Statistics of Extreme Spacings in Determinantal Random Point Processes

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1 Introduction

Determinantal (a.k.a. fermion) random point processes were introduced in probability theory by Macchi about thirty years ago ([13], [14], [3]). In the last ten years the subject has attracted a considerable attention due to its rich connections to Random Matrix Theory, Combinatorics, Representation Theory, Random Growth Models, Number Theory and several other areas of mathematics. We refer the reader to the recent surveys ([20], [9], [8]), and research papers on the subject ([1], [4], [5], [6], [7], [12], [10], [15], [17], [18], [19], [22], [23], [24]).

In this paper we shall consider determinantal random point processes on the real line with the translation-invariant correlation kernel. In other words, a one particle space X is given as $X = \mathcal{R}^1$, and the space of elementary outcomes Ω consists of the countable, locally finite particle configurations on the real line

$$\Omega = \{\xi = (x_i)_{i=-\infty}^{+\infty} : \ \#(x_i \in [-N, N]) < +\infty, \ \ \forall N > 0\},$$

where $x_i \in \mathcal{R}^1$, $i = 0, \pm 1, \pm 2, \dots$, and $\#(x_i \in [-N, N]) =: \#([-N, N])$ denotes the number of the particles in the interval $[-N, N]$. Let us denote the set of the non-negative integers by $\mathcal{Z}^1_+ = \{0, 1, 2, \dots\}$, and the set of the positive integers by $\mathcal{N} = \{1, 2, \dots\}$. The σ-algebra $\mathcal{F}$ of the measurable subsets of Ω is generated by the cylinder sets $C^{n_1, n_2, \dots n_k}_{I_1, I_2, \dots, I_k} = \{\xi : \ \#(I_j) = n_j, \ j = 1, \dots, k\}$, where k is an arbitrary positive integer, $k \in \mathcal{N}$, $I_1, \dots, I_k$ are arbitrary disjoint subintervals of the real line, and $n_1, n_2, \dots, n_k \in \mathcal{Z}^1_+$.

A probability measure $\mathcal{P}$ on the measurable space $(\Omega, \mathcal{F})$ defines a random point process $(\Omega, \mathcal{F}, \mathcal{P})$. A random point process is called determinantal if its k-point correlation functions have determinantal form

$$\rho_k(x_1, \dots, x_k) = \det(K(x_i, x_j))_{i,j=1,\dots,k}, \quad k = 1, 2, \dots, \tag{1}$$

where $K(x, y)$ is usually called the correlation kernel of the determinantal random point process. We remind the reader that k-point correlation functions are defined in such a way that

*Dedicated to Yakov Sinai with great respect and admiration on the occasion of his 70th birthday

$$E\prod_{l=1}^{k}\#(I_l) = \int_{I_1\times\ldots I_k}\rho_k(x_1,\ldots,x_k)dx_1\ldots dx_k \tag{2}$$

for the arbitrary disjoint intervals $I_1,\ldots,I_k$. Since the r.h.s. of (1) is non-negative, it follows that the correlation kernel $K(x,y)$ has non-negative minors. If, in addition, the integral operator $K: L^2(\mathcal{R}^1)\to L^2(\mathcal{R}^1),\ (Kf)(x) = \int_{-\infty}^{+\infty}K(x,y)f(y)dy$, is Hermitian, one can conclude that K is non-negative definite, i.e. $Spec(K)\in[0,+\infty)$. In the Hermitian case one can show that the necessary and sufficient condition on K to define a determinantal random point field (1) is

$$0\le K\le 1, \tag{3}$$

in other words both K and $1-K$ must be non-negative definite operators ([20], [13]).

In this paper we consider the translation-invariant kernel

$$K(x,y) = g(y-x),\quad \texttt{where}\quad g(x) = \int_{-\infty}^{+\infty}\exp(2\pi ixt)\phi(t)dt, \tag{4}$$

and $\phi(t)$ is an even real-valued integrable function

$$\phi(t) = \phi(-t),\qquad \phi\in L^1(R^1). \tag{5}$$

It follows from (3) that

$$0\le\phi(t)\le 1\quad (a.e.). \tag{6}$$

In addition, we assume that the following technical conditions are satisfied

$$\int_{-\infty}^{+\infty}t^2\phi(t)dt < +\infty, \tag{7}$$

$$|g(x)|\le\frac{C}{1+|x|^{\frac{1}{2}+\epsilon}}, \tag{8}$$

$$|g'(x)|\le\frac{C}{1+|x|^{\frac{1}{2}+\epsilon}}, \tag{9}$$

where C is a positive constant, and ϵ is an arbitrary small positive constant.

Let L be a large positive number. Consider a restriction of a configuration ξ to the interval $[0,L]$. Let us denote the points of $\xi\cap[0,L]$ by $x_1,x_2,\ldots,x_{N(L)}$, where $N(L)$ is the cardinality of $\xi\cap[0,L]$. We assume that the points in $\xi\cap[0,L]$ are ordered: $x_1<x_2<\ldots<x_{N(L)}$. It is a well known (see e.g. [20]) that with probability 1 no two particles of a determinantal random point process coincide. We are interested to study the nearest spacings $\theta_i = x_{i+1}-x_i,\ \ i=1,\ldots,N(L)-1$, between the neighboring particles. Functional Central Limit Theorem for the empirical distribution function of the nearest spacings of particles in $[0,L]$ (in the limit $L\to\infty$) was proven in [21] for $K(x,y) = \frac{\sin(\pi x)}{\pi x}$ (i.e. ϕ is the indicator of $[-\frac{1}{2},\frac{1}{2}]$), and for

similar kernels arising in Random Matrix Theory. It was shown in [20] that the result could be extended to a quite general class of translation-invariant correlation kernels.

