\,dz_2\\&\leqslant\int_1^L\frac{z_1^{2/3}}{(z_1-z_2)^2+1}\cdot z_2^{-4}\,dz_2=\underline{0}(z_1^{-4/3})\end{aligned}$$

Integrating over z_1 we arrive at (3.25). To integrate V_N^2 near the diagonal we observe that kernels $Q(z_1, z_2)$, $Q_{m,n}^{(i)}(z_1, z_2)$ are bounded in $[1, +\infty)\times[1, +\infty)$; therefore there exists some const'_N such that $|V_N(z_1, z_2)|\leqslant\text{const}'_N$ and $\iint_{|z_1-z_2|\leqslant(1/z_2^2)}(V_N(z_1, z_2))^2\,dz_1\,dz_2\leqslant\int_1^{+\infty}((2\cdot\text{const}'_N)/z_2^2)\,dz_2=\underline{0}(1)$. Lemma 6 is proven. ∎

Taking $L = (2/3\pi)\, T^{3/2}$ we deduce from Lemmas 4–6 and (3.5) that

$$\begin{aligned}
\operatorname{Var}&\left(\#\left(y_i \in \left(-T, -\left(\frac{3\pi}{2}\right)^{2/3}\right)\right)\right)\\
&= \int_{-T}^{-(3\pi/2)^{2/3}} K(y, y)\, dy - \int_{-T}^{-(3\pi/2)^{2/3}} \int_{-T}^{-(3\pi/2)^{2/3}} K^2(y_1, y_2)\, dy_1\, dy_2\\
&= \frac{2}{3\pi} T^{2/3} + \underline{0}(1) - \frac{2}{3\pi} T^{2/3} + \frac{11}{18\pi^2} \log\left(\frac{2}{3\pi} T^{3/2}\right) + \underline{0}(1)\\
&= \frac{11}{12\pi^2} \log T + \underline{0}(1)
\end{aligned}$$

This finishes the proof of Proposition 2 as well as the proof of the CLT for $\#(-T, +\infty)$.

In a very similar way one proves the CLT for arbitrary $\nu_k(T) = \#((-kT, -(k-1)\, T))$, $k > 1$. To prove the result for the joint distribution of $\{\nu_k(T)\}$ we note that the decay of $K(x, y)$ off the diagonal implies

$$\begin{aligned}
C_{1,1}(\nu_k(T), \nu_l(T)) &= \operatorname{Cov}(\nu_k(T), \nu_l(T))\\
&= \int_{-kT}^{-(k-1)\,T} \int_{-lT}^{(l-1)\,T} K^2(x_1, x_2)\, dx_1\, dx_2\\
&= -\operatorname{Trace} \chi_{[-lT, -(l-1)\,T)} \cdot K \cdot \chi_{[-kT, -(k-1)\,T)} \cdot K\\
&= \underline{0}(1) \qquad \text{if} \quad |k-l| > 1
\end{aligned}$$

This together with

$$\operatorname{Var}\left(\sum_{l=1}^{k} \nu_l(T)\right) = 11/(12\pi^2) \log T + \underline{0}(1)$$

$$\operatorname{Var}(\nu_k(T)) = 11/(12\pi^2) \log T + \underline{0}(1), \qquad k = 1, 2, \ldots$$

implies that

$$\begin{aligned}
C_{1,1}(\nu_k(T), \nu_l(T)) &= \operatorname{Cov}(\nu_k(T), \nu_l(T))\\
&= -11/(24\pi^2) \log T + \underline{0}(1)
\end{aligned}$$

for $|k-l| = 1$. Therefore as $T \to \infty$.

$$E\, \frac{\nu_k(T) - E\nu_k(T)}{\sqrt{\operatorname{Var} \nu_k(T)}} \cdot \frac{\nu_l(T) - E\nu_l(T)}{\sqrt{\operatorname{Var} \nu_l(T)}} \to \delta_{k,l} - 1/2\, \delta_{k,l-1} - 1/2\, \delta_{k,l+1}$$

