Janossy Densities II. Pfaffian Ensembles. *

Alexander Soshnikov†

Abstract

We extend the main result of the companion paper math-ph/0212063 to the case of the pfaffian ensembles.

1 Introduction and Formulation of Results

Let us consider a $2n$-particle pfaffian ensemble introduced by Rains in [9]: Let (X, λ) be a measure space, $\phi_1, \phi_2, \dots \phi_{2n}$ be complex-valued functions on X, and $\epsilon(x, y)$ be an antisymmetric kernel such that

$$p(x_1, \dots, x_{2n}) = (1/Z_{2n}) \det(\phi_j(x_k))_{j,k=1,\dots 2n}\ pf(\epsilon(x_j, x_k))_{j,k=1,\dots,2n} \tag{1}$$

defines the density of a $2n$-dimensional probability distribution on $X^{2n} = X \times \cdots \times X$ with respect to the product measure $\lambda^{\otimes 2n}$. Ensembles of this form were introduced in [9] and [11]. We recall (see e.g. [5]) that the pfaffian of a $2n \times 2n$ antisymmetric matrix $A = (a_{jk}), j, k = 1, \dots, 2n,\ a_{jk} = -a_{kj}$, is defined as $pf(A) = \sum_\tau (-1)^{sign(\tau)} a_{i_1 j_1} \times \cdots \times a_{i_n, j_n}$, where the summation is over all partitions of the set $\{1, \dots, 2m\}$ into disjoint pairs $\{i_1, j_1\}, \dots, \{i_n, j_n\}$ such that $i_k < j_k,\ k = 1, \dots n$, and $sign(\tau)$ is the sign of the permutation $(i_1, j_1, \dots, i_n, j_n)$. The normalization constant in (1) (usually called the partition function)

$$Z_{2n} = \int_{X^{2n}} \det(\phi_j(x_k))_{j,k=1,\dots 2n}\ pf(\epsilon(x_j, x_k))_{j,k=1,\dots,2n} \tag{2}$$

can be shown to be equal $(2n)! pf(M)$, where the $2n \times 2n$ antisymmetric matrix $M = (M_{jk})_{j,k=1,\dots,2n}$ is defined as

$$M_{jk} = \int_{X^2} \phi_j(x)\epsilon(x, y)\phi_k(y)\lambda(dx)\lambda(dy). \tag{3}$$

For the pfaffian ensemble (1) one can explicitly calculate k-point correlation functions
$\rho_k(x_1, \dots, x_k) := ((2n)!/(2n-k)!) \int_{X^{2n-k}} p(x_1, \dots, x_k, x_{k+1}, \dots, x_{2n}) d\lambda(x_{k+1}) \dots d\lambda(x_{2n}),\ k = 1, \dots, 2n$
and show that they have the pfaffian form ([9])

$$\rho_k(x_1, \dots, x_k) = pf(K(x_i, x_j))_{i,j=1,\dots k}, \tag{4}$$

*AMS 2000 subject classification: 15A52 ; 60G55 keywords and phrases: random matrices, orthogonal polynomials, Janossy densities, pfaffian ensembles

†Department of Mathematics, University of California at Davis, One Shields Ave., Davis, CA 95616, USA. Email address: soshniko@math.ucdavis.edu. Research was supported in part by the Sloan Research Fellowship and the NSF grant DMS-0103948.

where $K(x,y)$ is the antisymmetric matrix kernel

$$K(x,y)=\left(\begin{array}{cc}\sum_{1\le j,k\le 2n}\phi_j(x)M^{-t}_{jk}\phi_k(y) & \sum_{1\le j,k\le 2n}\phi_j(x)M^{-t}_{jk}(\epsilon\phi_k)(y)\\ \sum_{1\le j,k\le 2n}(\epsilon\phi_j)(x)M^{-t}_{jk}\phi_k(y) & -\epsilon(x,y)+\sum_{1\le j,k\le 2n}(\epsilon\phi_j)(x)M^{-t}_{jk}(\epsilon\phi_k)(y)\end{array}\right), \tag{5}$$

provided the matrix M is invertible (by definition $(\epsilon\phi)(x)=\int_X\epsilon(x,y)\phi(y)\lambda(dy)$). If $X\subset\mathbb{R}$ and λ is absolutely continuous with respect to the Lebesgue measure, then the probabilistic meaning of the k-point correlation functions is that of the density of probability to find an eigenvalue in each infinitesimal interval around points $x_1,x_2,\ldots x_k$. In other words

$$\begin{aligned}&\rho_k(x_1,x_2,\ldots x_k)\lambda(dx_1)\cdots\lambda(dx_k)=\\ &\Pr\{\text{ there is a particle in each infinitesimal interval }\quad (x_i,x_i+dx_i)\}.\end{aligned}$$

