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GAUSSIAN APPROXIMATION OF THE DISTRIBUTION OF STRONGLY REPELLING PARTICLES ON THE UNIT CIRCLE

В статье рассматривается модель n упорядоченных сильно отталкивающихся частиц $\{e^{i\theta_j}\}_{j=0}^{n-1}$ с плотностью

$$p(\theta_0,\dots,\theta_{n-1}) = Z_n^{-1}\exp\biggl\{-\frac{\beta}{2}\sum_{j\neq k}\sin^{-2}\biggl(\frac{\theta_j-\theta_k}{2}\biggr)\biggr\}, \qquad \beta>0.$$

Пусть $\theta_j = 2\pi j/n + x_j/n^2 + \mathrm{const}$ таково, что $\sum_{j=0}^{n-1} x_j = 0$. Определим $\zeta_n(2\pi j/n) = x_j/\sqrt{n}$ и продолжим ζ_n кусочно линейным образом на $[0,2\pi]$. Доказывается функциональная сходимость $\zeta_n(t)$ к $\zeta(t) = \sqrt{2/\beta}\,\mathrm{Re}(\sum_{k=1}^{\infty}(1/k)e^{ikt}Z_k)$, где Z_k — независимые одинаково распределенные комплексные стандартные гауссовские случайные величины.

Ключевые слова и фразы: система частиц с сильным отталкиванием, многомерное гауссовское распределение, сходимость конечномерных распределений, функциональная сходимость.

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1. Introduction. The study of random matrix theory (RMT) can be traced back to sample covariance matrices studied by J. Wishart in data analysis in 1920s–1930s. In 1951, E. Wigner associated the energy levels of heavy-nuclei atoms with Hermitian matrices whose components are i.i.d. random variables.

In 1960s, F. Dyson and M. Mehta introduced three archetypal types of matrix ensembles: circular orthogonal ensemble (COE), circular unitary ensemble (CUE), and circular symplectic ensemble (CSE) (see, e.g., [8]). In particular, the CUE is defined to be the ensemble of $n\times n$ unitary matrices equipped with the Haar measure. The ordered eigenvalues are denoted as $\{e^{i\theta_j}\}_{j=0}^{n-1}$, where $0\leqslant\theta_0\leqslant\cdots\leqslant\theta_{n-1}\leqslant 2\pi$. The joint probability density of

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the angles $\{\theta_j\}_{j=0}^{n-1}$ is given by

$$p_\beta^c(\theta_1,\dots,\theta_n) = \frac{1}{Z_{n,\beta}^c}\prod_{0\leqslant j<k\leqslant n-1}|e^{i\theta_j}-e^{i\theta_k}|^\beta \tag{1}$$

$$= \frac{1}{Z_{n,\beta}^c}\exp\biggl\{\frac{\beta}{2}\sum_{j\neq k}\ln\biggl|2\sin\frac{\theta_j-\theta_k}{2}\biggr|\biggr\}, \tag{2}$$

with $\beta=2$ and

$$Z_{n,\beta}^c = \frac{(2\pi)^n}{n!}\,\frac{\Gamma(\beta n/2+1)}{\Gamma(\beta/2+1)}. \tag{3}$$

Similarly, the joint probability density for the COE/CSE is given by (1)–(3) with $\beta=1$ for the COE and $\beta=4$ for the CSE. The generalized ensemble for $\beta>0$ is named the circular β-ensemble (see [9]). These random matrix ensembles were originally introduced in physics, but recently have played an important role in linking RMT with number theory, because of the connections with the Riemann zeta function. Recall that the Riemann zeta function is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p\in\mathcal{P}}\biggl(1-\frac{1}{p^s}\biggr)^{-1}$$

for $\operatorname{Re} s>1$ and can be analytically extended to the whole complex plane.

Many efforts have been made to find the deep connection between zeta function and random matrices (see e.g. [14]–[16], [2], [5], and [4]). In 2000, J. Keating and N. Snaith made an important contribution in connecting the characteristic polynomial of CUE and value distribution of $\zeta(z)$ on the critical line (see [13]). They showed that the distribution of values taken by $\ln\det(e^{is}-U_n)$ averaged over $U_n\in$ CUE is a good approximation to the value distribution of $\ln\zeta(1/2+it)$ for large n,t given the relation that $n=\ln(t/(2\pi))(1+o(1))$.

In 1988, K. Johansson [11] proved the Central Limit Theorem (CLT) for the linear statistics in the circular β-ensemble.

Theorem 1.1. *Let $f\in C^{1+\varepsilon}(S^1)$, $\varepsilon>0$. Then*

$$\sum_{j=0}^{n-1}f(\theta_j)-\frac{n}{2\pi}\int_0^{2\pi}f(x)\,dx \xrightarrow{\text{law}} N\biggl(0,\frac{2}{\beta}\sum_{k=-\infty}^{\infty}|k||c_k|^2\biggr),$$

where $c_k=(2\pi)^{-1}\int_0^{2\pi}f(x)e^{-ikx}\,dx$.

Remark 1.1. For $\beta=2$, the result holds under the optimal condition $\sum_{k=-\infty}^{\infty}|k||c_k|^2<\infty$ (see also [7], [17] and references therein).

To study the characteristic polynomial of CUE, one can write

$$\ln|\det(e^{is}-U_n)| = \sum_{j=1}^{n}\ln|e^{is}-e^{i\theta_j}|.$$

Due to the singularity of the logarithm function, we cannot use Theorem 1.1. T. Baker and P. Forrester proved in [1] that $\sqrt{2}\ln|\det(e^{is}-U_n)|/\sqrt{\ln n}$ converges in distribution to standard normal distribution for fixed s. It was proved in [10] that for the CUE ($\beta=2$), $\sqrt{2}\ln|\det(e^{is}-U_n)|$ converges in distribution to a generalized random function

$$T(s)=\mathrm{Re}\Bigg(\sum_{k=1}^{\infty}\frac{e^{iks}}{\sqrt{k}}Z_k\Bigg), \tag{4}$$

where Z_k are i.i.d. complex standard Gaussian variables (see also [1] and [6]). We refer the reader to [12] for a detailed exposition of random trigonometric series.

The generalized random function $T(s)$ makes another appearance in the circular β-ensemble as follows. One can show that the joint probability density of (2) obtains its maximum at the lattice configuration $\theta_j=2\pi j/n+\mathrm{const}$ $(0\leqslant j\leqslant n-1)$. Write

$$\theta_j=\frac{2\pi j}{n}+\frac{t_j}{n}+\mathrm{const}.$$

Let us choose the constant such that $\sum_{j=0}^{n-1}t_j=0$ and take the Taylor expansion of (2) around this critical configuration. If we ignore the cubic and higher terms, then we get, as an approximation, a multivariate Gaussian distribution on the hyperplane $\sum_{j=0}^{n-1}t_j=0$ with the density

$$\widetilde{p}_g(t)=\frac{1}{\widetilde{Z}_g}\exp\Bigg\{-\frac{\beta}{16}\sum_{j\neq k}\frac{1}{\sin^2(\pi(j-k)/n)}\frac{(t_j-t_k)^2}{n^2}\Bigg\}. \tag{5}$$

It can be shown that t_j from (5) can be expressed as

$$t_j=\frac{2}{\sqrt{\beta}}\mathrm{Re}\Bigg(\sum_{k=1}^{n}\frac{e^{2\pi ijk/n}}{\sqrt{k}}Z_k\Bigg)(1+\epsilon_n), \tag{6}$$

where ϵ_n is a negligible random error term with $\mathbf{D}\epsilon_n=o_n(1)$. Moreover, the linear statistics $\sum_{j=0}^{n-1}f(2\pi j/n+t_j/n)$ satisfies the same CLT as in Theorem 1.1.

Remark 1.2. We refer the reader to [18, Section 3.4] for a related discussion of the mesoscopic structure of CUE eigenvalues.

This indicates that the generalized random function $T(s)$ defined in (4) gives a good approximation of the eigenvalue statistics of CUE. However, it is not entirely clear in what sense we can ignore cubic and higher order terms of the Taylor expansion of (2). This motivated us to consider a new model of interacting particles on the unit circle with stronger repulsion than that in the circular β-ensembles. The purpose of this paper is to establish the Gaussian approximation for the distribution of strongly repelling particles on the unit circle.

Throughout this paper, the letters C_k, C'_k, c_k, and c'_k ($k \in \mathbf{N}$) denote positive constants whose values might change in different parts of the paper, but are always independent of n. We write $a_n \ll b_n$ or $a_n = o(b_n)$ if $a_n/b_n \to 0$ as $n \to \infty$, and $a_n = O(b_n)$ if there exists some positive constant C such that $|a_n| \leqslant C|b_n|$ as $n \to \infty$. If $a_n \to 0$ as $n \to \infty$ and the decay rate does not depend on other parameters, we write $a_n = o_n(1)$. Also, we denote $a_n \sim b_n$ if there exist positive constants c and C such that $cb_n \leqslant X \leqslant Ca_n$ as $n \to \infty$.

2. Set up and notation. Consider a strong repulsion model of particles distributed on

$$\mathbf{T}^n/S_n = \{\theta = (\theta_0, \dots, \theta_{n-1}) \in [0, 2\pi]^n \colon \theta_0 \leqslant \theta_1 \leqslant \dots \leqslant \theta_{n-1}\}.$$

The joint probability density is defined as

$$q(\theta) = \frac{1}{Z_n} e^{H_{n,\beta}(\theta)}, \tag{7}$$

where

$$H_{n,\beta}(\theta) = -\frac{\beta}{2} \sum_{i \neq j} \frac{1}{\sin^2((\theta_i - \theta_j)/2)} \tag{8}$$

and

$$Z_n = \int_{\mathbf{T}^n/S_n} e^{H_{n,\beta}(\theta)}\, d\theta. \tag{9}$$

For any measurable subset $A \in \mathbf{T}^n/S_n$, let $\mathbf{P}(A) = \int_A q(\theta)\, d\theta$. Note that the repulsion in $H_{n,\beta}(\theta)$ is stronger than the logarithmic one in (2). Let

$$\theta_i = \frac{2\pi i}{n} + \psi + \frac{x_i}{n^2}, \tag{10}$$

where ψ is a constant chosen so that

$$\sum_{i=0}^{n-1} x_i = 0. \tag{11}$$

Thus,

$$\psi = \frac{1}{n} \sum_{i=0}^{n-1} \theta_i - \frac{\pi(n-1)}{n}. \tag{12}$$

For notational simplicity, define $\alpha_i = 2\pi i/n + \psi$, $\alpha = (\alpha_0, \dots, \alpha_{n-1})$, $x = (x_0, \dots, x_{n-1})$, then $\theta = \alpha + x/n^2$. Further, we introduce some useful lemmas.

Lemma 2.1. *The probability density $q(\theta)$ in (7) obtains its maximum at $\theta = \alpha$, and*

$$\begin{aligned} &H_{n,\beta}(\alpha) - H_{n,\beta}(\theta) \\ &= \frac{\beta}{2} \sum_{i \neq j} \frac{(x_i - x_j)^2}{n^4} \int_0^1 \frac{1/2 + \cos^2(\pi(i-j)/n + \tau(x_i - x_j)/(2n^2))}{\sin^4(\pi(i-j)/n + \tau(x_i - x_j)/(2n^2))} (1 - \tau)\, d\tau. \end{aligned} \tag{13}$$

This implies that the lattice configuration $\theta_i = 2\pi j/n + \text{const}$ is the ground state of the strongly repelling particle system.

Lemma 2.2. *The following identities hold*:

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2(\pi k/n)} = \frac{n^2-1}{3}; \tag{14}$$

$$\sum_{k=1}^{n-1} \frac{1}{\sin^4(\pi k/n)} = \frac{(n^2-1)(n^2+11)}{45}, \tag{15}$$

$$\sum_{k=1}^{n-1} \frac{\sin^2(m\pi k/n)}{\sin^2(\pi k/n)} = m(n-m) \quad (1 \leqslant m \leqslant n-1); \tag{16}$$

$$\sum_{k=1}^{n-1} \frac{\sin^2(m\pi k/n)}{\sin^4(\pi k/n)} = \frac{m^2(n-m)^2}{3} + \frac{2}{3}m(n-m) \quad (1 \leqslant m \leqslant n-1). \tag{17}$$

The proof of Lemma 2.2 is a standard exercise in complex analysis and is left to the reader. The identity (14) implies that the maximum of $H_{n,\beta}(\theta)$ is

$$H_{n,\beta}(\alpha) = -\frac{\beta}{2} \sum_{i \neq j} \frac{1}{\sin^2(\pi(i-j)/n)} = \frac{(n^3-n)\beta}{6}. \tag{18}$$

The following lemma shows that typically $H_{n,\beta}(\theta)$ is not far from $H_{n,\beta}(\alpha)$.