In this paper we study the smallest nearest spacings. Our main result is the following

Theorem 1. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a determinantal random point process on the real line with the translation-invariant correlation kernel $K(x,y) = g(y-x)$ satisfying (4)-(9). Then the number of the nearest spacings less than $s/L^{1/3}$ in the interval $[0, L]$ converges in distribution to the Poisson random variable with the mean αs^3, in the limit $L \to \infty$, where*

$$\alpha = \frac{1}{3} g(0) g''(0) = \frac{4\pi^2}{3} \int_{-\infty}^{+\infty} \phi(t) dt \int_{-\infty}^{+\infty} t^2 \phi(t) dt. \tag{10}$$

Let $\eta(L) = L^{1/3} \min\{\theta_i, \ i = 1, \dots, N(L)-1\}$. In other words, $\eta(L)$ is the smallest nearest spacing in $[0, L]$, rescaled by $L^{1/3}$. Theorem 1 immediately implies

Theorem 2. *Let the conditions in Theorem 1 be satisfied. Then*

$$\lim_{L\to\infty} \Pr(\eta(L) > s) = \exp(-\alpha s^3). \tag{11}$$

The method of the proof of Theorems 1 and 2 relies on the detailed analysis of k-point correlation and cluster functions of the s-modified random point process, introduced in [20], [21].

The rest of the paper is organized as follows. Point correlation and cluster functions, and s-modified random point processes are discussed in Section 2. The proofs of Theorems 1 and 2 are given in Section 3.

We will use the notations $const$, $const_k$, $Const$, to denote various positive constants throughout this text. The values of these constants may be different in various parts of the paper. We shall use the notation $f = O(g)$ if the ratio f/g is bounded from above and below by some positive constants, and the notation $f = o(g)$ if the ratio f/g goes to zero.

2 Correlation and Cluster Functions

We start this section by recalling the definition of a k-point cluster function (sometimes also known as the Ursell factor). For additional information we refer the reader to [16], [11], [2], [21].

Definition. *The l-point cluster function $r_l(x_1, \dots, x_l)$, $l = 1, 2, \dots,$ of a random point field is defined in terms of the point correlation functions by the formula*

$$r_l(x_1, \dots, x_l) = \sum_G (-1)^{m-1} (m-1)! \prod_{j=1}^{m} \rho_{|G_j|}(\bar{x}(G_j)) \tag{12}$$

where the sum is over all partitions G of $[l] = \{1, 2, \ldots, l\}$ into subsets $G_1, \ldots, G_m,\ m = 1, \ldots, l,$ and $\bar{x}(G_j) = \{x_i : i \in G_j\},\ |G_j| = \#(G_j)$.

The point correlation functions can be expressed in terms of the point cluster functions as

$$\rho_l(x_1, \ldots, x_l) = \sum_G \prod_{j=1}^{m} r_{|G_j|}(\bar{x}(G_j)). \tag{13}$$

The reader can observe that (12) is the Möbius inversion formula applied to (13). In particular,

$$\rho_1(x) = r_1(x),$$
$$\rho_2(x_1, x_2) = r_2(x_1, x_2) + r_1(x_1)r_1(x_2),$$
$$\rho_3(x_1, x_2, x_3) = r_3(x_1, x_2, x_3) + r_2(x_1, x_2)r_1(x_3) + r_2(x_1, x_3)r_1(x_2) +$$
$$r_2(x_2, x_3)r_1(x_1) + r_1(x_1)r_1(x_2)r_1(x_3).$$

It follows from (13) and (1) that for determinantal random point fields

$$r_l(x_1, \ldots, x_l) = (-1)^{l-1} \sum_{\texttt{cyclic}\ \sigma \in S_l} K(x_1, x_{\sigma(1)})K(x_2, x_{\sigma(2)}) \ldots K(x_l, x_{\sigma(l)}), \tag{14}$$

where the sum in (14) is over all cyclic permutations. In other words, for determinantal random processes the difference between the formula (14) for the l-point cluster function and the formula

$$\rho_l(x_1, \ldots, x_l) = \sum_{\sigma \in S_l} (-1)^{\sigma} K(x_1, x_{\sigma(1)}) \cdot K(x_2, x_{\sigma(2)}) \cdot \ldots \cdot K(x_l, x_{\sigma(l)}) \tag{15}$$

for the l-point correlation function is that in (15) the summation is taken over all permutations in S_l, and in (14) the summation is over the cyclic permutations only. One can rewrite (14) as

$$r_l(x_1, \ldots, x_l) = (-1)^{l-1} \cdot \frac{1}{l} \sum_{\sigma \in s_l} K(x_{\sigma(1)}, x_{\sigma(2)})K(x_{\sigma(2)}, x_{\sigma(3)}) \ldots K(x_{\sigma(l)}, x_{\sigma(1)}). \tag{16}$$

It follows from (2) that the integral of the k-point correlation function over the k-dimensional cube $[0, L]^k$ is equal to the k-th factorial moment of the counting random variable $\#(I),\ I = [0, L]$, namely

$$E\#(I)(\#(I) - 1) \ldots (\#(I) - k + 1) = \int_{I^k} \rho_k(x_1, \ldots, x_k) dx_1 \ldots dx_k. \tag{17}$$

The integral of the k-point cluster function, in turn, can be expressed as a linear combination of the cumulants of $\#([0, L])$. Namely, let $C_k(L)$ denote the k-th cumulant of $\#([0, L])$, and $V_k(L) = \int_{[0,L]^k} r_k(x_1, x_2, \ldots x_k) dx_1 \times dx_2 \ldots dx_k$. Then (see e.g. [2], [21])

$$\sum_{n=1}^{\infty} \frac{C_n(L)}{n!} z^n = \sum_{n=1}^{\infty} \frac{V_n(L)}{n!} (e^z - 1)^n. \tag{18}$$

To apply the machinery of the cluster functions to the problem at hand, we consider a so-called s-modified random point process, which can be constructed in the following way. We start with a random configuration $\xi = (x_i)_{i=-\infty}^{+\infty}$ from the original random point field, and keep only those points x_i for which there is exactly one neighbor to the right within distance s, i.e. $x_{i+1} - x_i \leq s, \quad x_{i+2} - x_i > s$. The points x_i for which this conditions is not satisfied are thrown away. As a result, we obtain a new random configuration $\xi(s) \subset \xi$, such that $\xi(s) = \{x_i : x_{i+1} - x_i \leq s, \quad x_{i+2} - x_i > s\}$, where $\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$ are the points of the original configuration ξ. The number of spacings less than s of the original random point field in the interval $[0, L]$ is related to the number of points of the s-modified random point field in $[0, L]$. As will be shown later, for large L and $s \sim L^{-1/3}$, these two counting random variables coincide with probability very close to 1.