On the other hand, if μ is supported by a discrete set of points, then

$$\rho_k(x_1,x_2,\ldots x_k)\lambda(x_1)\cdots\lambda(x_k)=\Pr\{\text{ there is a particle at each of the points } x_i,\ i=1,\ldots,k\}.$$

In general, random point processes with the k-point correlation functions of the pfaffian form (4) are called pfaffian random point processes. ([8]). Pfaffian point processes include determinantal point processes ([10]) as a particular case when the matrix kernel has the form $\left(\begin{array}{cc}\epsilon & K\\ -K & 0\end{array}\right)$ where K is a scalar kernel and ϵ is an antisymmetric kernel.

So-called Janossy densities $\mathcal{J}_{k,I}(x_1,\ldots,x_k),\quad k=0,1,2,\ldots$, describe the distribution of the eigenvalues in any given interval I. If $X\subset\mathbb{R}$ and λ is absolutely continuous with respect to the Lebesgue measure then

$$\begin{aligned}\mathcal{J}_{k,I}(x_1,\ldots x_k)\lambda(dx_1)\cdots\lambda(dx_k)=\ &\Pr\{\text{ there are exactly } k \text{ particles in } I,\\ &\text{one in each of the } k \text{ distinct infinitesimal intervals}(x_i,x_i+dx_i)\}^{\cdot}\end{aligned}$$

If λ is discrete then

$$\mathcal{J}_{k,I}(x_1,\ldots x_k)=\Pr\{\text{ there are exactly } k \text{ particles in } I,\text{ one at each of the } k \text{ points } x_i,\ i=1,\ldots,k\}.$$

See [4] and [3] for details and additional discussion. For pfaffian point processes the Janossy densities also have the pfaffian form (see [9], [8]) with an antisymmetric matrix kernel L_I :

$$\mathcal{J}_{k,I}(x_1,\ldots,x_k)=const(I)pf(L_I(x_i,x_j))_{i,j=1,\ldots k}, \tag{6}$$

where

$$L_I=K_I(Id+JK_I)^{-1}, \tag{7}$$

$J=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)$ and $const(I)=pf(J-K_I)$ is the Fredholm pfaffian of the restriction of the operator K on the interval I, i.e. $const(I)=pf(J-K_I)=(pf(J+L_I))^{-1}=(\det(Id+J\times K_I))^{1/2}=(\det(Id-JL_I))^{-1/2}$. (we refer the reader to [9], section 8 for the treatment of Fredholm pfaffians).

Let us define three $2n\times 2n$ matrices $G^I,\ M^I,\ M^{X\setminus I}$:

$$G^I_{jk}=\int_I\phi_j(x)\int_X\epsilon(x,y)\phi_k(y)\lambda(dy)\lambda(dx), \tag{8}$$

$$M^I_{jk} = \int_{I^2} \phi_j(x)\epsilon(x,y)\phi_k(y)\lambda(dx)\lambda(dy), \tag{9}$$

$$M^{X\setminus I}_{jk} = \int_{(X\setminus I)^2} \phi_j(x)\epsilon(x,y)\phi_k(y)\lambda(dx)\lambda(dy) \tag{10}$$

(please compare (9)-(10) with the above formula (3) for M). Throughout the paper we will assume that the matrices M^I and $M^{X\setminus I}$ are invertible.

The main result of this paper is

Theorem 1.1 *The kernel L_I has a form similar to the formula (5) for K. Namely, L_I is equal to*

$$L_I(x,y) = \begin{pmatrix} \sum_{1\le j,k\le 2n} \phi_j(x)(M^{X\setminus I})^{-t}_{jk}\phi_k(y) & \sum_{1\le j,k\le 2n} \phi_j(x)(M^{X\setminus I})^{-t}_{jk}(\epsilon_{X\setminus I}\phi_k)(y) \\ \sum_{1\le j,k\le 2n}(\epsilon_{X\setminus I}\phi_j)(x)(M^{X\setminus I})^{-t}_{jk}\phi_k(y) & -\epsilon(x,y)+\sum_{1\le j,k\le 2n}(\epsilon_{X\setminus I}\phi_j)(x)(M^{X\setminus I})^{-t}_{jk}(\epsilon_{X\setminus I}\phi_k)(y) \end{pmatrix} \tag{11}$$

where $(\epsilon_{X\setminus I}\phi)(x) = \int_{X\setminus I}\epsilon(x,y)\phi(y)\lambda(dy)$.