Lemma 2.3. *For any* $C > 1$, *define*

$$\Theta = \{\theta \in \mathbf{T}^n/S_n \colon H_{n,\beta}(\alpha) - H_{n,\beta}(\theta) \leqslant Cn \ln n\}. \tag{19}$$

Then there exists some $c > 0$ *such that*

$$\mathbf{P}(\Theta) \geqslant 1 - n^{-cn}.$$

Remark 2.1. If we choose $C = 1$, then the condition on the set Θ should be modified as $H_{n,\beta}(\alpha) - H_{n,\beta}(\theta) \leqslant n \ln n - C'n$ for some $C' > 0$.

Using Lemmas 2.1 and 2.3, we have the following lemma.

Lemma 2.4. *For any* $C > 1$, *define* Θ *as in* (19). *If* $\theta \in \Theta$, *then there exists some positive constant* C_0 *such that*

$$\sum_{i \neq j} \frac{(x_i - x_j)^2}{n^4 \sin^4(\pi(i-j)/n)} \leqslant C_0 n \ln^3 n. \tag{20}$$

Moreover, for all $0 \leqslant i \neq j \leqslant n-1$,

$$|x_i - x_j| \leqslant C_0 |i-j|_o n^{1/2} \ln^{3/2} n, \tag{21}$$

where

$$|i-j|_o = \min\{|i-j|, n - |i-j|\}. \tag{22}$$

Taking the Taylor expansion of the joint probability function $q(\theta)$ around the critical configuration $\theta = \alpha$, we have that for some $\delta \in [0,1]$,

$$q(\theta) = \frac{1}{Z_n} e^{H_{n,\beta}(\alpha)} \exp\left\{ \frac{\beta}{4} \sum_{i \neq j} \frac{-3/2 + \sin^2(\pi(i-j)/n)}{n^4 \sin^4(\pi(i-j)/n)} (x_i - x_j)^2 \right\}$$
$$\times \exp\left\{ \frac{\beta}{12} \sum_{i \neq j} \left[\frac{3\cos(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))}{n^6 \sin^5(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))} \right.\right.$$
$$\left.\left. - \frac{\cos(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))}{n^6 \sin^3(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))} \right] (x_i - x_j)^3 \right\}. \tag{23}$$

Denote the quadratic term by

$$G(x) := \frac{\beta}{4} \sum_{i \neq j} \frac{-3/2 + \sin^2(\pi(i-j)/n)}{n^4 \sin^4(\pi(i-j)/n)} (x_i - x_j)^2, \tag{24}$$

and the cubic term by

$$F(x) := \frac{\beta}{12} \sum_{i \neq j} \left[\frac{3\cos(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))}{n^6 \sin^5(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))} \right.$$
$$\left. - \frac{\cos(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))}{n^6 \sin^3(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))} \right] (x_i - x_j)^3. \tag{25}$$

Using (10), consider the change of variable $\theta \to (x, \psi)$, where x is a degenerate vector on the hyperplane

$$\Gamma = \left\{ x \in \mathbf{R}^n \colon \sum_{i=0}^{n-1} x_i = 0 \right\}. \tag{26}$$

Let

$$f(x) = q(\theta(x, \phi)). \tag{27}$$

Note that the joint probability density f only depends on x. But the domain Ω depends on both x and ψ. If $\theta \in \mathbf{T}^n / S_n$, then

$$(x, \psi) \in \Omega$$
$$= \left\{ \Gamma \times \left[-\pi + \frac{\pi}{n}, \pi + \frac{\pi}{n} \right] \colon x_i - x_{i-1} \geqslant -2\pi n;\ -\frac{x_0}{n^2} \leqslant \psi \leqslant \frac{2\pi}{n} - \frac{x_{n-1}}{n^2} \right\}.$$

Thus, the marginal density function for x is

$$p(x) = \int_{-x_0/n^2}^{2\pi/n - x_{n-1}/n^2} f(x)\, d\psi = \left(\frac{2\pi}{n} - \frac{x_{n-1} - x_0}{n^2} \right) f(x), \tag{28}$$

where

$$x \in \Lambda = \{x \in \Gamma\colon x_i - x_{i-1} \geqslant -2\pi n;\ x_0 \geqslant -\pi n(n+1);\\ x_{n-1} \leqslant \pi n(n+1);\ x_{n-1} - x_0 \leqslant 2\pi n\}. \tag{29}$$

Otherwise, if $x \in \Lambda^c$, then $p(x) = 0$. For any measurable subset $A \subset \Gamma$, denote

$$\mathbf{P}_x(A) = \int_A p(x)\, dx. \tag{30}$$

It follows from Lemmas 2.3 and 2.4 that there exists a subset of Λ, namely

$$\Gamma_D = \Bigg\{ x \in \Gamma\colon \max_{i \neq j} \frac{|x_i - x_j|}{|i-j|_o} \leqslant D n^{1/2} \ln^{3/2} n;\\ x_0 \geqslant -\pi n(n+1);\ x_{n-1} \leqslant \pi n(n+1) \Bigg\}, \tag{31}$$

such that

$$\mathbf{P}_x(\Gamma_D^c) \leqslant n^{-cn}. \tag{32}$$

In addition, if $x \in \Gamma_D$, the probability density of x can be written as

$$p(x) = \frac{1}{\widetilde{Z}_n} e^{G(x)+F(x)} (1 + o_n(1)), \tag{33}$$

where

$$\widetilde{Z}_n = \int_{\Gamma_D} e^{G(x)+F(x)}\, dx. \tag{34}$$

Let us define a Gaussian distribution on the hyperplane Γ by its density

$$p_g(x) = \frac{1}{Z_g} e^{G(x)}, \tag{35}$$

where

$$Z_g = \int_\Gamma e^{G(x)}\, dx. \tag{36}$$

For any measurable subset $A \subset \Gamma$, denote

$$\mathbf{P}_g(A) = \int_A p_g(x)\, dx. \tag{37}$$

We use $\mathbf{E}_g$ and $\mathbf{D}_g$ to denote the expectation and variance taken under this Gaussian probability measure.

Remark 2.2. Refining the argument used in the proof of Lemma 2.4, we can also prove that there exists a subset $\Gamma_{D'} \subset \Gamma$ such that (1) and (2) hold where

1) if $x \in \Gamma_{D'}$, then we have $\max_j |x_j| \ll n$, and thus $\phi \sim 1/n$;
2) $\mathbf{P}_x(\Gamma_{D'}^c) = o_n(1)$.

In addition, let $\widetilde{\psi} = n\psi$. Then

$$p(\widetilde{\psi} \mid x) = \frac{f(x)}{\int_{-x_0/n^2}^{2\pi - x_{n-1}/n} f(x)\, d\widetilde{\psi}} = \frac{1}{2\pi - (x_{n-1} - x_0)/n} = \frac{1}{2\pi}(1 + o_n(1)).$$

One can show that $n\psi$ and x are asymptotically independent from each other and $n\psi$ converges to the uniform distribution on $[0, 2\pi]$. However, we are not going to use this in the paper.

3. Main theorems. In this section, we formulate our main results. We start with an auxiliary proposition. Recall that we have defined $\mathbf{P}_x$ and $\mathbf{P}_g$ in (33), (34), (30), and in (35)–(37), respectively. Also, $F(x)$ is defined in (25).

Proposition 3.1. *There exists a subset* $\Gamma' \subset \Gamma$ *such that*

$$\mathbf{P}_x(\Gamma') = 1 - o_n(1), \qquad \mathbf{P}_g(\Gamma') = 1 - o_n(1),$$

and

$$\sup_{x \in \Gamma'} F(x) = o_n(1).$$

Proposition 3.1 immediately implies that the total variation distance between $\mathbf{P}_x$ and $\mathbf{P}_g$ goes to zero as n goes to infinity.

Theorem 3.1.

$$\sup_{A \subset \Gamma} |\mathbf{P}_x(A) - \mathbf{P}_g(A)| = o_n(1),$$

where the supremum on the left-hand side is taken over all measurable subsets $A \subset \Gamma$.

Based on Theorem 3.1, we obtain the main theorem. For each fixed n, we construct a random function in $C[0, 2\pi]$, denoted as $\zeta_n(t)$, by letting $\zeta_n(2\pi j/n) = x_j/\sqrt{n}$ and then connecting these lattice points with straight segments. Define the limiting random function to be

$$\zeta(t) = \sqrt{\frac{2}{\beta}}\, \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{1}{k} e^{ikt} Z_k\right) = \sqrt{\frac{2}{\beta}} \sum_{k=1}^{\infty} \left(\frac{\cos(kt)}{k} X_k - \frac{\sin(kt)}{k} Y_k\right), \quad (38)$$

where X_k, Y_k are i.i.d. real standard Gaussian random variables, and Z_k are i.i.d. complex standard Gaussian random variables. Note that this is a well-defined random function since the variance is bounded. It can be viewed as an analogue of (4) in the CUE case. We have the following theorem.

Theorem 3.2. $\zeta_n(t)$ *converges to* $\zeta(t)$ *in finite dimensional distributions. Furthermore, the functional convergence takes place. In other words,* $\zeta_n(t)$ *converges to* $\zeta(t)$ *in distribution weakly on the space* $C[0, 2\pi]$.

Finally, we conclude this section by formulating two corollaries.

Corollary 3.1. *Consider periodic function g on S^1 with complex Fourier coefficients $\{c_k\}_{k\geqslant 0}$, where $c_k = (2\pi)^{-1}\int_0^{2\pi} g(x)e^{ikx}\,dx$, which satisfy the following condition:* $\sum_{k=-\infty}^{\infty}|k|^{3/2}|c_k| < \infty$. *Then*

$$\mathbf{E}\exp\Big\{it\Big(\sqrt{n}\sum_{j=0}^{n-1} g(\theta_j) - n^{3/2}c_0\Big)\Big\} = \exp\Big\{-\frac{t^2}{\beta}\sum_{k=1}^{\infty}|c_k|^2\Big\}(1+o_n(1)). \quad (39)$$

In other words,

$$\sqrt{n}\sum_{j=0}^{n-1} g(\theta_j) - n^{3/2}c_0 \xrightarrow{\text{law}} N\Big(0, \frac{2}{\beta}\sum_{k=1}^{\infty}|c_k|^2\Big).$$

Remark 3.1. Corollary 3.1 is expected to hold when $f \in C^{1+\varepsilon}(\mathbf{T})$.

Corollary 3.2. *The following convergence holds*:

$$\max_{0\leqslant i\leqslant n-1}\left|\frac{x_j}{\sqrt{n}}\right| \xrightarrow{\text{law}} \sup_{t\in[0,2\pi]}|\zeta(t)|.$$

To simplify the notation, the proofs of these results are written for $\beta = 2$. The general case $\beta > 0$ is essentially identical.

4. Proofs of the Lemmas in Section 2. We start this section by proving Lemma 2.1.

Proof of Lemma 2.1. Define $\phi(\tau) := H_{n,2}(\alpha + \tau x/n^2)$. We compute its first and second derivatives with respect to τ,

$$\phi'(\tau) = \sum_{i\neq j}\frac{\cos(\pi(i-j)/n + \tau(x_i-x_j)/(2n^2))}{\sin^3(\pi(i-j)/n + \tau(x_i-x_j)/(2n^2))}\,\frac{x_i-x_j}{n^2}; \quad (40)$$

$$\phi''(\tau) = -\sum_{i\neq j}\frac{1/2 + \cos^2(\pi(i-j)/n + \tau(x_i-x_j)/(2n^2))}{\sin^4(\pi(i-j)/n + \tau(x_i-x_j)/(2n^2))}\Big(\frac{x_i-x_j}{n^2}\Big)^2. \quad (41)$$

Note that

$$\begin{aligned}\phi'(0) &= \sum_{i\neq j}\frac{\cos(\pi(i-j)/n)}{\sin^3(\pi(i-j)/n)}\,\frac{x_i-x_j}{n^2}\\ &= \frac{1}{n^2}\Big(\sum_{i\neq j}\frac{\cos(\pi(i-j)/n)}{\sin^3(\pi(i-j)/n)}\,x_i - \sum_{i\neq j}\frac{\cos(\pi(i-j)/n)}{\sin^3(\pi(i-j)/n)}\,x_j\Big)\\ &= \frac{1}{n^2}\sum_i\Big(\sum_{k=1}^{n-1}\frac{\cos(\pi k/n)}{\sin^3(\pi k/n)}\Big)x_i - \frac{1}{n^2}\sum_j\Big(\sum_{l=1}^{n-1}\frac{\cos(\pi l/n)}{\sin^3(\pi l/n)}\Big)x_j = 0,\end{aligned}$$

and $\phi''(0) \leqslant 0$. Thus,

$$H_{n,2}(\alpha) - H_{n,2}(\theta) = \phi(0) - \phi(1) = -\int_0^1 (1-\tau)\phi''(\tau)\,d\tau = \sum_{i\neq j} \frac{(x_i - x_j)^2}{n^4}$$
$$\times \int_0^1 \frac{1/2 + \cos^2(\pi(i-j)/n + \tau(x_i - x_j)/(2n^2))}{\sin^4(\pi(i-j)/n + \tau(x_i - x_j)/(2n^2))}(1-\tau)\,d\tau \geqslant 0.$$

This implies that $H_{n,2}(\theta)$ obtains its maximum at $\theta = \alpha$. Lemma 2.1 is proved.