To take care of the joint cumulants of higher order it is enough to prove

Lemma 7. Let at least two indices in $(k_1,..., k_s)$ are non-zero. Then

$$C_{k_1,..., k_s}(v_1(T),..., v_s(T)) = \underline{0}(\log T)$$

Proof. According to Proposition 1 Lemma 7 follows from

Lemma 8.

$$\text{Trace}\, \chi_{l_1} K \chi_{l_2} K \cdots K \chi_{l_s} K \chi_{l_1} = \underline{0}(\log T)$$

where χ_{l_j} are the indicators of the intervals $(-l_j T, -(l_j - 1)\, T]$, $l_j \in Z^1_+$ and at least two intervals are disjoint.

Proof. This has been already established for $s = 2$. Let $s > 2$. Since not all indices coincide by cyclicity of the trace we may assume $l_1 \neq l_2$. Now if $l_1 = l_3$ we can use the positivity of $\chi_{l_1} K \chi_{l_2} K \chi_{l_1}$ to write

$$\begin{aligned} |\text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1} K \chi_{l_4} \cdots K \chi_{l_s} K \chi_{l_1})| &\leqslant \text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) \cdot \|K \chi_{l_4} \cdots K \chi_{l_s} K \chi_{l_1}\| \\ &\leqslant \text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) \end{aligned}$$

where we used $\|K\| \leqslant 1$, $\|\chi_{l_j}\| \leqslant 1$. Since $\text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) = \underline{0}(\log T)$ Lemma 8 is proven when $l_1 = l_3 \neq l_2$.

If $l_1 \neq l_3$ one more trick is needed. Let us denote

$$D_1 = \chi_{l_1} K \chi_{l_2}, \qquad D_2 = K \chi_{l_3} K \cdots \chi_{l_s} K \chi_{l_1}$$

Then

$$\begin{aligned} &\text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_3} \cdots \chi_{l_s} K \chi_{l_1}) \\ &\quad = \text{Trace}(D_1 D_2) \leqslant (\text{Trace}(B_1 B_1^*))^{1/2}\, (\text{Trace}(D_2 D_2^*))^{1/2} \end{aligned}$$

(see [RS, Vol. I, Section VI.6]). As before

$$\text{Trace}(D_1 D_1^*) = \text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) = \underline{0}(\log T)$$

To obtain a similar bound for $\text{Trace}(D_1 D_1^*)$ we define $1 < p \leqslant s$ as the maximal index such that $l_p \neq l_1$. Since we assume in Lemma 8 that there are at least two different indices, such p always exists. Then

$$\begin{aligned} &\text{Trace}(K \chi_{l_3} K \cdots \chi_{l_p} K \chi_{l_1} \cdots K \chi_{l_1} \cdots K \chi_{l_1}) \cdot (K \chi_{l_3} K \cdots \chi_{l_p} K \chi_{l_1} \cdots K \chi_{l_1} \cdots K \chi_{l_1})^* \\ &\quad = \text{Trace}(K \chi_{l_3} K \cdots \chi_{l_p} K \chi_{l_1} \cdots K \chi_{l_1} \cdots K \chi_{l_1}) \\ &\quad \cdot (\chi_{l_1} K \cdots \chi_{l_1} K \cdots K \chi_{l_1} K \chi_{l_p} \cdots K \chi_{l_3} K) \end{aligned}$$

Using the identity $\mathrm{Trace}(D_1 D_2) = \mathrm{Trace}(D_2 D_1)$ where

$$D_1 = K\chi_{l_3} K \cdots \chi_{l_{p-1}} K$$
$$D_2 = \chi_{l_p} K\chi_{l_1} K \cdots \chi_{l_1} K\chi_{l_1} \cdots \chi_{l_1} K\chi_{l_p} K\chi_{l_{p-1}} K \cdots K\chi_{l_3} K$$

we can rewrite and estimate the r.h.s. as

$$\begin{aligned}&|\mathrm{Trace}(\chi_{l_p} K\chi_{l_1} \cdots \chi_{l_1} K \cdots \chi_{l_1} K\chi_{l_p}) \cdot (K\chi_{l_{p-1}} \cdots K\chi_{l_3} KK\chi_{l_3} K \cdots \chi_{l_{p-1}} K)| \\ &\quad \leqslant \mathrm{Trace}(\chi_{l_p} K\chi_{l_1} \cdots K\chi_{l_1} K \cdots \chi_{l_1} K\chi_{l_p}) \cdot \|K\chi_{l_{p-1}} \cdots K\chi_{l_3} KK\chi_{l_3} K \cdots \chi_{l_{p-1}} K\|\end{aligned}$$