Comparing (11) with (5) one can see that the kernel L_I is constructed in the following way: 1) first it is constructed on $X\setminus I$ by the same recipe used to construct the kernel K on the whole X, 2) it is extended then to I (we recall that L_I acts on $L^2(I, d\lambda(x))$, not on $L^2(X\setminus I, d\lambda(x))$).

This result contains as a special case Theorem 1.1 from the companion paper [3]. The rest of the paper is organized as follows. We discuss some interesting special cases of the theorem, namely so-called polynomial ensembles ($\beta = 1, 2$ and 4) in section 2. The proof of the theorem is given in section 3.

2 Random Matrix Ensembles with $\beta = 1, 2, 4$.

We follow the discussion in [9] (see also [11] and [12]).

Biorthogonal Ensembles .

Consider the particle space to be the union of two identical measure spaces (V,μ) and (W,μ) : $X = V \cup W,\ V = W$. The configuration of $2n$ particles in X will consist of n particles $v_1,\dots,v_n$ in V and n particles $w_1,\dots,w_n$ in W in such a way that the configurations of particles in V and W are identical (i.e. $v_j = w_j, j = 1,\dots,n$). Let $\xi_j,\ \psi_j, j = 1,\dots,n$ be some functions on V. We define $\{\phi_j\}$ and ϵ in (1) so that $\phi_j(v) = 0,\ v\in V,\ \phi_j(w) = \xi_j(w),\ w \in W,\ j = 1,\dots n,\ \phi_j(v) = \psi_{j-n-1}(v),\ v \in V,\ \phi_j(w) = 0,\ w \in W,\ j = n+1,\dots,2n$, and $\epsilon(v_1,v_2) = 0,\ v_1,v_2\in V,\ \ \epsilon(w_1,w_2) = 0,\ w_1,w_2 \in W,\ \epsilon(v,w) = -\epsilon(w,v) = \delta_{vw},\ v\in V, w\in W$. The restriction of the measure λ on both V and W is defined to be equal to μ. Then (1) specializes into (see Corollary 1.5. in [9])

$$p(v_1,\dots,v_n) = const_n \det(\xi_j(v_i))_{i,j=1,\dots,n}\det(\psi_j(v_i))_{i,j=1,\dots,n}. \tag{12}$$

Ensembles of the form (12) are known as biorthogonal ensembles (see [7], [1]). The statement of the Theorem 1.1 in the case (12) has been proven in the companion paper [3]. The special case of the biorthogonal ensemble (12) when $V = \mathbb{R}, \xi_j(x) = \psi_j(x) = x^{j-1}$, and $V = \{\mathbb{C}|\,|z| = 1\}, \xi_j(z) = \overline{\psi}_j(z) = z^{j-1}$, such ensembles are well known in Random Matrix Theory as *unitary ensembles*, see [6] for details. An ensemble of the form (12) which is different from random matrix ensembles was studied in [7]. We specifically want to single out the polynomial ensemble with $\beta = 2$.

Polynomial $(\beta = 2)$ **Ensembles.**

Let $X = \mathbb{R}$ or $\mathbb{Z}$, $\phi_j(x) = x^{j-1}$, $j = 1, \ldots, 2n$, and $\lambda(dx)$ has a density $\omega(x)$ with respect to the reference measure on X (Lebesgue measure in the continuous case, counting measure in the discrete case). Then the formula (12) specializes into

$$p(v_1, \ldots, v_n) = const_n \prod_{1 \le i < j \le n} (v_i - v_j)^2 \prod_{1 \le j \le n} \omega(v_j). \tag{13}$$

Polynomial $(\beta = 1)$ **Ensembles.**

Let $X = \mathbb{R}$ or $\mathbb{Z}$, $\phi_j(x) = x^{j-1}$, $j = 1, \ldots, 2n$, $\epsilon(x, y) = \frac{1}{2} sgn(y - x)$ and $\lambda(dx)$ has a density $\omega(x)$ with respect to the reference measure on X (Lebesgue measure in the continuous case, counting measure in the discrete case). Then the formula (1) specializes into the formula for the density of the joint distribution of $2n$ particles in a so-called $\beta = 1$ polynomial ensemble (see [9], Remark 1):

$$p(x_1, \ldots, x_{2n}) = const_n \prod_{1 \le i < j \le 2n} |x_i - x_j| \prod_{1 \le j \le 2n} \omega(x_j). \tag{14}$$

In Random Matrix Theory the ensembles (14) in the continuous case are known as *orthogonal ensembles* , see [6].