Further, we turn our attention to proving Lemma 2.3.

Proof of Lemma 2.3. It follows from the definition of Θ in (19) and the trigonometric identity (18) that

$$\mathbf{P}(\Theta^c) = \frac{1}{Z_n}\int_{\Omega^c} e^{H_{n,2}(\theta)}\,d\theta \leqslant \exp\left\{\frac{n^3 - n}{3} - Cn\ln n\right\}\frac{1}{Z_n}\mu(\mathbf{T}^n/S_n)$$
$$= \frac{(2\pi)^n}{n!}\exp\left\{\frac{n^3 - n}{3} - Cn\ln n\right\}\frac{1}{Z_n}, \tag{42}$$

where μ denote the Lebesgue measure on $\mathbf{R}^n$. Choose any $0 < C' < C$ and define a subset of Θ,

$$\Theta' = \{\theta \in \mathbf{T}^n/S_n \colon H_{n,2}(\alpha) - H_{n,2}(\theta) \leqslant C'n\ln n\}. \tag{43}$$

Then

$$Z_n = \int_{\mathbf{T}^n/S_n} e^{H_{n,2}(\theta)}\,d\theta \geqslant \int_{\Theta'} \exp\{H_{n,2}(\alpha) - C'n\ln n\}\,d\theta$$
$$= \exp\left\{\frac{n^3 - n}{3} - C'n\ln n\right\}\mu(\Theta'). \tag{44}$$

Note that if there exists some constant $M > 0$ such that $|x_j| \leqslant M$, then

$$H_{n,2}(\alpha) - H_{n,2}(\theta) \leqslant \sum_{i\neq j} \frac{4M^2}{n^4\sin^4(\pi(i-j)/n)}$$
$$\leqslant 4M^2 \sum_{i\neq j} \frac{1}{\pi^4|i-j|^4} \leqslant C'n\ln n.$$

Therefore,

$$\left\{\theta \in \mathbf{T}^n/S_n \colon \theta_i - \alpha_i \leqslant \frac{M}{n^2}\ \forall\, 0 \leqslant i \leqslant n-1\right\} \subset \Theta',$$

and thus the Lebesgue measure of the set Θ' can be bounded from below as follows:

$$\mu(\Theta') \geqslant \left(\frac{M}{n^2}\right)^n = \exp\{n\ln M - 2n\ln n\}. \tag{45}$$

Therefore, by (44) and (45),

$$Z_n \geqslant \exp\left\{\frac{n^3-n}{3} - C'n\ln n + n\ln M - 2n\ln n\right\}. \tag{46}$$

Combining it with (18) and (42), we obtain

$$\begin{aligned}\mathbf{P}(\Theta^c) &\leqslant \frac{(2\pi)^n}{n!}\exp\{-(C-C')n\ln n - n\ln M + 2n\ln n\}\\ &\leqslant C''\exp\{-(C-C'-1)n\ln n\} = o_n(1),\end{aligned} \tag{47}$$

provided $C > 1$ and $0 < C' < C-1$. Lemma 2.3 is proved.

Using Lemma 2.3, we finish this section by giving the proof of Lemma 2.4.

Proof of Lemma 2.4. By Lemma 2.3, if $\theta \in \Theta$, then for some constant $C > 1$, we have

$$\sum_{i\neq j}\frac{(x_i-x_j)^2}{n^4}\int_0^1\frac{(1-\tau)\,d\tau}{\sin^4(\pi(i-j)/n+\tau(x_i-x_j)/(2n^2))} \leqslant Cn\ln n. \tag{48}$$

Let $I = \{(i,j)\colon 0 \leqslant i \neq j \leqslant n-1\}$, $I_1 = \{(i,j) \in I\colon |x_i-x_j| < n\eta_n|i-j|_o\}$, and $I_2 = I \setminus I_1$. Let $\eta_n \gg 1$. Then by (48)

$$\begin{aligned}Cn\ln n &\geqslant \sum_{I_1}\frac{(x_i-x_j)^2}{n^4}\int_0^1\frac{(1-\tau)\,d\tau}{\sin^4(\pi(i-j)/n+\tau(x_i-x_j)/(2n^2))}\\ &\geqslant \sum_{I_1}\frac{(x_i-x_j)^2}{n^4}\int_0^1\frac{(1-\tau)\,d\tau}{(\pi|i-j|_o/n+\tau|x_i-x_j|/(2n^2))^4}\\ &\geqslant C'\sum_{I_1}(x_i-x_j)^2\frac{1}{\eta_n^4|i-j|_o^4}.\end{aligned}$$

Thus,

$$\sum_{I_1}\frac{(x_i-x_j)^2}{|i-j|_o^4} \leqslant C''n\eta_n^4\ln n. \tag{49}$$

Next, it can be shown that $I_2 = \varnothing$ for η_n satisfying

$$\eta_n \geqslant M\ln^{1/2} n \tag{50}$$

for sufficient large $M > 0$. Note that

$$\begin{aligned}&\int_0^1\frac{(1-\tau)\,d\tau}{\sin^4(\pi(i-j)/n+\tau(x_i-x_j)/(2n^2))}\\ &\qquad\geqslant \int_0^1\frac{(1-\tau)\,d\tau}{(\pi|i-j|_o/n+\tau|x_i-x_j|/(2n^2))^4}\\ &\qquad= \frac{3\pi|i-j|_o/n+|x_i-x_j|/n^2}{6(\pi|i-j|_o/n)^3(\pi|i-j|_o/n+|x_i-x_j|/(2n^2))^2}.\end{aligned} \tag{51}$$

If $(i,j) \in I_2$, then

$$\frac{|x_i - x_j|}{n^2} \geqslant \eta_n \frac{|i-j|_o}{n},$$

and

$$\text{RHS of (51)} \geqslant \text{const}\, \frac{\eta_n |i-j|_o/n}{(|i-j|_o/n)^3(|x_i-x_j|/n^2)^2} \geqslant \text{const}\, \frac{\eta_n n^6}{(|i-j|_o)^2(|x_i-x_j|)^2}. \tag{52}$$

Note that by the triangle inequality, if $(i,j) \in I_2$, then there exists at least $|i-j|_o$ index pairs belonging to I_2, in the form of (i,k) or (k,j) where k is between i and j. Thus by (48), (51), and (52),

$$Cn\ln n \geqslant C' \sum_{I_2} \frac{\eta_n n^2}{|i-j|_o^2} \geqslant C' n^2 \eta_n \frac{1}{|i-j|_o}.$$

This implies that $|i-j|_o \geqslant C'' n\eta_n \ln^{-1} n$, and thus $|x_i - x_j| \geqslant C'' n^2 \eta_n^2 \ln^{-1} n$. Due to (50), we obtain

$$|x_i - x_j| \geqslant C'' M^2 n^2.$$

With a sufficiently large M, the last inequality contradicts that $|x_i - x_j| \leqslant 2\pi n^2$. Therefore $I_2 = \varnothing$ and there exists some positive constant C_0 such that

$$\sum_I \frac{(x_i - x_j)^2}{|i-j|_o^4} \leqslant C_0 n \ln^3 n. \tag{53}$$

Furthermore, denote $x_{n+i} = x_i$, $i \geqslant 0$, then for $0 \leqslant j \leqslant n-1$,

$$(x_{j+1} - x_j)^2 \leqslant C_0 n \ln^3 n.$$

Therefore, by the triangle inequality, we have

$$|x_i - x_j| \leqslant C_0 |i-j|_o n^{1/2} \ln^{3/2} n.$$

Finally, since $2/\pi \leqslant \sin(x/x) \leqslant 1$, $0 < x \leqslant \pi/2$, we have

$$\sum_{i \neq j} \frac{(x_i - x_j)^2}{n^4 \sin^4(\pi(i-j)/n)} \leqslant C_0' n \ln^3 n. \tag{54}$$

Lemma 2.4 is proved.

5. Estimate of the multivariate Gaussian distribution. Note that $G(x)$ defined in (24) can be written in the quadratic form of $-\frac{1}{2}x^\top Ax$, where

$$A_{i,j} = -\frac{3}{n^4 \sin^4(\pi(i-j)/n)} + \frac{2}{n^4 \sin^2(\pi(i-j)/n)} \qquad (i \neq j), \tag{55}$$

and, by the identities (14) and (15) in Lemma 2.2,

$$A_{i,i} = \frac{(n^2-1)(n^2+11)}{15n^4} - \frac{2n^2-2}{3n^4} = -\sum_{j\neq i} A_{i,j}. \tag{56}$$

Since A is not invertible, p_g defined in (35), (36) can be viewed as a degenerate Gaussian distribution on the hyperplane Γ defined in (26),

$$p_g(x) = \frac{1}{Z_g} e^{-x^\top Ax/2},$$

where $Z_g = \int_\Gamma e^{-x^\top Ax/2}\,dx$.

Next, we aim to explore the covariance structure of this Gaussian distribution. Note that A is a circular matrix generated by the vector $(A_{0,0}, A_{0,1}, \ldots, A_{0,n-1})$. Therefore its normalized eigenvectors can be chosen as

$$v_k = \frac{1}{\sqrt{n}}(\omega_k^0, \omega_k^1, \ldots, \omega_k^{n-1}) \qquad (k = 0, 1, \ldots, n-1), \tag{57}$$

where $\omega_k = e^{2\pi ik/n}$. By using the identities (16) and (17) in Lemma 2.2, the corresponding eigenvalues are given by

$$\begin{aligned}
\lambda_k &= A_{0,0} + A_{0,1}\omega_k + A_{0,2}\omega_k^2 + \cdots + A_{0,n-1}\omega_k^{n-1} \\
&= \frac{(n^2-1)(n^2+11)}{15n^4} - \frac{2n^2-2}{3n^4} - \frac{3}{n^4}\sum_{j=1}^{n-1}\frac{\cos(2\pi jk/n)}{\sin^4(j\pi/n)} + \frac{2}{n^4}\sum_{j=1}^{n-1}\frac{\cos(2\pi jk/n)}{\sin^2(j\pi/n)} \\
&= \frac{6}{n^4}\sum_{j=1}^{n-1}\frac{\sin^2(\pi jk/n)}{\sin^4(j\pi/n)} - \frac{4}{n^4}\sum_{j=1}^{n-1}\frac{\sin^2(\pi jk/n)}{\sin^2(j\pi/n)} \\
&= \frac{2k^2(n-k)^2}{n^4}, \qquad 0 \leqslant k \leqslant n-1.
\end{aligned} \tag{58}$$

Define

$$U = (v_0^\top, v_1^\top, \ldots, v_{n-1}^\top), \tag{59}$$

then we have $U^*U = 1$ and $A = U\Lambda U^*$, where Λ is the diagonal matrix generated by $(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$. Let $s = U^*x$. Then for $1 \leqslant k, j \leqslant n-1$,

$$\mathbf{E}_g s_k \overline{s}_k = \frac{1}{\lambda_k} = \frac{n^4}{2k^2(n-k)^2} \quad \text{and} \quad \mathbf{E} s_k \overline{s}_j = 0, \qquad k \neq j. \tag{60}$$

Note that $s_0 = 0$ because of the definition of Γ in (26). We also note that

$$s_j = \overline{s}_{n-j} \quad \text{if} \quad j > \frac{n-1}{2}. \tag{61}$$

For simplicity, we assume that n is odd (the even case can be treated in a similar way). Then $(s_1, \ldots, s_{(n-1)/2})$ are $(n-1)/2$ independent complex

Gaussian random variables. In particular, we can write

$$s_k = \frac{n^2}{2k(n-k)}X_k + i\frac{n^2}{2k(n-k)}Y_k \quad \text{for} \quad 1 \leqslant k \leqslant \frac{n-1}{2}, \tag{62}$$

where $\{X_k\}$ and $\{Y_k\}$ are independent real standard Gaussian variables.