Here we used the positivity of $\chi_{l_p} K\chi_{l_1} K\chi_{l_1} \cdots K\chi_{l_1} \cdots K\chi_{l_1}$. The norm of the last factor again is not greater than 1. Finally,

$$\begin{aligned}\mathrm{Trace}(\chi_{l_p} K\chi_{l_1} K\chi_{l_1} \cdots K\chi_{l_1} \cdots \chi_{l_1} K\chi_{l_p}) &= \mathrm{Trace}(\chi_{l_1} K\chi_{l_p} K\chi_{l_1} \cdots K\chi_{l_1} \cdots K\chi_{l_1}) \\ &\leqslant \mathrm{Trace}(\chi_{l_1} K\chi_{l_p} K\chi_{l_1}) \cdot 1 = \underline{0}(\log T)\end{aligned}$$

Here we also used cyclicity of the trace. Combining the estimates for Trace $D_1 D_1^*$ and Trace $D_1 D_1^*$ we finish the proof of the Lemmas 7 and 8. ▌

It follows from Lemma 7 that the higher joint cumulants of the normalized random variables go to zero which implies that the limiting distribution function is Gaussian with the known covariance function. Theorem 1 is proven. ▌

4. PROOF OF THEOREM 2 AND SIMILAR RESULTS FOR THE CLASSICAL COMPACT GROUPS

The Bessel kernel has the form (see Section 1)

$$K(y_1, y_2) = \frac{J_\alpha(\sqrt{y_1}) \cdot \sqrt{y_2} \cdot J'_\alpha(\sqrt{y_2}) - \sqrt{y_1} \cdot J'_\alpha(\sqrt{y_1}) \cdot J_\alpha(\sqrt{y_2})}{2(y_1 - y_2)}$$
$$y_1, y_2 \in (0, +\infty), \qquad \alpha > -1 \tag{4.1}$$

The level density is given by

$$\rho_1(y) = K(y, y) = \tfrac{1}{4} J_\alpha(\sqrt{y})^2 - \tfrac{1}{4} J_{\alpha+1}(\sqrt{y}) \cdot J_{\alpha-1}(\sqrt{y}) \tag{4.2}$$

The asymptotic formula for large y is well known in the case of Bessel functions (see asymptotic expansion in (4.5) below). In particular, one can see that

$$\rho_1(y) \sim \frac{1}{2\pi \sqrt{y_1}} \qquad \text{for} \quad y \to +\infty \tag{4.3}$$

The last formula suggests to make the (unfolding) change of variables $z_i = \sqrt{y_i/\pi}$, $i = 1, 2$. The kernel Q corresponding to the new evenly spaced random point field is given by

$$Q(z_1, z_2) = 2\pi y_1^{1/4} \cdot y_2^{1/4} \cdot K(y_1, y_2)$$
$$= z_1^{1/2} z_2^{1/2} \cdot \frac{J_\alpha(\pi z_1) \cdot \pi z_2 \cdot J'_\alpha(\pi z_2) - \pi z_1 \cdot J'_\alpha(\pi z_1) \cdot J_\alpha(\pi z_2)}{z_1^2 - z_2^2} \tag{4.4}$$

The asymptotic expansion of the Bessel function at infinity is given by (see, for example, [Ol])

$$J_\alpha(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cdot \left[\cos\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{A_{2s}(\alpha)}{z^{2s}} \right.$$
$$\left. - \sin\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{A_{2s+1}(\alpha)}{z^{2s+1}}\right] \tag{4.5}$$

where

$$A_0(\alpha) = 1, \qquad A_s(\alpha) = \frac{(4\alpha^2 - 1^2) \cdot (4\alpha^2 - 3^2) \cdots (4\alpha^2 - (2s-1)^2)}{s! \cdot 8^s} \tag{4.6}$$

Similarly,

$$J'_\alpha(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cdot \left[-\sin\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{B_{2s}(\alpha)}{z^{2s}} \right.$$
$$\left. - \cos\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{B_{2s+1}(\alpha)}{z^{2s+1}}\right] \tag{4.7}$$

The coefficients $B_s(\alpha)$ can be obtained from (4.5)–(4.6); for example, $B_0(\alpha) = 1$.