Polynomial $(\beta = 4)$ **Ensembles.**

Similar to the biorthogonal case $(\beta = 2)$ let us consider the particle space to be the union of two identical measure spaces $(Y, \mu), (Z, \mu)$, $X = Y \cup Z$, $Y = Z$, where $Y = \mathbb{R}$ or $Y = \mathbb{Z}$. The configuration of $2n$ particles $x_1, \ldots, x_{2n}$, in X will consist of n particles $y_1, \ldots, y_n$ in Y and n particles $z_1, \ldots, z_n$, in Z in such a way that the configurations of particles in Y and Z are identical. We define $\{\phi_j\}$ and ϵ so that $\phi_j(y) = y^j$, $\in Y$, $\phi_j(z) = jz^{j-1}$, $z \in Z$, $\epsilon(y_1, y_2) = 0$, $\epsilon(z_1, z_2) = 0$, $\epsilon(y, z) = -\epsilon(z, y) = \delta_{yz}$. As above we assume that the measure μ has a density ω with respect to the reference measure on Y. Then the formula (1) specializes into the formula for the density of the joint distribution of n particles in a $\beta = 4$ polynomial ensemble (see Corollary 1.3. in [9]))

$$p(y_1, \ldots, y_n) = const_n \prod_{1 \le i < j \le n} (y_i - y_j)^4 \prod_{1 \le j \le n} \omega(y_j). \tag{15}$$

In Random Matrix Theory the ensembles (15) are known as *symplectic ensembles*, see [6].

3 Proof of the Main Result

Consider matrix kernels

$$\mathcal{K}_I = -JK_I, \;\; \mathcal{L}_I = -JL_I. \tag{16}$$

The the relation (7) simplifies into

$$\mathcal{L}_I = \mathcal{K}_I (Id - \mathcal{K}_I)^{-1} \tag{17}$$

which is the same relation that is satisfied by the correlation and Janossy scalar kernels in the determinantal case ([4], [2]). The consideration of $\mathcal{K}_I$ and $\mathcal{L}_I$ is motivated by the fact that the pfaffians of the $2k \times 2k$ matrices with the antisymmetric matrix kernels K_I and L_I are equal to the quaternion determinants ([6]) of $2k \times 2k$ matrices with the kernels $\mathcal{K}_I$, $\mathcal{L}_I$ when the latter matrices are viewed as $k \times k$ quaternion matrices (i.e. each quaternion entry corresponds to a 2×2 block with complex entries). It follows from (5) and (16) that the kernel $\mathcal{K}_I$ is given by the formula

$$\mathcal{K}_I = \sum_{j,k=1,\dots 2n} M^{-t}_{jk} \begin{pmatrix} -(\epsilon\phi_j)\otimes\phi_k & -(\epsilon\phi_j)\otimes(\epsilon\phi_k) \\ \phi_j\otimes\phi_k & \phi_j\otimes(\epsilon\phi_k) \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}. \tag{18}$$

Let us denote by $\widetilde{\mathcal{L}}_I$ the following kernel

$$\widetilde{\mathcal{L}}_I(x,y) = \sum_{1\le j,k\le 2n} (M^{X\setminus I})^{-t}_{jk} \begin{pmatrix} -(\epsilon_{X\setminus I}\phi_j)\otimes\phi_k & -(\epsilon_{X\setminus I}\phi_j)\otimes\ (\epsilon_{X\setminus I}\phi_k) \\ \phi_j\otimes\phi_k & \phi_j\otimes(\epsilon_{X\setminus I}\phi_k) \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}. \tag{19}$$

As above, $\epsilon\phi$ stands for $\int_X \epsilon(x,y)\phi(y)$. We use the notation $\phi_j\otimes\phi_k$ is a shorthand for $\phi_j(x)\phi_k(y)$. To prove the main result of the paper we will show that $\widetilde{\mathcal{L}}_I = \mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ (in other words we are going to prove that $\widetilde{\mathcal{L}}_I = \mathcal{L}_I$, where $\mathcal{L}_I$ is defined in (17)). The proof relies on Lemmas 1 and 2 given below. Let us introduce the notation $(\epsilon_I\phi)(x) = \int_I \epsilon(x,y)\phi_s(y)d\lambda(y)$. We will show that the finite-dimensional subspace $\mathcal{H} = Span\left\{\begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix}, \begin{pmatrix} -\epsilon\phi_s \\ -\phi_s \end{pmatrix}, \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}\right\}_{s=1,\dots 2n}$ is invariant under $\mathcal{K}_I$ and $\widetilde{\mathcal{L}}_I$. The main part of the proof of the theorem is to show that $\widetilde{\mathcal{L}}_I = \mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ holds on $\mathcal{H}$.