Since the moments and the cumulants of the counting random variable $\#([0, L])$ can be expressed in terms of the integrals of point correlation and cluster functions (17), (18), it is essential to be able to calculate the point correlation and cluster functions of the s-modified random point process. We shall denote the k- point correlation and k-point cluster functions of the modified random process by $\rho_k(x_1, \ldots, x_k; s)$ and $r_k(x_1, \ldots, x_k; s)$, correspondingly. It follows from the inclusion-exclusion principle that, provided $|x_i - x_j| > s, \quad 1 \leq i \neq j \leq k,$ one obtains

$$\rho_k(x_1, \ldots, x_k; s) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \int_{x_1}^{x_1+s} \cdots \int_{x_k}^{x_k+s} \int_{I(x_1,\ldots,x_k;s)^m}$$
$$\rho_{2k+m}(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_m) dy_1 \ldots dy_k dz_1 \ldots dz_m, \tag{19}$$

where $I(x_1, \ldots, x_k; s) = \bigcup_{i=1}^m [x_i, x_i+s]$, and $I(x_1, \ldots, x_k; s)^m = I(x_1, \ldots, x_k; s) \times \ldots \times I(x_1, \ldots, x_k; s)$ stands for the m-th fold Cartesian product of $I(x_1, \ldots, x_k; s)$ (see e.g. [20], [21]).

In the determinantal case (1) the formula for the k-point cluster function of the s-modified random process has a somewhat similar structure ([20], [21]). Provided $|x_i - x_j| > s, \quad 1 \leq i \neq j \leq k,$ one has

$$r_k(x_1, \ldots, x_k; s) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \int_{x_1}^{x_1+s} \cdots \int_{x_k}^{x_k+s} \int_{I(x_1,\ldots,x_k;s)^m}$$
$$\rho_{2k+m}^{trun}(x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_m) dy_1 \ldots dy_k dz_1 \ldots dz_m, \tag{20}$$

where

$$\rho^{trun}_{2k+m}(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m) = \sum^{*}_{\sigma\in S_{2k+m}} (-1)^\sigma K(x_1,\sigma(x_1))\dots K(x_k,\sigma(x_k)) \times$$
$$K(y_1,\sigma(y_1))\dots K(y_k,\sigma(y_k))K(z_1,\sigma(z_1))\dots K(z_m,\sigma(z_m)), \tag{21}$$

where the summation in (21) is over the permutations $\sigma \in S_{2k+m}$ satisfying the property **A** described below (we note that σ acts on the set of $2k+m$ variables $x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots z_m$).

Property A

Consider an index $1 \le i \le k$. Define $X(i)$ to be the subset of the set of variables $\{x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots z_m\}$, that consists of x_i, y_i, and those of the variables $z_1,\dots,z_m$, that belong to $[x_i, x_i+s]$ (we remind the reader that each z_l, $1 \le l \le m$, belongs to exactly one interval $[x_j, x_j+s]$, $1 \le j \le k$). Then for any pair of disjoint integers $1 \le i \ne j \le k$ there exists a positive integer $r = r(i,j) > 0$, such that $\sigma^r(X(i)) \bigcap X(j) \ne \emptyset$.

We would like to bring to the reader's attention the fact that the relation between (19) and (20) is, in a sense, quite similar to the relation between (15) and (14).

3 Proof of the Main Result

The strategy of the proof is the following. We consider the rescaling $\tilde{s} = s \times L^{-\frac{1}{3}}$, (we shall show that the smallest spacings in the interval $[0, L]$ are of order $L^{-1/3}$). We shall keep s fixed as $L \to \infty$, so $\tilde{s}$ will be proportional to $L^{-\frac{1}{3}}$. We are interested in the asymptotics of the integrals

$$V_k(L) = \int_{[-L,L]^k} r_k(x_1,x_2,\dots x_k;\tilde{s})dx_1dx_2\dots dx_k. \tag{22}$$

We claim that $\lim_{L\to\infty} V_1(L) = \alpha s^3$, where α is defined in (10), and $\lim_{L\to\infty} V_k(L) = 0$, for $k > 1$.

Lemma 1. *Let $V_k(L)$ be defined as in (27), where $r_k(x_1,x_2,\dots x_k;\tilde{s})$ is the k-point cluster function of the $\tilde{s}$-modified random point process introduced above and $\tilde{s} = sL^{-\frac{1}{3}}$. Then*

$$\lim_{L\to\infty} V_k(L) = \begin{cases} \alpha s^3 \ if \ k = 1, \\ 0 \ \ if \ k > 1, \end{cases} \tag{23}$$

where α has been defined in (10).

The result of Lemma 1, combined with (18), implies that the number of the points of the $\tilde{s}$- modified random process in the interval $[0, L]$ converges in distribution to the Poisson law as $L \to \infty$.

Once Lemma 1 is proven , we shall show that the number of points in $[0, L]$ of the original determinantal process that have at least two neighbors to the right within distance $s/L^{1/3}$ is zero with probability very close to 1, provided that L is large and s stays finite.