Since $x = Us$, we can compute the covariance structure for x,

$$\mathbf{E}_g x_k \overline{x}_j = \frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{\lambda_m}e^{2\pi i m(k-j)/n} = \frac{1}{2n}\sum_{m=1}^{n-1}e^{2\pi i m(k-j)/n}\frac{n^4}{m^2(n-m)^2}. \tag{63}$$

In particular,

$$\mathbf{D}_g x_k \sim n. \tag{64}$$

In addition,

$$\begin{aligned} x_j &= \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}e^{2\pi ijk/n}s_k = \frac{1}{\sqrt{n}}\sum_{k=1}^{(n-1)/2}e^{2\pi ijk/n}s_k + \frac{1}{\sqrt{n}}\sum_{k=1}^{(n-1)/2}e^{-2\pi ijk/n}\overline{s}_k \\ &= \frac{2}{\sqrt{n}}\sum_{k=1}^{(n-1)/2}\mathrm{Re}(e^{2\pi ijk/n}s_k) \\ &= \frac{2}{\sqrt{n}}\sum_{k=1}^{(n-1)/2}\left[\cos\left(\frac{2\pi jk}{n}\right)\frac{n^2}{2k(n-k)}X_k - \sin\left(\frac{2\pi jk}{n}\right)\frac{n^2}{2k(n-k)}Y_k\right]. \end{aligned} \tag{65}$$

For $0 \leqslant j \leqslant n-1$, $0 \leqslant l \leqslant n-1$, let

$$\xi_j^{(l)} = x_{j+l} - x_j, \quad \text{where} \quad x_{n+i} = x_i,\ i \geqslant 0. \tag{66}$$

It is also useful for us to compute the covariance of $\xi_j^{(l)}$ and $\xi_k^{(l)}$.

Proposition 5.1. *There exists some positive constant C which is independent of other parameters such that*

$$|\mathbf{E}_g \xi_k^{(l)}\overline{\xi}_j^{(l)}| \leqslant C\min\left\{l, \frac{l^2}{|k-j|_o}\right\}. \tag{67}$$

In particular,

$$\mathbf{D}_g(\xi_k^{(l)}) \leqslant Cl. \tag{68}$$

Proof. By (63), we have

$$\begin{aligned} \mathbf{E}_g \xi_k^{(l)}\overline{\xi}_j^{(l)} &= \mathbf{E}_g(x_{k+l} - x_k)(\overline{x}_{j+l} - \overline{x}_j) \\ &= \mathbf{E}_g x_{k+l}\overline{x}_{j+l} + \mathbf{E}_g x_k\overline{x}_j - \mathbf{E}_g x_{k+l}\overline{x}_j - \mathbf{E}_g x_k\overline{x}_{j+l} \\ &= \frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{\lambda_m}e^{2\pi im(k-j)/n}(2 - e^{2\pi iml/n} - e^{-2\pi iml/n}) \\ &= \frac{2}{n}\sum_{m=1}^{n-1}\frac{\sin^2(\pi ml/n)}{(m/n)^2(1-m/n)^2}e^{2\pi i(k-j)m/n}. \end{aligned} \tag{69}$$

For the right-hand side of (69), we can find an upper bound as

$$
\begin{aligned}
|\mathbf{E}_g \xi_k^{(l)} \overline{\xi}_j^{(l)}| &\leqslant \frac{2}{n} \sum_{m=1}^{n-1} \frac{\sin^2(\pi m l/n)}{(m/n)^2(1-m/n)^2} = \frac{4}{n} \sum_{m=1}^{(n-1)/2} \frac{\sin^2(\pi m l/n)}{(m/n)^2(1-m/n)^2} \\
&\leqslant \frac{16l}{3} \sum_{m=1}^{(n-1)/2} \frac{\sin^2(\pi m l/n)}{(lm/n)^2} \frac{l}{n}. \qquad (70)
\end{aligned}
$$

The right-hand side of (70) can be viewed as a Riemann sum of the function $\sin^2(\pi x)/x^2$ corresponding to the evenly-spaced partition over $[0, l/2]$ with the subintervals of length $l/(n-1)$. Since the function $\sin^2(\pi x)/x^2$ can be bounded from above by a monotone function $m(x)$ defined by

$$
m(x) = \begin{cases} \pi^2 & \text{if } 0 \leqslant x \leqslant 1, \\ \dfrac{1}{x^2} & \text{if } x > 1, \end{cases}
$$

the right-hand side of (70) can be bounded by a Riemann sum of $m(x)$. Note that $m(x)$ is monotone, the error between its upper and lower Riemann sums is at most of the order l/n. Then there exists a universal constant $C > 0$ such that

$$
\begin{aligned}
|\mathbf{E}_g \xi_k^{(l)} \overline{\xi}_j^{(l)}| &\leqslant \frac{16}{3} l \int_0^{l/2} m(x)\, dx + O\left(\frac{l}{n}\right) \\
&\leqslant \frac{16l}{3} \int_0^{\infty} m(x)\, dx\, (1 + o(1)) \leqslant Cl.
\end{aligned}
$$

For large $|k-j|_o$, the heavy oscillation of the exponential term leads to cancellations between terms in the expression (69) for $\mathbf{E}_g \xi_k^{(l)} \overline{\xi}_j^{(l)}$. Thus the upper bound that we have obtained above is not sharp in this case. Let

$$
a_m = \frac{\sin^2(\pi m l/n)}{(m/n)^2(1-m/n)^2},
$$

and $b_m = e^{2\pi i (k-j) m/n}$. By summation by parts, we have

$$
\sum_{m=1}^{n-1} a_m b_m = \sum_{m=1}^{n-2} (a_m - a_{m+1}) \sum_{p=1}^{m} b_p + a_{n-1} \sum_{p=1}^{n-1} b_p.
$$

Then

$$
|\mathbf{E}_g \xi_k^{(l)} \overline{\xi}_j^{(l)}| \leqslant \frac{2}{n} \sum_{m=1}^{n-2} |a_m - a_{m+1}| \left| \frac{1 - e^{2\pi i (k-j) m/n}}{1 - e^{2\pi i (k-j)/n}} \right| + O\left(\frac{l^2}{n}\right).
$$

By differentiating the function $f(x) = \sin^2(\pi l x)/(x^2(1-x)^2)$, we find that the derivative is at most of the order l^2 and we have

$$|a_m - a_{m+1}| = \left| f\left(\frac{m}{n}\right) - f\left(\frac{m+1}{n}\right)\right| \leqslant \frac{C' l^2}{n}.$$

Using the inequality

$$\left|\frac{1 - e^{2\pi i(k-j)m/n}}{1 - e^{2\pi i(k-j)/n}}\right| \leqslant \left|\frac{2}{1 - e^{2\pi i(k-j)/n}}\right| \leqslant \frac{C' n}{|k-j|_o},$$

we have

$$|\mathbf{E}_g \xi_k^{(l)} \overline{\xi}_j^{(l)}| \leqslant \frac{C' l^2}{|k-j|_o}.$$

This finishes the proof of Proposition 5.1.

Combining (68) and (64) with (31), we have the following lemma.

Lemma 5.1. *There exist some positive constants c_1, c_2 such that*

$$\mathbf{P}_g(\Gamma_D^c) \leqslant 2e^{-c_1 n^3} + n^2 e^{-c_2 n \ln^3 n}. \tag{71}$$

6. Proof of Proposition 3.1 and Theorem 3.1. We start with some preliminary details. If $x = (x_0, \dots, x_{n-1}) \in \Gamma_D$, then by Lemma 2.4,

$$\begin{aligned} G(x) &= \frac{1}{2} \sum_{i \neq j} \frac{-3/2 + \sin^2(\pi(i-j)/n)}{n^4 \sin^4(\pi(i-j)/n)} (x_i - x_j)^2 \\ &\sim - \sum_{i>j} \frac{(x_i - x_j)^2}{|i-j|_o^4} = O(-n \ln^3 n), \end{aligned}$$

and

$$\frac{|x_i - x_j|}{|i-j|_o} = O(n^{1/2} \ln^{3/2} n). \tag{72}$$

Then $(x_i - x_j)/(2n^2) = O((\ln^{3/2} n)/n^{3/2})$ is negligible, and thus

$$\begin{aligned} F(x) = \frac{1}{6} \sum_{i \neq j} \Bigg[&\frac{3 \cos(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))}{n^6 \sin^5(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))} \\ &- \frac{\cos(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))}{n^6 \sin^3(\pi(i-j)/n + \delta(x_i - x_j)/(2n^2))} \Bigg] (x_i - x_j)^3 \\ \sim \sum_{i \neq j} \Bigg[&\frac{(x_i - x_j)^3}{2n^6 \sin^5(\pi(i-j)/n)} + \frac{(x_i - x_j)^3}{6n^6 \sin^3(\pi(i-j)/n)} \Bigg] \sim \sum_{i>j} \frac{(x_i - x_j)^3}{n|i-j|_o^5}. \end{aligned} \tag{73}$$

Comparing $F(x)$ with $G(x)$ and using (72), we have

$$|F(x)| = |G(x)| O\left(\max_{i\neq j} \frac{|x_i - x_j|}{n|i-j|_o}\right) = |G(x)| O\left(\frac{\ln^{3/2} n}{n^{1/2}}\right) = O(n^{1/2} \ln^{9/2} n). \tag{74}$$

In this section, we show that $F(x) = o_n(1)$ with probability $1 - o_n(1)$. To be more specific, by (73), we show that

$$F(x) \sim \sum_{l=1}^{n-1} \frac{1}{l^5} \left(\frac{1}{n} \sum_{j=0}^{n-l-1} (\xi_j^{(l)})^3 \right) = o_n(1), \tag{75}$$

where $\xi_j^{(l)}$ is defined in (66).

We divide the proof of (75) into three parts. The first step is to show that the normalized constant $\widetilde{Z}_n$ defined in (34) is not far from Z_g defined in (36). This implies that the probability distribution of x is not far from the Gaussian distribution $p_g(x)$ defined in (35), (36). The second step is to show that under the Gaussian distribution, $F(x) = o_n(1)$ with probability $1 - O(n^{-c})$. The last step is to combine the first two steps and obtain that $F(x) = o_n(1)$ with probability $1 - O(n^{-c})$ under the distribution defined in (33), (34).

6.1. Step 1: Comparing Z_g and $\widetilde{Z}_n$. For reader's convenience, recall the definition of the normalized constants $\widetilde{Z}_n$ and Z_g:

$$\widetilde{Z}_n = \int_{\Gamma_D} e^{G(x)+F(x)}\, dx, \qquad Z_g = \int_{\Gamma} e^{G(x)}\, dx.$$

We start with a lemma. Rescale the Gaussian distribution defined in (35) and define two new Gaussian distributions,

$$\begin{aligned} p_{g^+}(x) &= \frac{1}{Z_g^+} \exp\left\{ G(x) \left(1 + \frac{cB(n)}{n}\right) \right\}, \\ p_{g^-}(x) &= \frac{1}{Z_g^-} \exp\left\{ G(x) \left(1 - \frac{cB(n)}{n}\right) \right\}. \end{aligned} \tag{76}$$

Lemma 6.1. *If $B(n) \ll n$, then the normalized constants $Z_g^\pm$ satisfy*

$$Z_g^\pm = Z_g \exp\left\{ \mp \frac{c}{2} B(n)(1 + o_n(1)) \right\}.$$

Proof. Change the variable to $\widetilde{x} = \sqrt{1 - cB(n)/n}\, x$, then

$$\begin{aligned} Z_g^- &= \int_\Gamma \exp\left\{ -\frac{1}{2} x^\top A x \left(1 - \frac{cB(n)}{n}\right) \right\} dx \\ &= \int_\Gamma e^{-\widetilde{x}^\top A \widetilde{x}/2} \prod_{j=1}^n \frac{1}{\sqrt{1 - cB(n)/n}}\, d\widetilde{x} = Z_g \prod_{j=1}^n \frac{1}{\sqrt{1 - cB(n)/n}}. \end{aligned}$$

Note that

$$\left(1-\frac{cB(n)}{n}\right)^{-n/2}=\exp\left\{\frac{cB(n)}{2}(1+o_n(1))\right\}.$$

Similarly, we have

$$\left(1+\frac{cB(n)}{n}\right)^{-n/2}=\exp\left\{\frac{-cB(n)}{2}(1+o_n(1))\right\}.$$

Thus $Z_g^{\pm}=Z_g\exp\{\mp(c/2)B(n)(1+o_n(1))\}$. Lemma 6.1 is proved.