Lemma 1 *The operators $\mathcal{K}_I$, $\widetilde{\mathcal{L}}_I$ leave $\mathcal{H}$ invariant and $\widetilde{\mathcal{L}}_I = \mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ holds on $\mathcal{H}$.*

Below we give the proof of the lemma. Using the notations introduced above in (8)-(10) one can easily calculate

$$\mathcal{K}_I\begin{pmatrix} \epsilon\phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots 2n} -((G^I)^t M^{-1})_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} \tag{20}$$

$$\mathcal{K}_I\begin{pmatrix} 0 \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n} (G^I M^{-1})_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}. \tag{21}$$

Defining the $2n\times 2n$ matrix T as

$$T_{sk} = \int_I \phi_s(x)\int_{X\setminus I}\epsilon(x,y)\phi_k(y)d\lambda(y)d\lambda(x) \tag{22}$$

we compute

$$\mathcal{K}_I\begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots 2n}((G^I-T)M^{-1})_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix}, \tag{23}$$

where $(G^I-T)_{sk} = M^I_{sk} = \int_{I^2}\phi_s(x)\epsilon(x,y)\phi_k(y)\lambda(dx)\lambda(dy)$. One can rewrite the equations (20)-(21) as

$$\mathcal{K}_I\begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n}((G^I-(G^I)^t)M^{-1})^t_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}, \tag{24}$$

$$\mathcal{K}_I\begin{pmatrix} -\epsilon\phi_s \\ -\phi_s \end{pmatrix} = \sum_{j=1,\dots 2n}((G^I+(G^I)^t)M^{-1})_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} \tag{25}$$

We conclude that that the subspace $\mathcal{H}$ is indeed invariant under $\mathcal{K}_I$ and the matrix of the restriction of $\mathcal{K}_I$ on $\mathcal{H}$ has the following block structure in the basis $\left\{ \begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix}, \begin{pmatrix} -\epsilon\phi_s \\ -\phi_s \end{pmatrix}, \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} \right\}_{s=1,\ldots 2n}$:

$$\begin{pmatrix} (G^I-(G^I)^t)M^{-1} & (G^I+(G^I)^t)M^{-1} & (G^I-T)M^{-1} \\ 0 & 0 & 0 \\ -Id & -Id & 0 \end{pmatrix} \tag{26}$$

(in particular $Ran(\mathcal{K}_I|_{\mathcal{H}}) = Span\left\{ \begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix}, \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} \right\}_{s=1,\ldots 2n}$). Let us introduce some additional notations:

$$\begin{aligned} A &= (G^I-(G^I)^t)M^{-1}, && (27)\\ B &= (G^I+(G^I)^t)M^{-1}, && (28)\\ C &= (G^I-T)M^{-1}. && (29) \end{aligned}$$

When a matrix has a block form $\mathcal{M} = \begin{pmatrix} A & B & C \\ 0 & 0 & 0 \\ -Id & -Id & 0 \end{pmatrix}$ (as it is in our case) the matrix $\mathcal{M}\times(Id-\mathcal{M})^{-1}$ has the block form

$$\begin{pmatrix} (Id-A+C)^{-1}-Id & (B-C)(Id-A+C)^{-1} & C(Id-A+C)^{-1} \\ 0 & 0 & 0 \\ -(Id-A+C)^{-1} & -Id-(B-C)(Id-A+C)^{-1} & -C(Id-A+C)^{-1} \end{pmatrix} \tag{30}$$

As one can see from the formulas (31)-(33) the invertibility of $Id-\mathcal{M}$ follows from the invertibility of $M^{X\backslash I}$ which has been assumed throughout the paper. We have

$$\begin{aligned} (Id-A+C)^{-1} &= M(M+(G^I)^t-T)^{-1} = M(M^{X\backslash I})^{-1} && (31)\\ C(Id-A+C)^{-1} &= (G^I-T)(M+(G^I)^t-T)^{-1} = M^I(M^{X\backslash I})^{-1} && (32)\\ (B-C)(Id-A+C)^{-1} &= ((G^I)^t+T)(M+(G^I)^t-T)^{-1} = ((G^I)^t+T)(M^{X\backslash I})^{-1}. && (33) \end{aligned}$$