Proof of Lemma

We start with $V_1(L)$. Consider the one-point correlation function (intensity) of the $\tilde{s}$-modified point process $\rho_1(x;\tilde{s})$.

$$\rho_1(x;\tilde{s}) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \int_x^{x+\tilde{s}} \cdots \int_x^{x+\tilde{s}} \rho_{m+2}(x,y,z_1,\ldots,z_m)dyd^m z. \quad (24)$$

We claim that in the determinantal case

$$\rho_1(x;\tilde{s}) = \int_x^{x+\tilde{s}} \rho_2(x,y)D(x,y;\tilde{s})dy, \quad (25)$$

where $D(x,y;\tilde{s})$ is the Fredholm determinant of the integral operator $\tilde{K}$ on $L^2([x,x+\tilde{s}])$,

$$D(x,y;\tilde{s}) = \det(1-\tilde{K}), \quad \tilde{K} : L^2([x,x+\tilde{s}]) \to L^2([x,x+\tilde{s}]). \quad (26)$$

The kernel $\tilde{K}(u,v)$ in (26), (25) depends on x and y, and is given by the formula

$$\begin{aligned}\tilde{K}(u,v) = K(u,v) &- K(u,x)T_{11}(x,y)K(x,v) - K(u,x)T_{12}(x,y)K(y,v)\\ &- K(u,y)T_{21}(x,y)K(x,v) - K(u,y)T_{22}(x,y)K(y,v), \end{aligned} \quad (27)$$

where

$$\begin{pmatrix} T_{11}(x,y) & T_{12}(x,y) \\ T_{21}(x,y) & T_{22}(x,y) \end{pmatrix} = \begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{pmatrix}^{-1}.$$

Indeed, let us introduce the notation $K[x_1,\ldots,x_k] := \det(K(x_i,x_j))_{i,j=1,\ldots,k}$. Then,

$$K[x,y,z_1,\ldots,z_m] = K[x,y]\tilde{K}[z_1,\ldots,z_m] = \rho_2(x,y)\tilde{K}[z_1,\ldots,z_m].$$

In other words, the conditional distribution of a determinantal random point process with the correlation kernel K, given there are two particles at x and y is again a determinantal random point process (on $\mathcal{R}^1 \setminus \{x,y\}$) with the kernel $\tilde{K}$ (see e.g. [18]). This allows us to rewrite (24) as

$$\rho_1(x;\tilde{s}) = \int_x^{x+\tilde{s}} \rho_2(x,y)\Big(\sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \int_{[x,x+\tilde{s}]^m} \tilde{K}(z_1,\ldots,z_m)d^m z\Big)dy, \quad (28)$$

and (25) follows.

The intensity $\rho_1(x;\tilde{s})$ is constant (i.e. it does not depend on x) in the translation-invariant case. To estimate $\rho_1(x;\tilde{s}) = \rho_1(0;\tilde{s})$, we note that

$$\rho_{m+2}(x,y,z_1,\dots,z_m) = K[x,y,z_1,\dots,z_m] \le K(x,x)K(y,y)K(z_1,z_1)\dots K(z_m,z_m) \le g(o)^{m+2}, \tag{29}$$

since the determinant of a non-negative definite matrix is bounded from above by the product of the diagonal entries (for the generalization of this result see Lemma 2 below). Then

$$\rho_1(x;\tilde s) = \int_x^{x+\tilde s} \rho_2(x,y)dy - \int_x^{x+\tilde s}\int_x^{x+\tilde s} \rho_3(x,y,z_1)dydz_1 + \frac12 \int_x^{x+\tilde s}\int_x^{x+\tilde s}\int_x^{x+\tilde s} \rho_4(x,y,z_1,z_2)dydz_1dz_2 + \sum_{m=3}^{+\infty}\frac{(-1)^m}{m!}\int_x^{x+\tilde s}\dots\int_x^{x+\tilde s}\rho_{m+2}(x,y,z_1,\dots,z_m)dyd^mz = \int_x^{x+\tilde s} \rho_2(x,y)dy - \int_x^{x+\tilde s}\int_x^{x+\tilde s} \rho_3(x,y,z_1)dydz_1 + \frac12 \int_x^{x+\tilde s}\int_x^{x+\tilde s}\int_x^{x+\tilde s} \rho_4(x,y,z_1,z_2)dydz_1dz_2 + O(\tilde s^4).$$

To estimate $\rho_3(x,y,z_1) = K[x,y,z_1]$ and $\rho_4(x,y,z_1,z_2) = K[x,y,z_1,z_2]$ we recall that the point correlation functions are given by the determinants, and subtract the first column from the other columns both in $K[x,y,z_1]$ and $K[x,y,z_1,z_2]$. Since $y \in [x,x+\tilde s]$, $z_i \in [x,x+\tilde s]$, $i \ge 1$, and the first derivative of g is uniformly bounded, we observe that $\rho_3(x,y,z_1) = K[x,y,z_1] = O(\tilde s^2)$, $\rho_4(x,y,z_1,z_2) = K[x,y,z_1,z_2] = O(\tilde s^3)$, and, therefore

$$\rho_1(x;\tilde s) = \int_x^{x+\tilde s}\rho_2(x,y)dy + O(\tilde s^4) = \int_0^{\tilde s}(g^2(0)-g^2(t))dt + O(\tilde s^4) = \alpha\tilde s^3 + O(\tilde s^4), \tag{30}$$

where α has been defined in (10). It follows from (30) that $\lim_{L\to\infty} V_1(L) = \lim_{L\to\infty}\int_0^L \rho_1(x;\tilde s)dx = \lim_{L\to\infty}\rho_1(0;\tilde s)L = \alpha s^3$.

Next, we show that $\lim_{L\to\infty} V_k(L) = 0$ for $k > 1$. We remind the reader that $V_k(L)$ has been defined as $V_k(L) = \int_{[-L,L]^k} r_k(x_1,x_2,\dots x_k;\tilde s)\times dx_1dx_2\dots dx_k$. We start with the case $k = 2$. Recall (see (20)) that for $|x_1 - x_2| > \tilde s$

$$r_2(x_1,x_2;\tilde s) = \sum_{m=0}^{+\infty}\frac{(-1)^m}{m!}\int_{x_1}^{x_1+\tilde s}\int_{x_2}^{x_2+\tilde s}\int_{I(x_1,x_2;\tilde s)^m}\rho_{4+m}^{trun}(x_1,x_2,y_1,y_2,z_1,\dots,z_m) dy_1dy_2dz_1\dots dz_m, \tag{31}$$

where $\rho_{4+m}^{trun}(x_1,x_2,y_1,y_2,z_1,\dots,z_m)$ has been defined in (21).

