21A Midterm Exam Feb. 14, 2011

SOLUTIONS

Instructions: Read each question carefully and answer it in the space provided. **YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT!!!** Calculators, books or notes are not allowed.

Make sure that your exam contains 5 problems. Read through the entire exam before beginning work.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. Problem: Compute the derivative of the following functions: (You *do not* need to simplify your answers in Problem 1!)

(a) $f(x) = \dfrac{\sqrt[3]{x}}{\sin x}.$

$$f'(x) = \frac{\frac{1}{3}x^{-\frac{2}{3}}\sin x - \sqrt[3]{x}\cos x}{\sin^2 x}.$$

(b) $f(x) = e^x - 4x^3.$

$$f'(x) = e^x - 12x^2.$$

(c) $f(x) = (\cos x)(\sqrt{x}+2).$

$$f'(x) = (-\sin x)(\sqrt{x}+2) + (\cos x)(\frac{1}{2\sqrt{x}})$$

2. Problem: Use the definition of the derivative to compute the derivative of $f(x) = x^2 + 2x$. Note: no credit if you compute $f'(x)$ without using the definition of the derivative!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 2x + 2h - x^2 - 2x}{h} = \lim_{h \to 0} \frac{h^2 + 2hx + 2h}{h}$$

$$= \lim_{h \to 0} h + 2x + 2 \quad (\text{if } h \neq 0)$$

$$= \lim_{h \to 0} 2x = 2x.$$

3. Problem:
Compute the following limits (justify your answers!). Give each answer as a finite number, $+\infty$, $-\infty$ or as "limit does not exist".

(a) $\lim_{x \to 3} \frac{9 - x^2}{x - 3}$

$$\lim_{x \to 3} \frac{9 - x^2}{x - 3} = \lim_{x \to 3} \frac{-(x-3)(3+x)}{x - 3} = \lim_{x \to 3} -(3+x) = -6$$

(b) $\lim_{x \to 0} \frac{\sin 6x}{\sin 2x}$

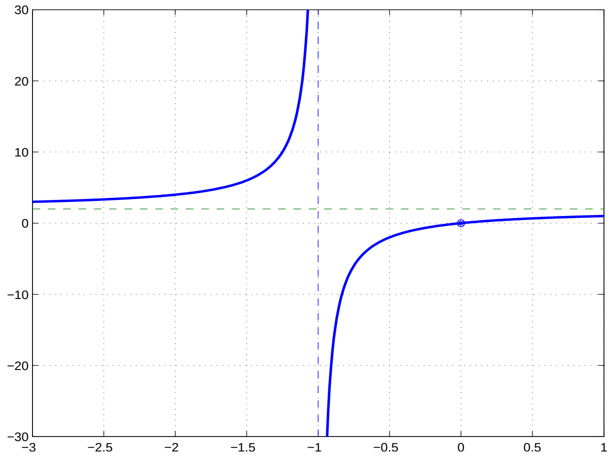
$$\lim_{x \to 0} \frac{\sin(6x)}{\sin(2x)} = \lim_{x \to 0} \frac{\sin(6x)}{6x} \frac{2x}{\sin(2x)} \frac{6x}{2x} = \frac{6}{2} \lim_{x \to 0} \frac{\sin(6x)}{6x} \lim_{x \to 0} \frac{2x}{\sin(2x)} = \frac{6}{2} \cdot 1 \cdot 1 = 3.$$

(c) $\lim_{x \to \infty} \frac{3x^2 - 2x + 10}{-4x^3 + 20}$

$$\lim_{x \to \infty} \frac{3x^2 - 2x + 10}{-4x^3 + 20} = 0$$

since the degree of the polynomial in the numerator is smaller than the degree of the polynomial in denominator.

4. Problem: Let $f(x) = \frac{2x}{x+1}$. Compute vertical and horizontal asymptotes, and sketch the graph of this function. On your graph clearly indicate all intercepts (those points, where the graph intersects the x-axis and the y-axis) and asymptotes.
Vertical asymptote: $\lim_{x\to\infty} \frac{2x}{x+1} = 2$ by comparing leading coefficients of the polynomials in numerator and denominator, hence the vertical asymptote is $y = 2$.
Horizontal asymptote: $x + 1 = 0$ implies $x = -1$, and $\lim_{x\to-1^+} \frac{2x}{x+1} = -\infty$, $\lim_{x\to-1^-} \frac{2x}{x+1} = \infty$, hence we have a horizontal asymptote at $x = -1$.
Intercept at $(0, 0)$ (not shown in my graph)



5. Problem: Assume a bullet fired straight up from the surface of the Earth reaches a height of $s = 320t - 16t^2$ feet after t seconds.
(a) How long will the bullet be in the air?
(b) How high will the bullet go?
(c) What is the speed of the bullet when it hits the ground?

(a) $s = 0$: $320t = 16t^2$, possible solutions are $t = 0$ or $t = 20$. Thus the bullet is 20 seconds in the air.

(b) $s' = 320 - 32t$, hence $s' = 0$ at $t = 10$

$s(10) = 3200 - 1600 = 1600$, the bullet will go 1600 feet high.

(c) $v = s' = 320 - 32t$ At $t = 20$: $v = 320 - 640 = -320$. Speed is $|v|$, hence the speed is 320 ft/second.