Let us now compute the matrix of the restriction of $\widetilde{\mathcal{L}}_I$ on $\mathcal{H}$. We have

$$\widetilde{\mathcal{L}}_I = \sum_{j,k=1,\ldots 2n} (M^{X\backslash I}_{jk})^t \begin{pmatrix} -(\epsilon_{X\backslash I}\phi_j)\otimes\phi_k & -(\epsilon_{X\backslash I}\phi_j)\otimes(\epsilon_{X\backslash I}\phi_k) \\ \phi_k\otimes\phi_k & \phi_j\otimes(\epsilon_{X\backslash I}\phi_k) \end{pmatrix} + \begin{pmatrix} 0 & \epsilon_{X\backslash I} \\ 0 & 0 \end{pmatrix}. \tag{34}$$

Similarly to the computations above one can see that $\mathcal{H}$ is invariant under $\widetilde{\mathcal{L}}_I$ and

$$\begin{aligned} \widetilde{\mathcal{L}}_I \begin{pmatrix} \epsilon\phi_s \\ -\phi_s \end{pmatrix} &= \sum_{j=1,\ldots 2n} ((T-(G^I)^t)(M^{X\backslash I})^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} \\ &- \sum_{1\le j\le 2n} ((T-(G^I)^t)(M^{X\backslash I})^{-1})_{sj} \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}, && (35)\\ \widetilde{\mathcal{L}}_I \begin{pmatrix} -\epsilon\phi_s \\ -\phi_s \end{pmatrix} &= \sum_{j=1,\ldots 2n} ((T+(G^I)^t)(M^{X\backslash I})^{-1})_{sj} \begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} \\ &- \sum_{1\le j\le 2n} ((T+(G^I)^t)(M^{X\backslash I})^{-1})_{sj} \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}, && (36) \end{aligned}$$

and

$$\widetilde{\mathcal{L}}_I\begin{pmatrix}\epsilon_I\phi_s\\0\end{pmatrix}=\sum_{j=1,\dots 2n}((G^I-T)(M^{X\setminus I})^{-1})_{sj}\begin{pmatrix}\epsilon\phi_j\\-\phi_j\end{pmatrix}-\sum_{j=1,\dots 2n}((G^I-T)(M^{X\setminus I})^{-1})_{sj}\begin{pmatrix}\epsilon_I\phi_j\\0\end{pmatrix}. \tag{37}$$

Therefore the restriction of $\widetilde{\mathcal{L}}_I$ to $\mathcal{H}$ in the basis $\left\{\begin{pmatrix}\epsilon\phi_s\\-\phi_s\end{pmatrix},\begin{pmatrix}-\epsilon\phi_s\\-\phi_s\end{pmatrix},\begin{pmatrix}\epsilon_I\phi_s\\0\end{pmatrix}\right\}_{s=1,\dots 2n}$ has the following block structure

$$\begin{pmatrix}(T-(G^I)^t)(M^{X\setminus I})^{-1} & (T+(G^I)^t)(M^{X\setminus I})^{-1} & (G^I-T)(M^{X\setminus I})^{-1}\\ 0 & 0 & 0\\ -Id-(T-(G^I)^t)(M^{X\setminus I})^{-1} & -Id-(T+(G^I)^t)(M^{X\setminus I})^{-1} & -(G^I-T)(M^{X\setminus I})^{-1}\end{pmatrix} \tag{38}$$

Comparing (30), (31)-(33) and (38) we see that $\widetilde{\mathcal{L}}_I=\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ on $\mathcal{H}$. Lemma is proven.

To show that $\widetilde{\mathcal{L}}_I=\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ also holds on the complement of $\mathcal{H}$ it is enough to prove it on the subspaces $\begin{pmatrix}(\mathcal{H}_1)^\perp\\0\end{pmatrix}$, and $\begin{pmatrix}0\\(\mathcal{H}_2)^\perp\end{pmatrix}$, where $\mathcal{H}_1=Span(\overline{\epsilon_I\phi_s})_{k=1,\dots,2n}$ and $\mathcal{H}_2=Span(\overline{\phi_s})_{k=1,\dots,2n}$. The inveribility of the matrix M_I implies that actually it is enough to prove $\widetilde{\mathcal{L}}_I=\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ on the subspaces $\begin{pmatrix}(\mathcal{H}_2)^\perp\\0\end{pmatrix}$, and $\begin{pmatrix}0\\(\mathcal{H}_1)^\perp\end{pmatrix}$. Here we use the standard notation $(\mathcal{H}_i)^\perp$ for the orthogonal complement in $L^2(I)$ with the standard scalar product $(f,g)_I=\int_I\times\overline{f(x)}g(x)d\lambda(x)$. We start with the first subspace.