As described in the Property A (right after the formula (21)), in order to define $\rho_{4+m}^{trun}(x_1,x_2,y_1,y_2,z_1,\dots,z_m)$ one introduces a partition

$X(1)\bigsqcup X(2)=\{x_1,x_2,y_1,y_2,z_1,\ldots,z_m\}$, where $X(1)$ consists of x_1,y_1, and those of the variables $z_1,\ldots,z_m$ that belong to $[x_1,x_1+s]$, and $X(2)$ consists of x_2,y_2, and those of the variables $z_1,\ldots,z_m$ that belong to $[x_2,x_2+s]$. Let $X(1)\bigcap\{z_1,\ldots,z_m\}=\{z_{i_1},\ldots,z_{i_l}\}$, and $X(2)\bigcap\{z_1,\ldots,z_m\}=\{z_{j_1},\ldots,z_{j_{m-l}}\}$. Then

$$\rho^{trun}_{4+m}(x_1,x_2,y_1,y_2,z_1,\ldots,z_m)=K[x_1,y_1,x_2,y_2,z_1,\ldots,z_m]-K[x_1,y_1,z_{i_1},\ldots,z_{i_l}]K[x_2,y_2,z_{j_1},\ldots,z_{j_{m-l}}]. \tag{32}$$

We claim that

$$|\rho^{trun}_{4+m}(x_1,x_2,y_1,y_2,z_1,\ldots,z_m)|\le(m+4)!(C\tilde{s})^{2+m}\frac{const}{1+|x_1-x_2|^{1+\epsilon}}, \tag{33}$$

where *const* is a constant that may depend on s, and C is the constant introduced after the formulas (8), (9).

The factor $(C\tilde{s})^{2+m}$ in (32) follows from the uniform bound on the derivative of g, and the fact that the $m+2$ variables $y_1,y_2,z_1,\ldots,z_m$ are within distance $\tilde{s}$ from either x_1 or x_2. In other words, one can subtract the first column in the matrices in $K[x_1,y_1,z_{i_1},\ldots,z_{i_l}]$ and $K[x_2,y_2,z_{j_1},\ldots,z_{j_{m-l}}]$ from the other columns, and subtract the first and the third column in the matrix in $K[x_1,y_1,x_2,y_2,z_1,\ldots,z_m]$ from the corresponding columns. Such linear operations do not change the value of the determinants, and the new matrices will contain the terms $g(u-w)-g(x_j-w)$, in all columns, except those corresponding to x_1 and x_2, where $j=1,2$, and $u\in[x_j,x_j+s]$. Such terms can be estimated from above by $\left(\max_{x\in[x_j-w,x_j+\tilde{s}-w]}|g'(x)|\right)\tilde{s}$.

It follows from the definition that $\rho^{trun}_{4+m}(x_1,x_2,y_1,y_2,z_1,\ldots,z_m)$ can be written as a sum over at most $(m+4)!$ permutations, each term being a product $m+4$ factors. As we just showed, $m+2$ out of those $m+4$ factors can be estimated in absolute value by $C\tilde{s}$. Moreover, Property A implies that at least two factors in each term must be given either by $g(x_1-v)$, or by $g(u-v)-g(x_1-v)$, or by $g(x_2-u)$, or by $g(v-u)-g(x_2-u)$, where $u\in[x_1,x_1+\tilde{s}]$, $v\in[x_2,x_2+\tilde{s}]$. The inequalities (8), (9) imply that these two factors each contribute an upper bound $\frac{C}{1+(|x_1-x_2|-\tilde{s})^{\frac{1}{2}+\epsilon}}$ and the desired estimate (33) follows.

We recall that we defined above $I(x_1,x_2;\tilde{s})=[x_1,x_1+\tilde{s}]\bigcup[x_2,x_2+\tilde{s}]$. Then

$$\left|\int_{x_1}^{x_1+\tilde{s}}\int_{x_2}^{x_2+\tilde{s}}\int_{I(x_1,x_2;\tilde{s})^m}\rho^{trun}_{4+m}(x_1,x_2,y_1,y_2,z_1,\ldots,z_m)dy_1dy_2dz_1\ldots dz_m\right|\le (m+4)!(C\tilde{s})^{2+m}(2\tilde{s})^{2+m}\frac{const}{1+|x_1-x_2|^{1+\epsilon}}, \tag{34}$$

and

$$|r_2(x_1,x_2;\tilde{s})| \le \sum_{m=0}^{+\infty} \frac{1}{m!}(m+4)!(C\tilde{s})^{2+m}(2\tilde{s})^{2+m}\frac{const}{1+|x_1-x_2|^{1+\epsilon}} \le$$

$$\frac{const(\tilde{s})^4}{1+|x_1-x_2|^{1+\epsilon}}. \tag{35}$$

We remind the reader that (35) has been derived for $|x_1 - x_2| > \tilde{s}$.

Since $\tilde{s} = sL^{-1/3}$, it follows from the above estimate that

$$\lim_{L\to\infty} \int_0^L \int_0^L r_2(x_1,x_2;\tilde{s})\chi_D(x_1,x_2)dx_1dx_2 = \lim_{L\to\infty} O(\tilde{s}^4 L) = 0, \tag{36}$$

where $D = \{(x_1,x_2): \ |x_1-x_2| > sL^{-1/3}\}$.

To estimate $r_2(x_1,x_2;\tilde{s})$ on D^c, we note that

$$r_2(x_1,x_2;\tilde{s}) = \rho_2(x_1,x_2;\tilde{s}) - \rho_1(x_1;\tilde{s})\rho_1(x_2;\tilde{s}).$$