Let $B(n)=Dn^{1/2}\ln^{3/2}n$. By the definition of Γ_D in (31), we have the inequality $\max_{i\neq j}(|x_i-x_j|/|i-j|_o)\leqslant B(n)$. Then by the first equation of (74), there exists a universal constant $c>0$ such that

$$\frac{cB(n)}{n}G(x)\leqslant F(x)\leqslant -\frac{cB(n)}{n}G(x).$$

Similarly to Lemma 6.1, one can show that

$$Z_ge^{-cB(n)}\leqslant \widetilde{Z}_n\leqslant Z_ge^{cB(n)}.$$

When we proceed to the Step 2 and Step 3 presented below, this estimate will not be sufficient for us to show $F(x)=o_n(1)$ because the exponential term $e^{cB(n)}$ grows faster than any polynomial. In order to get a better upper bound of $\max_{i\neq j}(|x_i-x_j|/|i-j|_o)$, we use iteration. We need the following two lemmas.

Lemma 6.2. *For some sufficiently large $M>0$, let $M\ln^{1/2}n\leqslant B\leqslant Dn^{1/2}\ln^{3/2}n$. If there exists some $\gamma>0$ such that*

$$\mathbf{P}_x\left(x\in\Gamma_D\colon \max_{i\neq j}\frac{|x_i-x_j|}{|i-j|_o}\geqslant B\right)\leqslant n^{-\gamma}, \tag{77}$$

then there exists a universal constant $c>0$ such that

$$Z_ge^{-cB}\leqslant \widetilde{Z}_n\leqslant Z_ge^{cB}. \tag{78}$$

Proof. Denote $A=\{x\in\Gamma_D\colon \max_{i\neq j}(|x_i-x_j|/|i-j|_o)\leqslant B\}$ and then $\mathbf{P}_x(\Gamma_D\setminus A)\leqslant n^{-\gamma}$. Because of (74), if $x\in A$, then there exists a universal constant $c>0$ such that

$$\frac{cB}{n}G(x)\leqslant F(x)\leqslant -\frac{cB}{n}G(x). \tag{79}$$

By (79), we have

$$\begin{aligned}\widetilde{Z}_n&=\int_{\Gamma_D}e^{G(x)+F(x)}\,dx=\int_A e^{G(x)+F(x)}\,dx+\mathbf{P}_x(\Gamma_D\setminus A)\widetilde{Z}_n\\&\leqslant\int_A e^{G(x)(1-cB/n)}\,dx+n^{-\gamma}\widetilde{Z}_n.\end{aligned}$$

Then

$$(1-n^{-\gamma})\widetilde{Z}_n \leqslant \int_\Gamma e^{G(x)(1-cB/n)}\,dx \leqslant e^{(cB/2)(1+o_n(1))}Z_g.$$

Thus

$$\widetilde{Z}_n \leqslant e^{(cB/2)(1+o_n(1))}Z_g(1+O(n^{-\gamma})) \leqslant e^{cB}Z_g. \tag{80}$$

For the lower bound, similarly,

$$\begin{aligned}\widetilde{Z}_n &\geqslant \int_A e^{G(x)+F(x)}\,dx \geqslant \int_A e^{G(x)(1+cB/n)}\,dx\\ &= \int_\Gamma e^{G(x)(1+cB/n)}\,dx - \int_{\Gamma\setminus A} e^{G(x)(1+cB/n)}\,dx\\ &\geqslant \int_\Gamma e^{G(x)(1+cB/n)}\,dx - \int_{\Gamma\setminus A} e^{G(x)}\,dx\\ &= \big(e^{-(cB/2)(1+o_n(1))} - \mathbf{P}_g(\Gamma\setminus A)\big)Z_g.\end{aligned}$$

By Lemma 5.1, we have

$$\begin{aligned}\mathbf{P}_g(\Gamma\setminus A) &\leqslant \mathbf{P}_g(\Gamma_D^c) + \mathbf{P}_g\bigg(\bigcup_{i\neq j}\frac{|x_i-x_j|}{|i-j|_o}\geqslant B\bigg)\\ &\leqslant 2e^{-c_1n^3} + n^2e^{-c_2n\ln^3 n} + n^2\max_{j,l}\mathbf{P}_g\bigg(\frac{|\xi_j^{(l)}|}{\sqrt{l}}\geqslant B\bigg).\end{aligned}$$

Since the variance of $\xi_j^{(j)}$ is at most of the order l by Proposition 5.1, we have

$$\mathbf{P}_g\bigg(\frac{|\xi_j^{(l)}|}{\sqrt{l}}\geqslant B\bigg) \leqslant e^{-c'B^2}, \tag{81}$$

where c' is a universal positive constant. Thus,

$$\mathbf{P}_g(\Gamma\setminus A) \leqslant 2e^{-c_1n^3} + n^2e^{-c_2n\ln^3 n} + n^2e^{-c'B^2},$$

and

$$\widetilde{Z}_n \geqslant \big(e^{-(cB/2)(1+o_n(1))} - n^2e^{-c'B^2} - 2e^{-c_1n^3} - n^2e^{-c_2n\ln^3 n}\big)Z_g.$$

If $B \geqslant M\ln^{1/2} n$ with some sufficiently large $M>0$, then $n^2e^{-c'B^2} + 2e^{-c_1n^3} + n^2e^{-c_2n\ln^3 n}$ is much smaller than $e^{-cB/2}$. We have

$$\widetilde{Z}_n \geqslant e^{-(cB/2)(1+o_n(1))}Z_g(1-o_n(1)) \geqslant Z_ge^{-cB}. \tag{82}$$

Lemma 6.2 is proved.

Using the result of Lemma 6.2, we can prove the following lemma.

Lemma 6.3. *For some sufficiently large $M > 0$, let $M\ln^{1/2} n \leqslant B_k \leqslant Dn^{1/2}\ln^{3/2} n$. If there exists some $\gamma > 0$ such that $kn^{-10} \leqslant n^{-\gamma}$ and*

$$\mathbf{P}_x\biggl(x \in \Gamma_D\colon \max_{i\neq j}\frac{|x_i - x_j|}{|i-j|_o} \geqslant B_k\biggr) \leqslant kn^{-10}, \tag{83}$$

then

$$\mathbf{P}_x\biggl(x \in \Gamma_D\colon \max_{i\neq j}\frac{|x_i - x_j|}{|i-j|_o} \geqslant B_{k+1}\biggr) \leqslant (k+1)n^{-10}, \tag{84}$$

with

$$B_{k+1} = \sqrt{\frac{4cB_k + 24\ln n}{c'}}. \tag{85}$$

Here c, c' are universal constants that do not depend on n, k.

Proof. Denote $A_k = \{x \in \Gamma_D\colon \max_{i\neq j}(|x_i - x_j|/|i-j|_o) \leqslant B_k\}$ and $A_k^{\mathrm{com}} = \Gamma_D \setminus A_k$. Then $\mathbf{P}_x(A_k^{\mathrm{com}}) \leqslant kn^{-10}$ by (83). Since $B_{k+1} \leqslant B_k$, we see that $A_{k+1} \subset A_k \subset \Gamma_D$ and $A_k^{\mathrm{com}} \subset A_{k+1}^{\mathrm{com}}$. By Lemma 6.2, if $x \in A_k$, then there exists a universal constant $c > 0$ such that

$$\frac{Z_g}{\widetilde{Z}_n} \leqslant e^{cB_k}.$$

Thus,

$$\begin{aligned}\mathbf{P}_x(A_{k+1}^{\mathrm{com}}) &= \mathbf{P}_x(A_k^{\mathrm{com}}) + \mathbf{P}_x(A_k \cap A_{k+1}^{\mathrm{com}})\\ &\leqslant kn^{-10} + \frac{1}{\widetilde{Z}_n}\int_{A_{k+1}^{\mathrm{com}}} e^{G(x)(1-cB_k/n)}\,dx\\ &= kn^{-10} + \frac{Z_g}{\widetilde{Z}_n}\frac{1}{Z_g}\int_{A_{k+1}^{\mathrm{com}}} e^{G(x)(1-cB_k/n)}\,dx\\ &\leqslant kn^{-10} + e^{cB_k}\biggl(\frac{1}{Z_g}\int_{A_{k+1}^{\mathrm{com}}} e^{G(x)(1-cB_k/n)}\,dx\biggr). \end{aligned}\tag{86}$$

Let $\widetilde{x} = \sqrt{1 - cB_k/n}x$. Then

$$\begin{aligned}\frac{1}{Z_g}\int_{A_{k+1}^{\mathrm{com}}} e^{G(x)(1-cB_k/n)}\,dx &= \biggl(1 - \frac{cB_k}{n}\biggr)^{-n/2}\frac{1}{Z_g}\int_{\widetilde{A}_{k+1}^{\mathrm{com}}} e^{G(\widetilde{x})}\,d\widetilde{x}\\ &\leqslant e^{cB_k}\frac{1}{Z_g}\int_{\widetilde{A}_{k+1}^{\mathrm{com}}} e^{G(\widetilde{x})}\,d\widetilde{x} = e^{cB_k}\mathbf{P}_g(\widetilde{A}_{k+1}^{\mathrm{com}}), \end{aligned}\tag{87}$$

where $\widetilde{A}_{k+1}^{\mathrm{com}} = \{x \in \Gamma_D\colon \max_{i\neq j}(|\widetilde{x}_i - \widetilde{x}_j|/|i-j|_o) > B_{k+1}\sqrt{1 - cB_k/n}\}$.

Note that

$$\begin{aligned}\mathbf{P}_g(\widetilde{A}_{k+1}^{\mathrm{com}}) &\leqslant \mathbf{P}_g\biggl(\max_{i\neq j}\frac{|x_i - x_j|}{|i-j|_o} \geqslant B_{k+1}\sqrt{1 - \frac{cB_k}{n}}\biggr)\\ &\leqslant n^2\max_{j,l}\mathbf{P}_g\biggl(|\xi_j^{(l)}| \geqslant B_{k+1}l\sqrt{1 - \frac{cB_k}{n}}\biggr)\end{aligned}$$

$$\leqslant n^2 \max_{j,l} \mathbf{P}_g\left(\frac{|\xi_j^{(l)}|}{\sqrt{l}} \geqslant B_{k+1}\sqrt{1-\frac{cB_k}{n}}\right)$$
$$\leqslant n^2 \exp\left\{-c' B_{k+1}^2\left(1-\frac{cB_k}{n}\right)\right\},$$

where $c' > 0$ is the universal constant introduced in (81). Thus

$$\text{LHS of (87)} \leqslant n^2 \exp\left\{-c' B_{k+1}^2\left(1-\frac{cB_k}{n}\right)+cB_k\right\}$$
$$\leqslant n^2 \exp\left\{-\frac{c'}{2}B_{k+1}^2+cB_k\right\} = \exp\left\{-\frac{c'}{2}B_{k+1}^2+cB_k+2\ln n\right\}. \quad (88)$$

Therefore, combining (86), (87), and (88), we have

$$\mathbf{P}_x(A_{k+1}^{\text{com}}) \leqslant kn^{-10} + \exp\left\{-\frac{c'}{2}B_{k+1}^2+2cB_k+2\ln n\right\}. \quad (89)$$

By letting the right-hand side of (89) equal $(k+1)n^{-10}$, we can solve $-c'B_{k+1}^2 + 4cB_k + 4\ln n = -20\ln n$ for B_{k+1}. We obtain $B_{k+1} = \sqrt{(4cB_k + 24\ln n)/c'}$. Lemma 6.3 is proved.

Combining Lemmas 6.2 and 6.3, we have the following result.

Proposition 6.1. *There exist some constants $C_1, C_2 > 0$ such that for sufficiently large n,*

$$\mathbf{P}_x\left(x \in \Gamma_D\colon \max_{i\neq j} \frac{|x_i - x_j|}{|i-j|_o} \geqslant C_1 \ln^{1/2} n\right) \leqslant n^{-9}, \quad (90)$$

and

$$Z_g e^{-C_2 \ln^{1/2} n} \leqslant \widetilde{Z}_n \leqslant Z_g e^{C_2 \ln^{1/2} n}. \quad (91)$$

Proof. The definition of Γ_D in (31) indicates that for $k=0$, (83) is satisfied with $B_0 = Dn^{1/2}\ln^{3/2} n$. Then we use Lemmas 6.2 and 6.3 to proceed the iteration by setting

$$B_{k+1} = \sqrt{\frac{4cB_k + 24\ln n}{c'}}. \quad (92)$$

Note that in Lemma 6.3, c, c' are universal constants.