It follows from (4.5)–(4.7) that for $\alpha = \pm\frac{1}{2}$

$$Q(z_1, z_2) = \frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} + \frac{\sin \pi(z_1 + z_2 - \alpha - 1/2)}{\pi(z_1 + z_2)} \tag{4.8}$$

which can be further simplified as

$$\frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} \mp \frac{\sin \pi(z_1 + z_2)}{\pi(z_1 + z_2)}$$

For general values of α a small remainder term appears at the r.h.s. of (4.8). To write the asymptotic expansion, we represent $Q(z_1, z_2)$ as the sum of six kernels: $Q(z_1, z_2) = \sum_{i=1}^{6} Q^{(i)}(z_1, z_2)$, where

$$Q^{(1)}(z_1, z_2) \sim \frac{\sin \pi(z_1 - z_2)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n, m=0}^{\infty} (-1)^{n+m} \cdot A_{2n}(\alpha)\, B_{2m}(\alpha) \cdot (z_1^{1-2n} \cdot z_2^{-2m} + z_1^{-2m} \cdot z_2^{1-2n}) \tag{4.9}$$

$$Q^{(2)}(z_1, z_2) \sim \frac{\sin \pi(z_1 + z_2 - \alpha - 1/2)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n, m=0}^{\infty} (-1)^{n+m} \cdot A_{2n}(\alpha) \cdot B_{2m}(\alpha) \cdot (z_1^{1-2n} \cdot z_2^{-2m} - z_1^{-2m} \cdot z_2^{1-2n}) \tag{4.10}$$

$$Q^{(3)}(z_1, z_2) \sim \frac{2\cos(\pi z_1 - (\pi\alpha/2) - (\pi/4)) \cdot \cos(\pi z_2 - (\pi\alpha/2) - (\pi/4))}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n, m=0}^{\infty} (-1)^{n+m+1} \cdot A_{2n}(\alpha) \cdot B_{2m+1}(\alpha) \cdot (z_1^{-2n} \cdot z_2^{-2m} + z_1^{-2m} \cdot z_2^{-2n}) \tag{4.11}$$

$$Q^{(4)}(z_1, z_2) \sim \frac{2\sin(\pi z_1 - (\pi\alpha/2) - (\pi/4)) \cdot \sin(\pi z_2 - (\pi\alpha/2) - (\pi/4))}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n, m=0}^{\infty} (-1)^{n+m} \cdot A_{2n+1}(\alpha) \cdot B_{2m}(\alpha) \cdot (z_1^{-1-2n} \cdot z_2^{1-2m} + z_1^{1-2m} \cdot z_2^{-1-2n}) \tag{4.12}$$

$$Q^{(5)}(z_1, z_2) \sim \frac{\sin \pi(z_1 - z_2)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n, m=0}^{\infty} (-1)^{n+m+1} \cdot A_{2n+1}(\alpha) \cdot B_{2m+1}(\alpha) \cdot (z_1^{-1-2n} \cdot z_2^{-2m} + z_1^{-2m} \cdot z_2^{-1-2n}) \tag{4.13}$$

$$Q^{(6)}(z_1, z_2) \sim \frac{\sin \pi(z_1 + z_2 - \alpha - 1/2)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n, m=0}^{\infty} (-1)^{n+m+1} \cdot A_{2n+1}(\alpha) \cdot B_{2m+1}(\alpha) \cdot (z_1^{-1-2n} \cdot z_2^{-2m} - z_1^{-2m} \cdot z_2^{-1-2n}) \tag{4.14}$$