It follows then from (30) that $\rho_1(x_1;\tilde{s}) = \rho_1(x_2;\tilde{s}) \le const s^3 L^{-1}$. To estimate $\rho_2(x_1,x_2;\tilde{s})$ we can assume without loss of generality that $x_2 \le x_1 \le x_2 + \tilde{s}$. Then

$$\rho_2(x_1,x_2;\tilde{s}) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \int_{x_1}^{x_1+s} \int_{I(x_1,x_2;\tilde{s})^m} \rho_{3+m}(x_1,x_2,y_1,z_1,\dots,z_m)$$

$$dy_1 dz_1 \dots dz_m, \tag{37}$$

where $I(x_1,x_2;\tilde{s}) = [x_1, x_1+\tilde{s}]\bigcup[x_2,x_2+\tilde{s}] = [x_2, x_1+\tilde{s}]$. Subtracting the first column in $\rho_{3+m}(x_1,x_2,y_1,z_1,\dots,z_m) = K[x_1,x_2,y_1,z_1,\dots,z_m]$ from the other columns and using (9), we see that $\rho_{3+m}(x_1,x_2,y_1,z_1,\dots,z_m) \le (m+3)!(const * \tilde{s})^{m+2}$. Integrating over $y_1, z_1, \dots, z_m$ and summing over m we obtain

$$\rho_2(x_1,x_2;\tilde{s}) \le const\tilde{s}^3, \tag{38}$$

which implies

$$\lim_{L\to\infty} \int_0^L \int_0^L r_2(x_1,x_2;\tilde{s})\chi_{D^c}(x_1,x_2)dx_1dx_2 =$$
$$\lim_{L\to\infty} \int_0^L \int_0^L \rho_2(x_1,x_2;\tilde{s})\chi_{D^c}(x_1,x_2)dx_1dx_2 -$$
$$\lim_{L\to\infty} \int_0^L \int_0^L \rho_1(x_1;\tilde{s})\rho_1(x_1;\tilde{s})\chi_{D^c}(x_1,x_2)dx_1dx_2 =$$
$$\lim_{L\to\infty} \big(O(\tilde{s}^4)L - O(\tilde{s}^7)L\big) = 0. \tag{39}$$

Combining (36) and (39) one obtaines $\lim_{L\to\infty} V_2(L) = 0$.

The argument in the case of general $k > 2$ is quite similar. Again, we first estimate $r_k(x_1,x_2,\dots,x_k;\tilde{s})$ on $D = \{(x_1,x_2,\dots,x_k) : |x_i -$

$x_j| > \tilde{s},\ 1 \le i \ne j \le k\}$. We will use formulas (20) and (21). To estimate $\rho_{2k+m}^{trun}(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m)$ we consider the partition $X(1)\bigsqcup\dots\bigsqcup X(k) = \{x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m\}$, where $X(i) = \{x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m\}\bigcap[x_i,x_i+\tilde{s}]$. Let $X(i)\bigcap\{z_1,\dots,z_m\} = \{z_1^{(i)},\dots,z_{ni}^{(i)}\}$. Then $X(i) = \{x_i,y_i,z_1^{(i)},\dots,z_{ni}^{(i)}\}$.

It follows from Property A and the inclusion-exclusion principle that

$$\rho_{2k+m}^{trun}(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m) = \sum_G(-1)^j(j-1)!\prod_{l=1}^{j}K_l, \tag{40}$$

where the summation is over all partitions $G = G_1\bigsqcup\dots\bigsqcup G_j$ of $[k] = \{1,2,\dots,k\}$, $j = 1,\dots,k$, and $K_l = K[x_i,y_i,z_1^{(i)},\dots,z_{ni}^{(i)} : i\in G_l]$; in other words, K_l depends on the variables from $\bigsqcup_{i\in G_l}X(i)$, and it is given by the determinant of the matrix built from the correlation kernel $K(x,y)$. We claim that

$$|\rho_{2k+m}^{trun}(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m)| \le (m+2k)!(C\tilde{s})^{k+m}\times \\ \left(\frac{const}{1+|x_1-x_2|^{\frac{1}{2}+\epsilon}}\frac{const}{1+|x_2-x_3|^{\frac{1}{2}+\epsilon}}\cdots\frac{const}{1+|x_k-x_1|^{\frac{1}{2}+\epsilon}}+\ \dots\right), \tag{41}$$

where the summation in the last factor of the r.h.s. of (41) is over all $(k-1)!$ cyclic permutations (for example, the first term in the sum corresponds to the cyclic permutation $1\to2\to3\to\dots\to k\to1$). We claim that the estimate (41) follows from (8), (9), (40) and Property A. As in the case $k=2$ discussed above, we use the fact that each of the $k+m$ variables $y_1,\dots,y_k,z_1,\dots,z_m$ lies within distance $\tilde{s}$ from one of the x_i's, $i=1,\dots,k$. In each $K_l = K[x_i,y_i,z_1^{(i)},\dots,z_{ni}^{(i)} : i\in G_l]$ in (40) we subtract for each $i\in G_l$ the column corresponding to x_i from the column corresponding to y_i and from the other columns corresponding to the variables from $X(i)$. These linear operations do not change the values of determinants, and, therefore, do not change the value of $\rho_{2k+m}^{trun}(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m)$. Now, according to the Property A, we observe that ρ_{2k+m}^{trun} is a sum of at most $(m+2k)!$ terms. Each term is a product of $m+2k$ factors. Property A assures that each term in the sum can be put into correspondence with a cyclic permutation σ on the set of k variables $x_1,x_2,\dots,x_k$, in such a way that k out of $m+2k$ terms in the product are of the form $g(x_{\sigma(i)}-v)$, or $g(u-v)-g(x_{\sigma(i)}-v)$, $i=1,\dots,k$, where $v\in[x_{\sigma(i+1)},x_{\sigma(i+1)}+\tilde{s}]$, and $\sigma(k+1)=\sigma(1)$. The bounds (8), (9) then imply (41) in the same manner as has been shown in the case $k=2$. Therefore,

$$|\int_{x_1}^{x_1+\tilde{s}}\dots\int_{x_k}^{x_k+\tilde{s}}\int_{I(x_1,\dots,x_k;\tilde{s})^m}\rho_{2k+m}^{trun}(x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_m)dy_1\dots dy_k \\ dz_1\dots dz_m| \le (m+2k)!(C\tilde{s})^{k+m}(ks)^{k+m}\times \\ \left(\frac{const}{1+|x_1-x_2|^{\frac{1}{2}+\epsilon}}\frac{const}{1+|x_2-x_3|^{\frac{1}{2}+\epsilon}}\cdots\frac{const}{1+|x_k-x_1|^{\frac{1}{2}+\epsilon}}+\dots\right), \tag{42}$$

provided $\boldsymbol{x} = (x_1, \dots, x_k) \in D$, i.e. $|x_i - x_j| > \tilde{s}$ for $i \neq j$. Then on D we have an estimate