The fixed point of the iteration (92) is

$$-c'B_f^2 + 4cB_f + 4\ln n = -20\ln n$$
$$\Longrightarrow \quad B_f = \frac{2c + \sqrt{4c^2 + 24c'\ln n}}{c'} \sim \ln^{1/2} n.$$

Recall that $B_0 \sim n^{1/2}\ln^{2/3} n$. Moreover, if $B_k \geqslant C\ln n$, then $B_{k+1} \sim \sqrt{B_k}$. This implies that for B_k to reach the value of order $\ln^{1/2} n$, one needs about

$\ln\ln n$ iteration steps. In other words, the sequence $\{B_k\}$ will starts from $B_0 = Dn^{1/2}\ln^{3/2} n$ and after $C_1' \ln\ln n$ number of iterations, B_k decreases below the level

$$B_\infty = C_1 \ln^{1/2} n.$$

Finally, we still need to check if the conditions of Lemmas 6.2, 6.3 are satisfied. To satisfy the first condition of Lemma 6.3 (also Lemma 6.2), i.e., $M \ln^{1/2} n \leqslant B_k \leqslant Dn^{1/2}\ln^{3/2} n$, we need to modify the stopping time of the iteration process. We will end the iteration right before B_k falls below $M \ln^{1/2} n$. But the result remains the same. Note that the number of iteration steps is of the order $\ln\ln n$, so the second condition of Lemma 6.3 also holds, i.e., $kn^{-10} \leqslant n^{-\gamma}$ for some $\gamma > 0$.

Therefore, we can find a subset, denoted as

$$A_\infty = \left\{ x \in \Gamma_D\colon \max_{i\neq j} \frac{|x_i - x_j|}{|i-j|_o} \leqslant C_1 \ln^{1/2} n \right\}, \tag{93}$$

such that

$$\mathbf{P}_x(\Gamma_D \setminus A_\infty) \leqslant n^{-9}. \tag{94}$$

Moreover, by Lemma 6.2, there exists some constant $C_2 > 0$ such that

$$Z_g e^{-C_2 \ln^{1/2} n} \leqslant \widetilde{Z}_n \leqslant Z_g e^{C_2 \ln^{1/2} n}.$$

Proposition 6.1 is proved.

Combining (94) and (32), we have the following corollary.

Corollary 6.1. *There exists a subset A_∞ defined in (93) such that*

$$\mathbf{P}_x(A_\infty^c) \leqslant n^{-8}. \tag{95}$$

6.2. Step 2: Estimate of $F(x)$ under Gaussian distribution. If x belongs to the set A_∞, which is defined in (93), then

$$\max_{i\neq j} \frac{|x_i - x_j|}{|i-j|_o} \leqslant C_1 \ln^{1/2} n, \tag{96}$$

and by (73), we obtain that

$$F(x) = O\left(\sum_{i>j} \frac{(x_i - x_j)^3}{n|i-j|_o^5} \right) = O(\ln^{3/2} n). \tag{97}$$

Note that

$$\sum_{i>j;\, |i-j|_o > \ln^3 n} \frac{|x_i - x_j|^3}{n|i-j|_o^5} \leqslant C \ln^{3/2} n \sum_{l > \ln^3 n} \frac{1}{l^2} \leqslant C \ln^{-3/2} n = o_n(1). \tag{98}$$

Thus, it is sufficient for us to estimate

$$\sum_{i>j;\,|i-j|_o\leqslant \ln^3 n} \frac{(x_i-x_j)^3}{n|i-j|_o^5} = \sum_{k=1}^{\ln^3 n} \frac{1}{l^5}\left(\frac{1}{n}\sum_{j=0}^{n-1}(\xi_j^{(l)})^3\right), \tag{99}$$

where $\xi_j^{(l)}$ is defined in (66). Denote

$$\Omega_l = \left\{\left|\frac{1}{n}\sum_{i=1}^{n-1}(\xi_i^{(l)})^3\right| \leqslant n^{-1/4}\right\}, \qquad 1\leqslant l\leqslant \ln^3 n, \tag{100}$$

and $\Omega_\infty = \bigcap_{l\leqslant \ln^3 n}\Omega_l$. Note that if $x\in\Omega' := \Omega_\infty\cap A_\infty$, then

$$\sum_{l=1}^{\ln^3 n}\frac{1}{l^5}\left|\frac{1}{n}\sum_{j=0}^{n-1}(\xi_j^{(l)})^3\right| \leqslant n^{-1/4}\sum_{l=1}^{\ln^3 n}\frac{1}{l^5} = O(n^{-1/4}). \tag{101}$$

Combining (99), (101), and (98), we have

$$F(x) = O(\ln^{-3/2} n). \tag{102}$$

Using Proposition 5.1 and Lemma 5.1, one can show that

$$\begin{aligned}\mathbf{P}_g(A_\infty^c) &\leqslant \mathbf{P}_g\left(\max_{i\neq j}\frac{|x_i-x_j|}{|i-j|_o}\geqslant C_1\ln^{1/2} n\right) + \mathbf{P}_g(\Gamma_D^c)\\ &\leqslant n^2 e^{-C_1' l\ln n} + 2e^{-c_1 n^3} + n^2 e^{-c_2 n\ln^3 n} \leqslant n^{-1/4},\end{aligned} \tag{103}$$

provided that C_1 and n are chosen sufficiently large.

Next, we want to show that $\mathbf{P}_g(\Omega_\infty^c) = o_n(1)$. The following lemma is useful.

Lemma 6.4. *There exists some constant $C>0$ such that*

$$\mathbf{P}_g\left(\left|\frac{1}{n}\sum_{i=1}^{n-1}(\xi_i^{(l)})^3\right| > n^{-1/4}\right) \leqslant \frac{Cl^4\ln n}{n^{1/2}}. \tag{104}$$

Proof. By the Wick formula, we have

$$\mathbf{E}_g(\xi_j^{(l)})^3(\xi_k^{(l)})^3 = 9\mathbf{E}_g(\xi_j^{(l)})^2\mathbf{E}_g(\xi_k^{(l)})^2\mathbf{E}_g\xi_j^{(l)}\xi_k^{(l)} + 6(\mathbf{E}_g\xi_j^{(l)}\xi_k^{(l)})^3.$$

By Proposition 5.1, we have $\mathbf{E}_g(\xi_k^{(l)})^2\leqslant Cl$ and $\mathbf{E}_g\xi_k^{(l)}\mathbf{E}_g\xi_j^{(l)}\leqslant Cl^2/|k-j|_o$ for $|j-k|_o\geqslant l$. Then when $|j-k|_o\geqslant l$, we have

$$\mathbf{E}_g(\xi_j^{(l)})^3(\xi_k^{(l)})^3 \leqslant 9C^3\frac{l^4}{|j-k|_o} + 6C^3\frac{l^6}{|j-k|_o^3} \leqslant \frac{C'l^4}{|j-k|_o}.$$

Similarly, if $|j-k|_o \leqslant l$, then $\mathbf{E}_g\xi_k^{(l)}\mathbf{E}_g\xi_j^{(l)} \leqslant Cl$, and thus $\mathbf{E}_g(\xi_j^{(l)})^3(\xi_k^{(l)})^3 \leqslant 15C^3l^3 \leqslant C'l^4/|j-k|_o$. Therefore, there exists some constant $C'>0$ such that

$$\mathbf{E}_g\left(\sum_{i=1}^{n-1}(\xi_i^{(l)})^3\right)^2 = n\mathbf{E}_g(\xi_k^{(l)})^6 + 4\sum_{j\neq k}\mathbf{E}_g(\xi_k^{(l)})^3(\xi_j^{(l)})^3 \leqslant C'l^4 n\ln n.$$

It follows from the Markov inequality that

$$\mathbf{P}_g\left(\left|\frac{1}{n}\sum_{i=1}^{n-1}(\xi_i^{(l)})^3\right| > n^{-1/4}\right) \leqslant \frac{\mathbf{E}_g\left(\sum_{i=1}^{n-1}(\xi^{(l)}_k)^3\right)^2}{n^{3/2}} \leqslant \frac{C'l^4\ln n}{n^{1/2}}. \tag{105}$$

Lemma 6.4 is proved.

By using Lemma 6.4, it can be shown directly that, for sufficiently large n, we have

$$\mathbf{P}_g(\Omega_\infty^c) \leqslant \mathbf{P}_g\left(\bigcup_{l\leqslant \ln^3 n}\Omega_l^c\right) \leqslant \frac{C'l^4\ln^3 n}{n^{1/2}} \leqslant \frac{C'\ln^{15} n}{n^{1/2}} \leqslant n^{-1/4}. \tag{106}$$

Let $x\in\Omega' = \Omega_\infty\cap A_\infty$. Combining (103) and (106), we see that

$$\mathbf{P}_g(\Omega'^c) \leqslant \mathbf{P}_g(\Omega_\infty^c) + \mathbf{P}_g(A_\infty^c) \leqslant 2n^{-1/4}. \tag{107}$$

Therefore, we have the following lemma.

Lemma 6.5. *There exists a subset of $\Omega'\subset\Gamma$ such that*

$$\mathbf{P}_g(\Omega'^c) = o_n(1),$$

and if $x\in\Omega'$, then $F(x) = o_n(1)$.

6.3. Step 3: Combining Step 1 and Step 2. In this subsection, we finish the proofs of Proposition 3.1 and Theorem 3.1 by combining the results in Step 1 and Step 2. In Step 1, we have showed that $\mathbf{P}_x(A_\infty^c) = o_n(1)$. In Step 2, we have obtained that $\mathbf{P}_g(\Omega_\infty^c) = o_n(1)$ and $\mathbf{P}_g(A_\infty^c) = o_n(1)$.

Proof of Proposition 3.1 and Theorem 3.1. We start by showing $\mathbf{P}_x(\Omega_\infty^c) = o_n(1)$. Recall that $\Omega_l = \left\{\left|(1/n)\sum_{i=1}^{n-1}(\xi_i^{(l)})^3\right| \leqslant n^{-1/4}\right\}$, $1\leqslant l\leqslant \ln^3 n$, and $\Omega_\infty = \bigcap_{l\leqslant\ln^3 n}\Omega_l$. Then

$$\mathbf{P}_x(\Omega_l^c) \leqslant \mathbf{P}_x(\Gamma_D^c) + \mathbf{P}_x(A_\infty^c) + \mathbf{P}_x(\Omega_l^c\cap A_\infty\cap\Gamma_D).$$

By (32) and (95),

$$\begin{aligned}\mathbf{P}_x(\Omega_l^c) &\leqslant n^{-cn} + n^{-8} + \frac{1}{\widetilde{Z}_n}\int_{\Omega_l^c}\exp\left\{G(x)\left(1-\frac{C_1\ln^{1/2} n}{n}\right)\right\}dx \\ &\leqslant 2n^{-8} + \frac{Z_g}{\widetilde{Z}_n}\frac{1}{Z_g}\int_{\Omega_l^c}\exp\left\{G(x)\left(1-\frac{C_1\ln^{1/2} n}{n}\right)\right\}dx. \end{aligned}\tag{108}$$

Changing the variable $\widetilde{x} = \sqrt{1 - C_1(\ln^{1/2} n)/n}\, x$, by Lemma 6.1, we have

$$\mathbf{P}_x(\Omega_l^c) \leqslant 2n^{-8} + e^{C_1 \ln^{1/2} n} \mathbf{P}_g(\widetilde{\Omega}_l^c), \tag{109}$$

where $\widetilde{\Omega}_l^c = \{ |(1/n) \sum_{i=1}^{n-1} (\widetilde{\xi_i^{(l)}})^3| \geqslant n^{-1/4}(1 - C_1(\ln^{1/2} n)/n)^{3/2} \}$. Using the Markov inequality and following the arguments in (105) in Lemma 6.4, we have

$$\mathbf{P}_g(\widetilde{\Omega}_l^c) = \frac{1}{Z_g} \int_{\widetilde{\Omega}_l^c} e^{G(\widetilde{x})}\, d\widetilde{x} \leqslant \frac{C l^4 \ln(n-l)}{n^{1/2}(1 - C_1(\ln^{1/2} n)/n)^3} \leqslant \frac{C' l^4 \ln n}{n^{1/2}}. \tag{110}$$

Combining (110), (109), and (91), we obtain

$$\mathbf{P}_x(\Omega_l^c) \leqslant 2n^{-8} + \frac{C' l^4 (\ln n) e^{(C_1 + C_2) \ln^{1/2} n}}{n^{1/2}}.$$

Thus,

$$\mathbf{P}_x(\Omega_\infty^c) = \mathbf{P}_x\Bigl(\bigcup_{l \leqslant \ln^3 n} \Omega_l^c \Bigr) \leqslant 2n^{-8} \ln^3 n + \frac{C'(\ln^{13} n) e^{C_3 \ln^{1/2} n}}{n^{1/2}} = O(n^{-1/4}). \tag{111}$$