Lemma 2 *The relation* $\widetilde{\mathcal{L}}_I=\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}$ *holds on* $\begin{pmatrix}0\\(\mathcal{H}_1)^\perp\end{pmatrix}$.

The proof is a straightforward check. The notations are slightly simplified when the functions $\{\epsilon_I\phi_k,\ \epsilon\phi_k,\ k=1,\dots,2n\}$ are linearly independent in $L^2(I)$. The degenerate case is left to the reader. Consider $f_s\in(\mathcal{H}_1)^\perp,\ s=1,\dots,2n$ such that

$$(\overline{\epsilon\phi_k},f_s)_I=(\overline{\epsilon\phi_k},\phi_s)_I,\ k=1,\dots,2n. \tag{39}$$

We are going to establish the relation for $\begin{pmatrix}0\\f_s\end{pmatrix}$, which then immediately extends by linearity to the linear combinations of $\begin{pmatrix}0\\f_s\end{pmatrix}$. We write

$$\begin{aligned}\mathcal{K}_I\begin{pmatrix}0\\-f_s\end{pmatrix}&=\sum_{j,k=1,\dots 2n}M^{-t}_{jk}(\overline{\epsilon\phi_k},-f_s)_I\begin{pmatrix}-\epsilon\phi_j\\\phi_j\end{pmatrix}-\begin{pmatrix}\epsilon_If_s\\0\end{pmatrix}\\&=\sum_{j,k=1,\dots 2n}M^{-t}_{jk}(\overline{\epsilon\phi_k},-\phi_s)_I\begin{pmatrix}-\epsilon\phi_j\\\phi_j\end{pmatrix}-\begin{pmatrix}\epsilon_If_s\\0\end{pmatrix}\\&=\sum_{j=1,\dots 2n}(G^IM^{-1}_{sj})\begin{pmatrix}\epsilon\phi_j\\-\phi_j\end{pmatrix}-\begin{pmatrix}\epsilon_If_s\\0\end{pmatrix}\end{aligned} \tag{40}$$

(we have used (39) in the second equality) and

$$\mathcal{K}_I\begin{pmatrix}\epsilon_I\phi_s\\-f_s\end{pmatrix}=\sum_{j=1,\dots 2n}((G^I-(G^I)^t)M^{-1})_{sj}\begin{pmatrix}\epsilon\phi_j\\-\phi_j\end{pmatrix}-\begin{pmatrix}\epsilon_If_s\\0\end{pmatrix}. \tag{41}$$

Combining (40) and (41) we get

$$\mathcal{K}_I\begin{pmatrix} -\epsilon_I\phi_s \\ -f_s \end{pmatrix} = \sum_{j=1,\dots 2n}((G^I+(G^I)^t)M^{-1})_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}, \tag{42}$$

Similarly to (23) we compute

$$\mathcal{K}_I\begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix} = \sum_{j=1,\dots 2n}((G^I-T)M^{-1})_{sj}\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} \tag{43}$$

It should be noted that $\mathcal{K}_I\begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix} = 0$ because $\int_I(\epsilon_I\phi_s)(x)\phi_j(x)d\lambda(x) = -\int_I f_s(x)(\epsilon_I\phi_j)(x)\times d\lambda(x) = 0$ for all $j=1,\dots,2n$. This together with (39) allows us to conclude that the calculation of $\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}\begin{pmatrix} 0 \\ f_s \end{pmatrix}$ is almost identical to the calculation of $\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}\begin{pmatrix} 0 \\ \phi_s \end{pmatrix}$ with the only difference that in the former one we have to replace the term $-\begin{pmatrix} \epsilon_I\phi_s \\ 0 \end{pmatrix}$ by $-\begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}$ (see the last equation of (44)). Namely

$$\begin{aligned}
\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}\begin{pmatrix} 0 \\ -f_s \end{pmatrix} &= \mathcal{K}_I(Id-\mathcal{K}_I)^{-1}\left(\frac{1}{2}\begin{pmatrix} \epsilon\phi_s \\ -f_s \end{pmatrix} + \begin{pmatrix} -\epsilon\phi_s \\ -f_s \end{pmatrix}\right) \\
&= \sum_{j=1,\dots,2n}(1/2)\left((A+B)(Id-A+C)^{-1}\right)_{sj}\begin{pmatrix} \epsilon_I\phi_j \\ -\phi_j \end{pmatrix} \\
&- \sum_{j=1,\dots,2n}(1/2)\left((A+B)(Id-A+C)^{-1}\right)_{sj}\begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix} \\
&= \sum_{j=1,\dots,2n}\left[G^I(M^{X\setminus I})^{-1}\right]_{sj}\left[\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix}\right] - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}
\end{aligned} \tag{44}$$