The analysis of (4.9)–(4.14) is very similar to Section 3. One can see that the only contribution to the leading term of the variance comes from

$Q^{(1)}_{0,0}(z_1, z_2) + Q^{(2)}_{0,0}(z_1, z_2)$ which is exactly the r.h.s. of (4.8). It can be shown by a straightforward calculation that

$$\int_0^L (Q^{(1)}_{0,0}(z, z) + Q^{(2)}_{0,0}(z, z))\, dz \sim L + \underline{0}(1)$$

$$\int_0^L \int_0^L (Q^{(1)}_{0,0}(z_1, z_2) + Q^{(2)}_{0,0}(z_1, z_2)))^2\, dz_1\, dz_2 \sim L - \frac{1}{2\pi^2} \log L + \underline{0}(1)$$

Taking into account that $L = T^{1/2}/\pi$ we finish the proof. ▮

The kernels $(\sin \pi(x-y)/\pi(x-y)) \pm (\sin \pi(x+y)/\pi(x+y))$ are well known in Random Matrix Theory. For one, they are the kernels of restrictions of the sine-kernel integral operator to the subspaces of even and odd functions and play an important role in spacings distribution in G.O.E. and G.S.E. [Me]. They also appear as the kernels of limiting correlation functions in orthogonal and symplectic groups near $\lambda = 1$ [So1]. Let us start with the even case. Consider the normalized Haar measure on $SO(2n)$. The eigenvalues of matrix M can be arranged in pairs:

$$\exp(i\theta_1), \exp(-i\theta_1), \ldots, \exp(i\theta_n), \exp(-i\theta_n), \qquad 0 \leqslant \theta_1, \theta_2, \ldots, \theta_n \leqslant \pi \tag{4.15}$$

In the rescaled coordinates near the origin $x_i = (2n-1) \cdot (\theta_i/2\pi)$, $i = 1, \ldots, n$, the k-point correlation functions are equal to

$$R_{n,k}(x_1, \ldots, x_k) = \det\left(\frac{\sin \pi(x_i - x_j)}{(2n-1) \cdot \sin(\pi \cdot (x_i - x_j)/(2n-1))} + \frac{\sin \pi(x)\, i + x_j)}{(2n-1) \cdot \sin(\pi(x_i + x_j)/(2n-1))}\right)_{i, j = 1, \ldots, k} \tag{4.16}$$

In the limit $n \to \infty$ the kernel in (4.16) becomes $(\sin \pi(x_i - x_j)/\pi(x_i - x_j)) + (\sin \pi(x_i + x_j)/\pi(x_i + x_j))$. If we consider resealing near arbitrary $0 < \theta < \pi$ the limiting kernel will be just the sine kernel.

Let us now consider the $SO(2n+1)$ case. The first $2n$ eigenvalues of $M \in SO(2n+1)$ can be arranged in pairs as in (4.15). The last one equals 1. In the resealed coordinates near $\theta = 0$

$$x_i = \frac{n\theta_i}{\pi}, \qquad i = 1, \ldots, n$$

the k-point correlation functions are given by the formula

$$R_{n,k}(x_1,..., x_k) = \det\left(\frac{\sin \pi(x_i - x_j)}{2n \cdot \sin(\pi \cdot (x_i - x_j)/2n)} - \frac{\sin \pi(x_i + x_j)}{2n \cdot \sin(\pi \cdot (x_i + x_j)/2n)}\right)_{i,j=1,...,k} \tag{4.17}$$

In the limit $n \to \infty$ the kernel in (4.17) becomes $(\sin \pi(x_i - x_j)/\pi \cdot (x_i - x_j)) - (\sin \pi(x_i + x_j)/\pi \cdot (x_i + x_j))$. If we again consider rescaling near $0 < \theta < \pi$, the limiting kernel appears to be the sine kernel. The case of symplectic group is very similar. Let $M \in Sp(n)$. Rescaled k-point correlation functions are given by

$$R_{n,k}(x_1,..., x_k) = \det\left(\frac{\sin \pi(x_i - x_j)}{(2n+1) \cdot \sin(\pi(x_i - x_j)/(2n+1)} - \frac{\sin \pi(x_i + x_j)}{(2n+1) \cdot \sin(\pi(x_i + x_j)/(2n+1))}\right)_{i,j=1,...,k} \tag{4.18}$$

One can then deduce the following result from the Costin–Lebowitz Theorem.