$$|r_k(x_1, \dots, x_k; \tilde{s})| \leq \sum_{m=0}^{+\infty} \frac{1}{m!}(2k+m)!(C\tilde{s})^{k+m}(k\tilde{s})^{k+m} \times$$
$$\left(\frac{const}{1+|x_1-x_2|^{\frac{1}{2}+\epsilon}} \times \frac{const}{1+|x_2-x_3|^{\frac{1}{2}+\epsilon}} \times \cdots \frac{const}{1+|x_k-x_1|^{\frac{1}{2}+\epsilon}} + ...\right) \leq$$
$$Const_k(\tilde{s})^{2k}\left(\frac{const}{1+|x_1-x_2|^{\frac{1}{2}+\epsilon}} \times \frac{const}{1+|x_2-x_3|^{\frac{1}{2}+\epsilon}} \times \cdots \frac{const}{1+|x_k-x_1|^{\frac{1}{2}+\epsilon}} + ...\right),$$

and

$$|\int_0^L \dots \int_0^L r_k(x_1, \dots, x_k; \tilde{s}) \chi_D(x_1, \dots, x_k) dx_1 \dots dx_k| \leq Const_k(L^{1+(k-1)(\frac{1}{2}-\epsilon)}) s^{2k} L^{-\frac{2k}{3}}. \quad (43)$$

The last estimate implies

$$\lim_{L\to\infty} \int_0^L \dots \int_0^L r_k(x_1, \dots, x_k; \tilde{s}) \chi_D(x_1, \dots, x_k) dx_1 \dots dx_k = 0, \quad (44)$$

for $k \geq 3$. Our next goal is to show that

$$\lim_{L\to\infty} \int_0^L \dots \int_0^L r_k(x_1, \dots, x_k; \tilde{s}) \chi_D^c(x_1, \dots, x_k) dx_1 \dots dx_k = 0. \quad (45)$$

To estimate $r_k(x_1, \dots, x_k; \tilde{s})$ on D^c, we rewrite the formula (12) that expresses the k-point cluster function in terms of point correlation functions:

$$r_k(x_1, \dots, x_k; \tilde{s}) = \rho_k(x_1, \dots, x_k; \tilde{s}) - \rho_1(x_1; \tilde{s})\rho_{k-1}(x_2, x_3, \dots, x_k; \tilde{s}) -$$
$$\rho_1(x_2; \tilde{s})\rho_{k-1}(x_1, x_3, \dots, x_k; \tilde{s}) \dots - \rho_1(x_k; \tilde{s})\rho_{k-1}(x_1, x_2, \dots, x_{k-1}; \tilde{s}) +$$
$$2\rho_2(x_1, x_2; \tilde{s})\rho_{k-2}(x_3, \dots, x_k; \tilde{s}) + 2\rho_2(x_1, x_3; \tilde{s})\rho_{k-2}(x_2, x_4, \dots, x_k; \tilde{s}) +$$
$$\dots \rho_2(x_{k-1}, x_k; \tilde{s})\rho_{k-2}(x_1, x_2, \dots, x_{k-2}; \tilde{s}) - \dots \quad (46)$$

We claim that the integral of each of the terms in (46) over $[0, L]^m \bigcap D^c$ has a zero limit as $L \to \infty$. To prove it, we consider an arbitrary term in (46),

$$\rho_{k_1}(x_1, x_2, \dots, x_{k_1}; \tilde{s})\rho_{k_2}(x_{k_1+1}, x_{k_1+2}, \dots, x_{k_1+k_2}; \tilde{s}) \cdots \rho_{k_l}(x_{k_1+\dots k_{l-1}+1}, \dots, x_k; \tilde{s}), \quad (47)$$

where $k_1 + k_2 \dots + k_l = k$, $k_i \geq 1$, $i = 1, \dots, k$. We shall estimate $\rho_{k_1}(x_1, x_2, \dots, x_{k_1}; \tilde{s})$, the other $l-1$ factors are estimated in the same way.

First assume that none of the variables $x_1, x_2, \dots, x_{k_1}$ are within distance $\tilde{s}$ from each other. Then one can clearly estimate $\rho_{k_1}(x_1, x_2, \dots, x_{k_1}; \tilde{s})$ from above as

$$\rho_{k_1}(x_1, x_2, \dots, x_{k_1}; \tilde{s}) \leq \int_{x_1}^{x_1+\tilde{s}} \dots \int_{x_{k_1}}^{x_{k_1}+\tilde{s}} \rho_{2k_1}(x_1, \dots, x_{k_1}, y_1, \dots y_{k_1}) dy_1 \dots dy_{k_1}. \quad (48)$$

Now, since $\rho_{2k_1}(x_1,\ldots,x_{k_1},y_1,\ldots y_{k_1}) = K[x_1,\ldots,x_{k_1},y_1,\ldots,y_{k_1}]$, and $K[x_1,\ldots,x_{k_1},y_1,\ldots,y_{k_1}]$ is the determinant of a $(2k_1)$-dimensional (non-negative definite) real symmetric matrix, we can estimate the determinant from above by the product of the determinants of the 2×2 diagonal blocks

$$K[x_1,\ldots,x_{k_1},y_1,\ldots,y_{k_1}] = K[x_1,y_1,\ldots,x_{k_1},y_{k_1}] \le \prod_{i=1}^{k_1} K[x_i,y_i]. \quad (49)$$

The bound (49) follows from the Fischer inequality which we state below as Lemma 2.

Lemma 2. *Let* $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a block matrix, let* A *and* C *be* $n\times n$ *and, respectively,* $m\times m$ *non-negative definite matrices, and* B *be a* $m\times n$ *matrix. Then*

$$\det\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \le \det A \det C. \quad (50)$$

Proof

To prove Lemma 2, it is enough to reduce it to the obvious case $M = \begin{pmatrix} Id & B \\ B^* & Id \end{pmatrix}$ by appropriate rotations and dilations in C^n and C^m (see e.g. [18]).