Repeating the arguments from Step 2, we have that (97), (98), and (99) hold for $x \in A_\infty$ (recall that A_∞ is defined in (93)). If $x \in \Omega_\infty$, then (101) also holds. Therefore, if $x \in \Omega' = A_\infty \cap \Omega_\infty$, then the bound (102) on $F(x)$ still holds. In addition, by (95) and (111), we have

$$\mathbf{P}_x(\Omega'^c) \leqslant \mathbf{P}_x(\Omega_\infty^c) + \mathbf{P}_x(A_\infty^c) = O(n^{-1/4}). \tag{112}$$

Thus, we have proved Proposition 3.1. Combining (112), (107), and (102), one can show that

$$(1 - C' \ln^{-3/2} n) Z_g \leqslant \widetilde{Z}_n \leqslant (1 + C' \ln^{-3/2} n) Z_g.$$

If A is a measurable subset of Ω', then

$$\begin{aligned} \mathbf{P}_x(A) &= \frac{1}{\widetilde{Z}_n} \int_A e^{G(x) + F(x)}\, dx \leqslant (1 + C'' \ln^{-3/2} n) \frac{1}{Z_g} \int_A e^{G(x)}\, dx \\ &= \mathbf{P}_g(A)\bigl(1 + O(\ln^{-3/2} n)\bigr). \end{aligned}$$

Similarly, we have

$$\mathbf{P}_x(A) \geqslant (1 - C'' \ln^{-3/2} n) \frac{1}{Z_g} \int_A e^{G(x)}\, dx = \mathbf{P}_g(A)\bigl(1 - O(\ln^{-3/2} n)\bigr).$$

Combining with (112) and (107), we conclude that for any measurable set $A \subset \Gamma$,

$$|\mathbf{P}_x(A) - \mathbf{P}_g(A)| = O(\ln^{-3/2} n). \tag{113}$$

Theorem 3.1 is proved.

7. Proof of Theorem 3.2. In this section, we prove the functional convergence in distribution of $\zeta_n(t)$ to $\zeta(t)$.

First, we establish the convergence of finite dimensional distributions. Fix finitely many $0 \leqslant t_1, \ldots, t_m \leqslant 2\pi$. Let $j_l = \lfloor nt_l/(2\pi) \rfloor$, $l = 1, \ldots, m$. Because of (96) and the construction of $\zeta_n(t)$, with probability $1 - o_n(1)$, we have

$$\left| \zeta_n(t_l) - \frac{x_{j_l}}{\sqrt{n}} \right| \leqslant \left| \frac{x_{j_l}}{\sqrt{n}} - \frac{x_{j_l+1}}{\sqrt{n}} \right| = o_n(1).$$

Using the definition of $\zeta(t)$, one can also show that $|\zeta(t_l) - \zeta(2\pi j_l/n)| = o_n(1)$ with high probability. It is sufficient to prove that $x_j/\sqrt{n} = \zeta(2\pi j_l/n) + o_n(1)$ with probability $1 - o_n(1)$. By Theorem 3.1, the finite dimensional distribution of $(x_{j_1}, \ldots, x_{j_m})$ can be approximated by the finite dimensional distribution of the Gaussian law defined in (35). Without loss of generality, assume that n is odd. For even case, similar considerations hold. Using the representation (65) for x_j, we have

$$\begin{aligned}\frac{x_j}{\sqrt{n}} &= \frac{2}{n} \sum_{k=1}^{(n-1)/2} \left[\cos\left(\frac{2\pi jk}{n} \right) \frac{n^2}{2k(n-k)} X_k - \sin\left(\frac{2\pi jk}{n} \right) \frac{n^2}{2k(n-k)} Y_k \right] \\ &= \sum_{k=1}^{(n-1)/2} \left(\frac{\cos(2\pi jk/n)}{k} X_k - \frac{\sin(2\pi jk/n)}{k} Y_k \right) \\ &\quad + \sum_{k=1}^{(n-1)/2} \left(\frac{\cos(2\pi jk/n)}{n-k} X_k - \frac{\sin(2\pi jk/n)}{n-k} Y_k \right) \\ &= \sum_{k=1}^{\infty} \left(\frac{\cos(2\pi jk/n)}{k} X_k - \frac{\sin(2\pi jk/n)}{k} Y_k \right) + e_n \quad \text{for} \quad 0 \leqslant j \leqslant n-1,\end{aligned} \tag{114}$$

where $\{X_k\}$ and $\{Y_k\}$ are i.i.d. real standard normal random variables. Here

$$\begin{aligned}e_n = &\sum_{k=(n+1)/2}^{\infty} \left(\frac{\cos(2\pi jk/n)}{k} X_k - \frac{\sin(2\pi jk/n)}{k} Y_k \right) \\ &+ \sum_{k=1}^{(n-1)/2} \left(\frac{\cos(2\pi jk/n)}{n-k} X_k - \frac{\sin(2\pi jk/n)}{n-k} Y_k \right)\end{aligned}$$

is negligible because

$$\mathbf{D}_g e_n = \sum_{k=(n+1)/2}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{(n-1)/2} \left(\frac{1}{n-k} \right)^2 = O\left(\frac{1}{n} \right).$$

Therefore, by (113), for $1 \leqslant l \leqslant m$,

$$\zeta_n\left(\frac{2\pi j_l}{n}\right) = \frac{x_{j_l}}{\sqrt{n}} = \sum_{k=1}^{\infty}\left(\frac{\cos(2\pi j_l k/n)}{k}X_k - \frac{\sin(2\pi j_l k/n)}{k}Y_k\right) + e_n$$
$$= \zeta\left(\frac{2\pi j_l}{n}\right) + e_n, \tag{115}$$

where $\{X_k\}$, $\{Y_k\}$ are i.i.d. real Gaussian variables, e_n is a random error term with $\mathbf{D}e_n = o_n(1)$. Therefore, one proves that $\zeta_n(t)$ converges in finite dimensional distribution to $\zeta(t)$.

Now, we turn our attention to functional convergence. Note that the sequence of the distributions of $\zeta_n(t)$ gives a family of probability measures on the space $C[0, 2\pi]$. Because of the finite dimensional distribution convergence, it is sufficient for us to show the tightness of the distribution sequence. A sequence of probability measures $\{\mathbf{P}_n\}$ is tight if and only if the following two conditions hold [3]:

1) For any small $\eta > 0$, there exist corresponding a and n_0, such that

$$\mathbf{P}_n\bigl(f : |f(0)| \geqslant a\bigr) \leqslant \eta \quad \text{for} \quad n \geqslant n_0;$$

2) for any small $\varepsilon, \eta > 0$, there exist corresponding δ_0 and n_0 such that

$$\mathbf{P}_n\bigl(f : \omega_f(\delta_0) \geqslant \varepsilon\bigr) \leqslant \eta \quad \text{for} \quad n \geqslant n_0,$$

where $\omega_f(\delta) = \sup\{|f(s) - f(t)| : 0 \leqslant s, t \leqslant 2\pi,\ |s - t| < \delta\}$.

To check the first condition, by (115), we note that

$$\zeta_n(0) = \frac{x_0}{\sqrt{n}} = \sum_{k=1}^{\infty} \frac{1}{k} X_k + e_n, \tag{116}$$

and thus there exists sufficiently large n_0 such that if $n > n_0$, then

$$\mathbf{D}\zeta_n(0) = \sum_{k=1}^{\infty} \frac{1}{k^2} + o_n(1) \leqslant \frac{\pi^2}{3}. \tag{117}$$

Then

$$\mathbf{P}_x(|\zeta_n(0)| \geqslant a) \leqslant \frac{\pi^2}{3a^2}, \tag{118}$$

we choose $a = \sqrt{\pi^2/(3\eta)}$.

Further, to check the second condition, we need the following lemma.

Lemma 7.1. *There exist positive constants c_k $(1 \leqslant k \leqslant 6)$ such that*

$$\mathbf{P}_x\bigl(|\zeta_n(t) - \zeta_n(s)| \leqslant c_1|t - s|^{1/20} + c_2 n^{-1/10} \ln n \ \forall\, t, s \in [0, 2\pi] : |t - s| \leqslant \delta\bigr)$$
$$> 1 - (c_3 e^{-c_4 n^{4/5}} + c_5 \delta^{4/5} + c_6 \ln^{-3/2} n).$$

Assuming that Lemma 7.1 is proved, we can finish the proof of Theorem 3.2 by choosing δ_0 and n_0 such that

$$c_1\delta_0^{1/20} \leqslant \frac{\varepsilon}{2}, \qquad c_2 n_0^{-1/10}\ln n_0 \leqslant \frac{\varepsilon}{2}; \tag{119}$$

$$c_3 e^{-c_4 n_0^{4/5}} \leqslant \frac{\eta}{3}, \qquad c_5\delta_0^{4/5} \leqslant \frac{\eta}{3}, \qquad c_6 \ln^{-3/2} n_0 \leqslant \frac{\eta}{3}. \tag{120}$$

Finally, we need to prove Lemma 7.1.

Proof of Lemma 7.1. Because of (113), it is sufficient to prove that

$$\begin{aligned}\mathbf{P}_g\big(|\zeta_n(t)-\zeta_n(s)| \leqslant c_1|t-s|^{1/20} + c_2 n^{-1/10}\ln n \ \forall\, t,s\in[0,2\pi]\colon |t-s|\leqslant\delta\big)\\ > 1-(c_3e^{-c_4n^{4/5}}+c_5\delta^{4/5}).\end{aligned}$$

Without loss of generality, assume that $s<t$.

Case 1: Let us fix $C_0>0$ and assume that $|t-s|\leqslant C_0/n$. Then there exist i, j with $|i-j|\leqslant C_0+2$ such that

$$s\in\left[\frac{2\pi(i-1)}{n},\frac{2\pi i}{n}\right], \qquad t\in\left[\frac{2\pi j}{n},\frac{2\pi(j+1)}{n}\right]. \tag{121}$$

By the triangle inequality, we have

$$\begin{aligned}&\mathbf{P}_g(|\zeta_n(t)-\zeta_n(s)|>\varepsilon)\\ &\quad\leqslant \sum_{i\leqslant k\leqslant j+1}\mathbf{P}_g\left(\left|\zeta_n\left(\frac{2\pi(k-1)}{n}\right)-\zeta_n\left(\frac{2\pi k}{n}\right)\right| > \frac{\varepsilon}{|i-j|+2}\right).\end{aligned}$$

Note that $\zeta_n(2\pi k/n)=x_k/\sqrt{n}$ and $\mathbf{D}(x_k-x_{k-1})\sim 1$ for all k. Then for some positive constant C_2, C_3 depending on C_0, we have

$$\begin{aligned}\mathbf{P}_g\left(\left|\zeta_n\left(\frac{2\pi(k-1)}{n}\right)-\zeta_n\left(\frac{2\pi k}{n}\right)\right|\geqslant C_2\varepsilon\right) &= \mathbf{P}_g(|x_{k-1}-x_k|\geqslant C_2\varepsilon\sqrt{n})\\ &\leqslant C_3e^{-c'C_2^2\varepsilon^2 n}.\end{aligned}$$

Thus

$$\mathbf{P}_g(|\zeta_n(t)-\zeta_n(s)|>\varepsilon)\leqslant (C_0+2)C_3e^{-c'C_2^2\varepsilon^2 n}. \tag{122}$$

Furthermore, there exist positive constants C_4, C_5 which only depend on C_0, such that

$$\begin{aligned}\mathbf{P}_g\left(\exists\, s,t\in[0,2\pi]\colon |t-s|\leqslant\frac{C_0}{n}\ \text{and}\ |\zeta_n(t)-\zeta_n(s)|>\varepsilon\right) &\leqslant n^2C_0C_3e^{-c'C_2^2\varepsilon^2n}\\ &\leqslant C_5e^{-C_4\varepsilon^2 n}.\end{aligned} \tag{123}$$

Case 2: If $|t-s| \gg 1/n$, we introduce a new partition of $[s,t]$. We start by dividing the interval $[0,2\pi]$ into 2^k disjoint subintervals

$$\Delta_l^{(k)} = \left[\frac{2\pi}{2^k} l, \frac{2\pi}{2^k}(l+1)\right], \qquad l = 0,1,\dots,2^k-1. \tag{124}$$