Theorem 3. Consider the normalized Haar measure on $SO(n)$ or $Sp(n)$. Let $\theta \in [0, \pi)$, δ_n be such that $0 < \delta_n < \pi - \theta - \varepsilon$ for some $\varepsilon > 0$ and $n \cdot \delta_n \to +\infty$. Denote by v_n the number of eigenvalues in $[\theta, \theta + \delta_n]$. We have $Ev_n = (n/\pi) \cdot \delta_n + \underline{0}(1)$,

$$\text{Var } v_n = \begin{cases} \frac{1}{\pi^2} \log(n \cdot \delta_n) + \underline{0}(1) & \text{if } \theta > 0 \\ \frac{1}{2\pi^2} \log(n \cdot \delta_n) + \underline{0}(1) & \text{if } \theta = 0 \end{cases}$$

and the normalized random variable $(v_n - Ev_n)/\sqrt{\text{Var } v_n}$ converges in distribution to the normal law $N(0, 1)$.

Proof. Let $K_n(x, y)$ be the kernel in (4.16), (4.17), or (4.18). We observe that $0 \leqslant \chi_I K_n \chi_I \leqslant Id$ as a composition of projection, Fourier transform, another projection and inverse Fourier transform. To check the asymptotics of Var v_n is an excercise which is left to the reader. ∎

The case of several intervals is treated in a similar fasion.

Theorem 4. Let $\delta_n > 0$ be such that $\delta_n \to 0$, $n\delta_n \to \infty$ and $\nu_{k,n} = \#((k-1)\,\delta_n, k\delta_n])$. Then a sequence of normalized random variables $(\nu_{k,n} - E\nu_{k,n})/\sqrt{\operatorname{Var} \nu_{k,n}}$ converges in distribution to the centalized gaussian sequence $\{\{\xi_k\}_{k=1}^{\infty}\}$ with the covariance function

$$E\xi_k\xi_l = \begin{cases} \delta_{k,l} - 1/2\,\delta_{k,l+1} - 1/2\,\delta_{k,l-1}, & \text{if } \; k > 0, \quad l > 0 \\ \delta_{0,l} - 1/\sqrt{2}\,\delta_{1,l}, & \text{if } \; \theta = 0. \end{cases}$$

Finally we discuss the unitary group $U(n)$.

The eigenvalues of matrix M can be written as:

$$\exp(i\theta_1), \exp(i\theta_2), ..., \exp(i\theta_n) \cdot d, \qquad 0 \leqslant \theta_1, \theta_2, ..., \theta_n \leqslant 2\pi \tag{4.19}$$

In the rescaled coordinates $x_i = n \cdot (\theta_i/2\pi)$, $i = 1, ..., n$, the k-point correlation functions are equal to

$$R_{n,k}(x_1, ..., x_k) = \det \left(\frac{\sin \pi(x_i - x_j)}{n \cdot \sin(\pi \cdot (x_i - x_j)/n)} \right)_{i,\, j = 1, ..., k} \tag{4.20}$$

In the limit $n \to \infty$ the kernel in (4.20) becomes the sine kernel. We finish with the analoques of the last two theorems for $U(n)$.

Theorem 5. Consider the normalized Haar measure on $U(n)$. Let $\theta \in [0, 2\pi)$, δ_n be such that $0 < \delta_n < 2\pi - \theta - \varepsilon$ for some $\varepsilon > 0$ and $n \cdot \delta_n \to +\infty$. Denote by ν_n the number of eigenvalues in $[\theta, \theta + \delta_n]$. We have $E\nu_n = (n/2\pi) \cdot \delta_n$,

$$\operatorname{Var} \nu_n = \frac{1}{\pi^2} \log(n \cdot \delta_n) + \underline{0}(1)$$

and the normalized random variable $(\nu_n - E\nu_n)/\sqrt{\operatorname{Var} \nu_n}$ converges in distribution to the normal law $N(0, 1)$.