As was shown above (see calculations around formula (30)

$$\int_{x_i}^{x_i+\tilde{s}} K[x_i,y_i]dy_i \le const(\tilde{s})^3, \quad (51)$$

which then implies that

$$\rho_{k_1}(x_1,x_2,\ldots,x_{k_1}) \le const^{k_1}(\tilde{s})^{3k_1}. \quad (52)$$

If none of the variables are within $\tilde{s}$ from each other in all factors in (47), We infer from (52) that

$$\rho_{k_1}(x_1,x_2,\ldots,x_{k_1};\tilde{s})\rho_{k_2}(x_{k_1+1},x_{k_1+2},\ldots,x_{k_1+k_2};\tilde{s}) \times \ldots$$
$$\rho_{k_l}(x_{k_1+\ldots k_{l-1}+1},\ldots,x_k;\tilde{s}) \le Const(\tilde{s})^{3k} = O(L^{-k}), \quad (53)$$

and the integral of the l.h.s. of (53) over $[0,L]^k \bigcap D^c$ goes to zero as $L\to\infty$, since $vol([0,L]^k \bigcap D^c) = O(L^{k-1})$.

If some of the variables in $\rho_{k_1}(x_1,\ldots,x_{k_1})$ are within the distance $\tilde{s}$ from one another, the analysis is quite similar. Let us assume, for example, that $x_1 \le x_2 \le \ldots \le x_{k_1}$, and that $x_i \le x_{i+1} \le x_i + \tilde{s}, \;\; i = 1,\ldots,p,$ and that the rest of the variables $x_{p+1},\ldots,x_{k_1}$ are not within the distance $\tilde{s}$ from each other. Then

$$\rho_{k_1}(x_1, x_2, \ldots, x_{k_1}; \tilde{s}) \le \int_{x_{p+1}}^{x_{p+1}+\tilde{s}} \cdots \int_{x_{k_1}}^{x_{k_1}+\tilde{s}} \rho_{2k_1-p}(x_1, \ldots, x_{k_1}, y_{p+1}, \ldots y_{k_1}) dy_{p+1} \ldots dy_{k_1}. \tag{54}$$

One can write

$$\rho_{2k_1-p}(x_1, \ldots, x_{k_1}, y_{p+1}, \ldots y_{k_1}) = K[x_1, \ldots, x_{k_1}, y_{p+1}, \ldots, y_{k_1}] \le K[x_1, x_2, \ldots, x_{p+1}, y_{p+1}] \times \prod_{i=p+2}^{k_1} K[x_i, y_i]. \tag{55}$$

As before,

$$\int_{x_i}^{x_i+\tilde{s}} K[x_i, y_i] dy_i \le const(\tilde{s})^3, \quad i = p+1, \ldots, k_1. \tag{56}$$

As for the term $K[x_1, \ldots, x_{p+1}, y_{p+1}]$, one can substract the first column from all other columns, and obtain

$$K[x_1, \ldots, x_{p+1}, y_{p+1}] \le Const_k(\tilde{s})^{2p+2}, \tag{57}$$

since $|g(x)-g(y)| = O(\tilde{s^2})$ for $0 \le x, y \le \tilde{s}$ (we used the fact that $g'(0) = 0$). Combining (56) and (57), and integrating over the y's we obtain

$$\rho_{k_1}(x_1, x_2, \ldots, x_{k_1}; \tilde{s}) \le Const(\tilde{s})^{3k_1-p+2} = O(L^{-k_1+\frac{p}{3}-\frac{2}{3}}). \tag{58}$$

Note, however, that

$$Vol\{(x_1, \ldots, x_{k_1}) : x_i \le x_{i+1} \le x_i+\tilde{s}, \ i = 1, \ldots, p\} \bigcap [0, L]^{k_1} = O(L^{k_1-p}\tilde{s}^p), \tag{59}$$

and the product of the right hand sides of (58) and (59) goes to zero.

If there are several factors in (53) for which there are variables within distance $\tilde{s}$ from each other, the analysis is very similar, and we leave the details to the reader. Combining all the estimate together, one concludes the integral of the l.h.s. of (53) over $[0, L]^m \bigcap D^c$ goes to zero as Ł $\to \infty$. This finishes the proof of Lemma.

The result of Lemma 1 and formula (18) imply that $\lim_{L\to\infty} \sum_{n=1}^{+\infty} \frac{C_n(L)}{n!} \times z^n = \alpha s^3(e^z - 1)$, where $\{C_n(L)\}_{n=1}^{+\infty}$ is the sequence of the cumulants of the counting random variable $N(L)$, where $N(L)$ is the number of the points of the $\tilde{s}$-modified random pont field in the interval $[0, L]$. It follows from the definition of the $\tilde{s}$-modified random pont field that $N(L) = N_1(L)+N_2(L)$, where $N_1(L)$ counts the number of particles of the original random point field that have exactly one neighbor within distance $\tilde{s}$ to the right, and $N_2(L)$ counts the number of particles of the original random point field that have more than one neighbor within distance $\tilde{s}$ to the right. We claim that the probability that $N_2(L) \ne 0$ is going to zero as $L \to \infty$. Specifically, we establish

Lemma 3.

$$\lim_{L\to\infty} EN_2(L) = 0, \tag{60}$$

where E denotes the mathematical expectation.

Since $N_2(L)$ is a non-negative, integer-valued random variable, (60) implies that $\Pr(N_2(L) \neq 0) \to 0$ as $L \to \infty$.

The proof of Lemma 3 is elementary. We use the estimate

$$EN_2(L) \le \int_0^L \left(\int_x^{x+\tilde{s}} \int_x^{x+\tilde{s}} \rho_3(x, y_1, y_2) dy_1 dy_2 \right) dx. \tag{61}$$

As before one can show that $\rho_3(x, y_1, y_2) = K[x, y_1, y_2] = O(\tilde{s}^4)$, and thus $EN_2(L) \le (\tilde{s})^6 L = o(1)$.

Theorem 1 is proven. Theorem 2 immediately follows from Theorem 1. Indeed, the event $\{\eta(L) > s\}$ is exactly the event that there are no nearest spacings smaller than $sL^{-\frac{1}{3}}$ between the particles in $[0, L]$.

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