There exist the smallest $k = k_0$, and related l_0, such that $\Delta_{l_0}^{(k_0)} \subset [s,t]$. Note that k_0 and l_0 are unique. Let $S_0 = \Delta_{l_0}^{(k_0)}$. If $[s,t] \neq S_0$, then there exists a unique smallest $k_1 > k_0$, and one or two values of l_1, such that $\Delta_{l_1}^{(k_1)} \subset [s,t] \setminus S_0$ (we could potentially add $\Delta_{l_1}^{(k_1)}$ on the left of $\Delta_{l_0}^{(k_0)}$ or add one on the right). If there is only one value of l_1, let $\Delta_{a_1}^{(k_1)} = \Delta_{l_1}^{(k_1)}$ and $\Delta_{b_1}^{(k_1)} = \varnothing$. If there are two values of l_1, let a_1 be the smallest of the two and b_1 the largest. Set $S_1 = S_0 \cup \Delta_{b_1}^{(k_1)} \cup \Delta_{a_1}^{(k_1)}$. We continue this process. For each $m \geqslant 2$, find a unique smallest $k_m > k_{m-1}$ such that $\Delta_{a_m}^{(k_m)}, \Delta_{b_m}^{(k_m)} \subset [s,t] \setminus S_{m-1}$ (recall that $\Delta_{b_m}^{(k_m)}$ might be empty). Let $S_m = S_{m-1} \cup \Delta_{b_m}^{(k_m)} \cup \Delta_{a_m}^{(k_m)}$. We will stop at $m = r$, when either $[s,t] = S_r$ or the length of $S_{r+1} := [s,t] \setminus S_r$ is not greater than C_0/n, where $C_0 > 0$ is some fixed constant. Thus,

$$[s,t] = S_0 \cup S_{r+1} \cup \left(\bigcup_{m=1}^{r} \Delta_{b_m}^{(k_m)}\right) \cup \left(\bigcup_{m=1}^{r} \Delta_{a_m}^{(k_m)}\right). \tag{125}$$

Note that

$$\frac{1}{2^{k_0}} \leqslant |s-t| \leqslant \frac{3}{2^{k_0}}, \qquad \frac{1}{2^{k_r}} \leqslant \frac{C_0}{n} \leqslant \frac{1}{2^{k_r-1}}, \tag{126}$$

i.e., $k_0 \sim -\ln|s-t|$, $k_r \sim \ln n$.

For any interval $I = [a,b]$, define $D_n(I) := |\zeta_n(a) - \zeta_n(b)|$. For example, $D_n(\Delta_l^{(k)}) := |\zeta_n((2\pi/2^k)(l+1)) - \zeta_n((2\pi/2^k)l)|$. Let $a_0 = l_0$. Then by the triangle inequality, we have

$$\begin{aligned} D_n([s,t]) &\leqslant D_n(\Delta_{l_0}^{(k_0)}) + \sum_{m=1}^{r} D_n(\Delta_{a_m}^{(k_m)}) + \sum_{m=1}^{r} D_n(\Delta_{b_m}^{(k_m)}) + D_n(S_{r+1}) \\ &\leqslant 2\sum_{m=0}^{r} D_n(\Delta_{a_m}^{(k_m)}) + D_n(S_{r+1}). \end{aligned}$$

Since $|S_{r+1}| \leqslant C_0/n$, using the result in Case 1 and letting $\varepsilon = n^{-1/10}$, we have for some constants $C_1, C_2 > 0$,

$$\mathbf{P}_g\left(D_n(S_{r+1}) > n^{-1/10},\ \exists\, S_{r+1} \subset [0,2\pi] \colon |S_{r+1}| \leqslant \frac{C_0}{n}\right) \leqslant C_1 e^{-C_2 n^{4/5}}. \tag{127}$$

To estimate $D_n(\Delta_{a_m}^{(k_m)})$, we need the following lemma.

Lemma 7.2. *Fix $s, t \in [0, 2\pi]$. Then there exist some constants C_1, C_2, $C_4 > 0$ such that*

$$\mathbf{P}_g\big(|\zeta_n(t) - \zeta_n(s)| > |t-s|^{1/20} + 2n^{-1/10}\big) \leqslant 2C_1 e^{-C_2 n^{4/5}} + C_4|t-s|^{9/5}.$$

Proof. Fix s, t. Then there exist i, j such that

$$s \in \left[\frac{2\pi(i-1)}{n}, \frac{2\pi i}{n}\right], \qquad t \in \left[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right].$$

Thus, we have

$$\begin{aligned}
&\mathbf{P}_g\big(|\zeta_n(t) - \zeta_n(s)| > |t-s|^{1/20} + 2n^{-1/10}\big) \\
&\quad \leqslant \mathbf{P}_g\left(\left|\zeta_n\left(\frac{2\pi(i-1)}{n}\right) - \zeta_n\left(\frac{2\pi i}{n}\right)\right| > n^{-1/10}\right) \\
&\qquad + \mathbf{P}_g\left(\left|\zeta_n\left(\frac{2\pi j}{n}\right) - \zeta_n\left(\frac{2\pi(j+1)}{n}\right)\right| > n^{-1/10}\right) \\
&\qquad + \mathbf{P}_g\left(\left|\zeta_n\left(\frac{2\pi i}{n}\right) - \zeta_n\left(\frac{2\pi j}{n}\right)\right| > |t-s|^{1/20}\right).
\end{aligned}$$

Note that $\zeta_n(2\pi k/n) = x_k/\sqrt{n}$, and $\mathbf{D}(x_k - x_l) \sim |k-l|$. Thus, we have

$$\mathbf{P}_g\left(\left|\zeta_n\left(\frac{2\pi i}{n}\right) - \zeta_n\left(\frac{2\pi j}{n}\right)\right| > |t-s|^{1/20}\right) \leqslant \frac{C_3|i-j|^2}{n^2|t-s|^{1/5}} \leqslant C_4|t-s|^{9/5},$$

where the last inequality comes from $2\pi|i-j|/n \leqslant |t-s|$. Therefore,

$$\mathbf{P}_g\big(|\zeta_n(t) - \zeta_n(s)| > |t-s|^{1/20} + 2n^{-1/10}\big) \leqslant 2C_1 e^{-C_2 n^{4/5}} + C_4|t-s|^{9/5}.$$

Lemma 7.2 is proved.

By the result of Lemma 7.2, we have

$$\begin{aligned}
&\mathbf{P}_g\left(\bigcup_{l=0}^{2^k-1}\left\{D(\Delta_l^{(k)}) > \left(\frac{2\pi}{2^k}\right)^{1/20} + 2n^{-1/10}\right\}\right) \\
&\quad \leqslant 2^k\left(2C_1 e^{-C_2 n^{4/5}} + C_4\left(\frac{2\pi}{2^k}\right)^{9/5}\right) \leqslant 2^{k+1}C_1 e^{-C_2 n^{4/5}} + \frac{C_5}{(2^k)^{4/5}}.
\end{aligned} \tag{128}$$

Therefore,

$$\begin{aligned}
&\mathbf{P}_g\left(\bigcup_{k\geqslant k_0}^{k_r}\bigcup_{l=0}^{2^k-1}\left\{D(\Delta_l^{(k)}) > \left(\frac{2\pi}{2^k}\right)^{1/20} + 2n^{-1/10}\right\}\right) \\
&\quad \leqslant C_6 \cdot 2^{k_r} e^{-C_2 n^{4/5}} + \frac{C_7}{(2^{k_0})^{4/5}}.
\end{aligned} \tag{129}$$

Combining with (127), we have

$$\mathbf{P}_g\biggl(\bigcap_{k\geqslant k_0}^{k_r}\bigcap_{l=0}^{2^k-1}\biggl(\biggl\{D(\Delta_l^{(k)})\leqslant\biggl(\frac{2\pi}{2^k}\biggr)^{1/20}+2n^{-1/10}\biggr\}$$

$$\cap\biggl\{D(S_{r+1})\leqslant n^{-1/10}\ \forall\, S_{r+1}\subset[0,2\pi]\colon |S_{r+1}|\leqslant\frac{C_0}{n}\biggr\}\biggr)\biggr)$$

$$\geqslant 1-\biggl((C_1+C_6)\cdot 2^{k_r}e^{-C_2n^{4/5}}+\frac{C_7}{(2^{k_0})^{4/5}}\biggr). \tag{130}$$

Due to $r\leqslant k_r$ and (125), then

LHS of (130)

$$\leqslant\mathbf{P}_g\biggl(|\zeta_n(s)-\zeta_n(t)|\leqslant C_8\biggl(\frac{1}{2^{k_0}}\biggr)^{1/20}+C_9k_rn^{-1/10}\ \forall\, s,t\in[0,2\pi]\colon |t-s|\leqslant\delta\biggr).$$

Therefore, combining with (126), we have

$$\mathbf{P}_g\bigl(|\zeta_n(s)-\zeta_n(t)|\leqslant c_1|s-t|^{1/20}+c_2n^{-1/10}\ln n\ \forall\, s,t\in[0,2\pi]\colon |t-s|\leqslant\delta\bigr)$$
$$\geqslant 1-(c_3e^{-c_4n^{4/5}}+c_5\delta^{4/5}).$$

Lemma 7.1 is proved.

The proof of Corollary 3.1 is rather straightforward and left to the reader.

REFERENCES

1. T. H. Baker, P. J. Forrester, “Finite-N fluctuation formulas for random matrices”, *J. Statist. Phys.*, **88**:5-6 (1997), 1371–1386.
2. M. V. Berry, J. P. Keating, “The Riemann zeros and eigenvalues asymptotics”, *SIAM Rev.*, **41**:2 (1999), 236–266.
3. П. Биллингсли, *Сходимость вероятностных мер*, Наука, М., 1977, 351 с.; пер. с англ.: P. Billingsley, *Convergence of probability measures*, John Wiley & Sons, Inc., New York–London–Sydney, 1968, xii+253 pp.
4. E. B. Bogomolny, J. P. Keating, “Random matrix theory and the Riemann zeros. II. n-point correlations”, *Nonlinearity*, **9**:4 (1996), 911–935.
5. P. Bourgade, J. P. Keating, “Quantum chaos, random matrix theory, and the Riemann ζ-function”, *Chaos*, Proceedings of the 14th Poincaré seminar (Paris, 2010), Prog. Math. Phys., **66**, Birkhäuser/Springer, Basel, 2013, 125–168.
6. P. Diaconis, S. N. Evans, “Linear functionals of eigenvalues of random matrices”, *Trans. Amer. Math. Soc.*, **353**:7 (2001), 2615–2633.
7. P. Diaconis, M. Shahshahani, “On the eigenvalues of random matrices”, *J. Appl. Probab.*, **31A** (1994), 49–62.
8. F. J. Dyson, “Correlations between eigenvalues of a random matrix”, *Comm. Math. Phys.*, **19** (1970), 235–250.
9. P. J. Forrester, *Log-gases and random matrices*, London Math. Soc. Monogr. Ser., **34**, Princeton Univ. Press, Princeton, NJ, 2010, xiv+791 pp.

10. C. P. Hughes, J. P. Keating, N. O'Connell, "On the characteristic polynomial of a random unitary matrix", *Comm. Math. Phys.*, **220**:2 (2001), 429–451.
11. K. Johansson, "On Szegő's asymptotic formula for Toeplitz determinants and generalizations", *Bull. Sci. Math.* (2), **112**:3 (1988), 257–304.
12. Ж.-П. Кахан, *Случайные функциональные ряды*, Мир, М., 1973, 302 с.; пер. с англ.: J.-P. Kahane, *Some random series of functions*, D. C. Heath and Co. Raytheon Education Co., Lexington, MA, 1968, viii+184 pp.
13. J. P. Keating, N. C. Snaith, "Random matrix theory and $\zeta(1/2+it)$", *Comm. Math. Phys.*, **214**:1 (2000), 57–89.
14. H. L. Montgomery, "The pair correlation of zeros of the zeta function", *Analytic number theory* (St. Louis Univ., St. Louis, MO, 1972), Proc. Sympos. Pure Math., **24**, Amer. Math. Soc., Providence, RI, 1973, 181–193.
15. A. M. Odlyzko, *The* 10^{20}*-th zero of the Riemann zeta function and* 175 *million of its neighbors*, unpubl. manuscript, 1992, 163 pp., http://www.dtc.umn.edu/~odlyzko/unpublished/index.html.
16. Z. Rudnick, P. Sarnak, "Zeros of principal L-functions and random matrix theory", *Duke Math. J.*, **81**:2 (1996), 269–322.
17. A. Soshnikov, "The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities", *Ann. Probab.*, **28**:3 (2000), 1353–1370.
18. T. Tao, *Topics in random matrix theory*, Grad. Stud. Math., **132**, Amer. Math. Soc., Providence, RI, 2012, x+282 pp.

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