Theorem 6. Let $\delta_n > 0$ be such that $\delta_n \to 0$, $n\delta_n \to \infty$ and $\nu_{k,n} = \#((k-1)\,\delta_n, k\delta_n])$. Then a sequence of normalized random variables $(\nu_{k,n} - E\nu_{k,n})/\sqrt{\operatorname{Var} \nu_{k,n}}$ converges in distribution to the centalized gaussian sequence $\{\{\xi_k\}_{k=1}^{\infty}\}$ with the covariance function

$$E\xi_k\xi_l = \delta_{k,l} - 1/2\,\delta_{k,l+1} - 1/2\,\delta_{k,l-1}$$

Remark 5. Results similar to Theorems 5 and 6 in the regime $\delta_n = \delta > 0$ have been also established by K. Wieand [W].

Remark 6. For the results about smooth linear statistics we refer the reader to [DS, Jo1, So3].

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REFERENCES

[BF] T. H. Baker and P. J. Forrester, Finite N fluctuation formulas for random matrices, *J. Stat. Phys.* **88**:1371–1385 (1997).

[Ba] E. Basor, Distribution functions for random variables for ensembles of positive Hermitian matrices, *Commun. Math. Phys.* **188**:327–350 (1997).

[BaW] E. Basor and H. Widom, Determinants of Airy operators and applications to random matrices, *J. Stat. Phys.* **96**(1–2):1–20 (1999).

[BI] P. Bleher and A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and universality in the matrix model, *Ann. of Math.* **150**:185–266 (1999).

[BO1] A. Borodin and G. Olshanski, Point processes and the infinite symmetric group, *Math. Res. Lett.* **5**:799–816 (1998).

[BO2] A. Borodin and G. Olshanski, Z-Measures on partitions, Robinson–Schensted–Knuth correspondence and $\beta=2$ Random Matrix ensembles, available via http://xxx.lanl.gov/abs/math/9905189.

[BOO] A. Borodin, A. Okounkov, and G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, available via http://xxx.lanl.gov/abs/math/9905032.

[BdMK] A. Boutet de Monvel and A. Khorunzhy, Asymptotic distribution of smoothed eigenvalue density. I. Gaussian random matrices. II. Wigner random matrices, to appear in *Rand. Oper. Stoch. Equ.*

[BZ] E. Brézin and A. Zee, Universality of the correlations between eigenvalues of large random matrices, *Nucl. Phys. B* **402**:613–627 (1993).

[Br] B. V. Bronk, Exponential ensemble for random matrices, *J. Math. Phys.* **6**:228–237 (1965).

[CL] O. Costin and J. Lebowitz, Gaussian fluctuations in random matrices, *Phys. Rev. Lett.* **75**(1):69–72 (1995).

[DKMVZ] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Commun. Pure Appl. Math.* **52**:1335–1425 (1999).

[DS] P. Diaconis and M. Shahshahani, On the eigenvalues of random matrices, Studies in Appl. Prob., Essays in honour of Lajos Takacs, *J. Appl. Probab. A* **31**:49–62 (1994).

[E] A. Erdelyi (ed.), *Higher Transcendental Functions*, Vol. 2 (New York, McGraw–Hill, 1953).

[F] P. J. Forrester, The spectrum edge of random matrix ensembles, *Nucl. Phys. B* **402**:709–728 (1994).

[GK] I. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Non-selfadjoint Operators in Hilbert Space* (Amer. Math. Soc., Providence, R.I., 1969).

[Jo1] K. Johansson, On fluctuation of eigenvalues of random Hermitian matrices, *Duke Math. J.* **91**:151–204 (1998).

[Jo2] K. Johansson, Universality of local spacing distribution in certain Hermitian Wigner matrices, available via http://xxx.lanl.gov/abs/math.PR/0006145.

[Jo3] K. Johansson, Shape fluctuations and random matrices, available via http://xxx.lanl.gov/abs/math/9903134.