where $A,\ B,\ C$ are defined in (27)-(29). At the same time

$$\begin{aligned}
\widetilde{\mathcal{L}}_I\begin{pmatrix} 0 \\ -f_s \end{pmatrix} &= \sum_{j,k=1,\dots,2n}(M^{X\setminus I})^{-t})_{jk}\begin{pmatrix} \epsilon_{X\setminus I}\phi_j \\ -\phi_j \end{pmatrix}(\overline{\epsilon_{X\setminus I}\phi_k}, f_s)_I - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix} \\
&= \sum_{j,k=1,\dots,2n}(M^{X\setminus I})^{-t})_{jk}\begin{pmatrix} \epsilon_{X\setminus I}\phi_j \\ -\phi_j \end{pmatrix}(\overline{\epsilon\phi_k}, f_s)_I - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix} \\
&= \left[G^I(M^{X\setminus I})^{-1}\right]_{sj}\left[\begin{pmatrix} \epsilon\phi_j \\ -\phi_j \end{pmatrix} - \begin{pmatrix} \epsilon_I\phi_j \\ 0 \end{pmatrix}\right] - \begin{pmatrix} \epsilon_I f_s \\ 0 \end{pmatrix}.
\end{aligned} \tag{45}$$

Therefore $\widetilde{\mathcal{L}}_I\begin{pmatrix} 0 \\ -f_s \end{pmatrix} = \mathcal{K}_I(Id-\mathcal{K}_I)^{-1}\begin{pmatrix} 0 \\ -f_s \end{pmatrix}$, $\quad s=1,\dots 2n$. By linearity result follows for all $\begin{pmatrix} 0 \\ f \end{pmatrix}$ such that $(\overline{\epsilon_I\phi_k}, f)_I = \int_I(\epsilon_I\phi_k)(x)f(x)d\lambda(x) = 0, \quad k,j=1,\dots 2n$. Lemma 2 is proven.

To check (17) on $\begin{pmatrix} (\mathcal{H}_2)^\perp \\ 0 \end{pmatrix}$ we note that $\mathcal{K}_I(Id-\mathcal{K}_I)^{-1}\begin{pmatrix} g \\ 0 \end{pmatrix} = \widetilde{\mathcal{L}}_I\begin{pmatrix} g \\ 0 \end{pmatrix} = 0$ for g such that $\int_I g(x)\phi_k(x)d\lambda(x) = 0, \quad k=1,\dots,2n$, which together with the invertibility of M finishes the proof. Theorem is proven.

Acknowledgements. It is a great pleasure to thank Alexei Borodin and Eric Rains for useful discussions. The author would like to thank the anonimous referee for pointing out to a number of misprints.

References

[1] A.Borodin, Biorthogonal ensembles, *Nucl. Phys.*, **B 536**, 704-732, (1999).

[2] A.Borodin and G.Olshanski, Distributions on partitions, point processes, and the hypergeometric kernel, *Commun. Math. Phys.* , **211**, 335-358, (2000).

[3] A.Borodin and A.Soshnikov, Janossy densities I. Determinantal ensembles, available at arXiv:math-ph/0212063, to appear in *J. Stat. Phys.*, **113, No. 3/4**, 595-610, (2003).

[4] D.J.Daley, D.Vere-Jones, **An introduction to the Theory of Point Processes**, Springer-Verlag, New York, 1988.

[5] C.D.Godsil, **Algebraic Combinatorics**, Chapman & Hall, New York, 199?.

[6] M.L.Mehta, **Random Matrices**, Academic Press, New York, 1991.

[7] K.A.Muttalib, Random matrix models with additional interactions, *J. Phys. A*, **285**, L159, (1995).

[8] G.Olshanski, Pfaffian processes, unpublished note (2002).

[9] E.Rains, Correlation functions for symmetrized increasing subsequences, available at arXiv:math.CO/0006097.

[10] A.Soshnikov, Determinantal random point fields, *Russian Math. Surveys*, **55**, 923-975, (2000).

[11] C.A. Tracy and H. Widom, Correlation functions, Cluster functions, and spacing distributions for random matrices, *J.Stat. Phys.* **92, No. 5/6**, 809-835, (1998).

[12] H.Widom, On the relation between orthogonal, symplectic and unitary matrix ensembles, *J.Stat.Phys.* **94, No.3/4**, 347-363, (1999).