[Jo4] K. Johansson, Discrete orthogonal polynomials and the Plancherel measure, available via http://xxx.lanl.gov/abs/math/9906120.

[Jo5] K. Johansson, Transversal fluctuations for increasing subsequences on the plane, available via http://xxx.lanl.gov/abs/math/9910146.

[KKP] A. M. Khorunzhy, B. A. Khoruzhenko, and L. A. Pastur, Asymptotic properties of large random matrices with independent entries, *J. Math. Phys.* **37**:5033–5059 (1996).

[L1] A. Lenard, Correlation functions and the uniqueness of the state in classical statistical mechanics, *Commun. Math. Phys.* **30**:35–44 (1973).

[L2] A. Lenard, States of classical statistical mechanical system of infinitely many particles, I, II, *Arch. Rational Mech. Anal.* **59**:219–256 (1975).

[Ma] O. Macchi, The coincidence approach to stochastic point processes, *Advances in Appl. Probability* **7**:83–122 (1975).

[Me] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, New York, 1991).

[NW] T. Nagao and M. Wadati, Correlation functions of random matrix ensembles related to classical orthogonal polynomials, *J. Phys. Soc. Japan* **60**:3298–3322 (1991).

[Ok1] A. Okounkov, Random matrices and random permutations, available via http://xxx.lanl.gov/abs/math/9903176.

[Ok2] A. Okounkov, Infinite wedge and measures on partitions, available via http://xxx.lanl.gov/abs/math/9907127.

[Ol] F. W. J. Oliver, *Asymptotics and Special Functions* (Wellesley, Massachusetts, A.K. Peters, 1997).

[PS] L. A. Pastur and M. Shcherbina, Universality of the local eigenvalue statistics for a class of unitary invariant random matrices, *J. Stat. Phys.* **86**:109–147 (1997).

[PR] M. Plancherel and W. Rotach, Sur les valeurs asymptotiques des polynomes d'Hermite, *Comm. Math. Helv.* **1**:227–254 (1929).

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vols. II–V (Academic Press, New York, 1980, 1975, 1979, 1978).

[SiSo1] Ya. Sinai and A. Soshnikov, A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices, *Funct. Anal. Appl.* **32**(2):114–131 (1998).

[SiSo2] Ya. Sinai and A. Soshnikov, Central limit theorem for traces of large random symmetric matrices with independent entries, *Bol. Soc. Brasil. Mat.* **29**(1):1–24 (1998).

[So1] A. Soshnikov, Level spacing distribution for large random matrices: Gaussian fluctuations, *Ann. of Math.* **148**:573–617 (1998).

[So2] A. Soshnikov, Universality at the edge of the spectrum in Wigner random matrices, *Commun. Math. Phys.* **207**:697–733 (1999).

[So3] A. Soshnikov, Central limit theorem for local statistics in the classical compact groups and related combinatorial identities, available via http://xxx.lanl.gov/abs/math/9908063, to appear in *Ann. Prob.*

[So4] A. Soshnikov, Determinantal random point fields, available via http://xxx.lanl.gov/abs/math/0002099, to appear in *Russian Math. Surv.*

[Sp] H. Spohn, Interacting Brownian particles: A study of Dyson's model, in *Hydrodynamic Behavior and Interacting Particle Systems*, G. Papanicolau, ed. (Springer-Verlag, New York, 1987).

[TW1] C. A. Tracy and H. Widom, Level-spacing distribution and the Airy kernel, *Commun. Math. Phys.* **159**:151–174 (1994).

[TW2] C. A. Tracy and H. Widom, Level spacing distribution and the Bessel kernel, *Commun. Math. Phys.* **161**:289–309 (1994).

[Wig1] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* **62**:548–564 (1955).

[Wig2] E. Wigner, On the distribution of the roots of certain symmetric matrices, *Ann. of Math.* **67**:325–328 (1958).

[W] K. Wieand, Eigenvalue distribution of random matrices in the permutation group and compact Lie groups (Ph.D. thesis, Dept. Math. Harvard, 1998).