Solutions 21A Sample Final Exam

Note: The solutions presented here may not always show all necessary steps to obtain the final result.

1. Problem: Find the limits.

(a) $\lim\limits_{x\to 0}\frac{3^x-1}{2^x-1}$ using L'Hopital

$$= \lim_{x\to 0}\frac{3^x\ln 3}{2^x\ln 2} = \lim_{x\to 0}\left(\frac{3}{2}\right)^x\frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$$

(b) $\lim\limits_{x\to 0}\frac{1-\cos x}{2x}$

0 (for example via H'ospital's rule)

(c) $\lim\limits_{x\to\infty}\frac{2x^4+x\cos x^3}{x^4+x^2}$

2 (compare the degrees of the leading coefficients of the polynomials)

2. Problem:

(a) Determine a so that $f(x)$ is continuous at every point:

$$f(x) = \begin{cases} x^3+a-2 & \text{if } x \le 2, \\ ax^2 & \text{if } 2 < x. \end{cases}$$

$a = 2$

(b) Let $f(x) = \frac{x}{x^2-16}$. Verify that $\lim_{x\to\infty} f(x) = 0$. Then find an *integer* M which guarantees that $f(x) < 0.001$ if $x \ge M$.

$\lim_{x\to\infty} f(x) = 0$ since degree of poly in numerator is smaller than degree of poly in denominator.

$f(x) = \frac{1}{x-16/x}$, when x is large $16/x \approx 0$. If $x > 16$ then $x/16 < 1$, hence $x-1 < x-16/x$ and therefore $f(x) > \frac{1}{x-1}$. Thus it is sufficient to find an integer M such that

$$\frac{1}{x-1} < \frac{1}{1000}$$

This holds if $x > 1001$. Take $M = 1002$ ($M = 1001$ would also work).

3. Problem: $f(x) = x^3 - 3x^2 - 1$.

(a) Determine all local maxima and minima of f.

(b) On which intervals is f concave upward/downward?

(c) Determine the inflection points of f.

(d) Sketch the graph of the function.

(a) find critical points: $f'(x) = 3x^2 - 6x = 3x(x-2) = 0$, hence $x_1 = 0, x_2 = 2$.

$f(0) = -1, f(2) = -5$.

$f''(x) = 6x - 6$. $f''(0) = -6 < 0$, hence f has a local max. at $(0, -1)$.

$f''(2) = 12 - 6 = 6 > 0$, hence f has a local min. at $(2, -5)$.

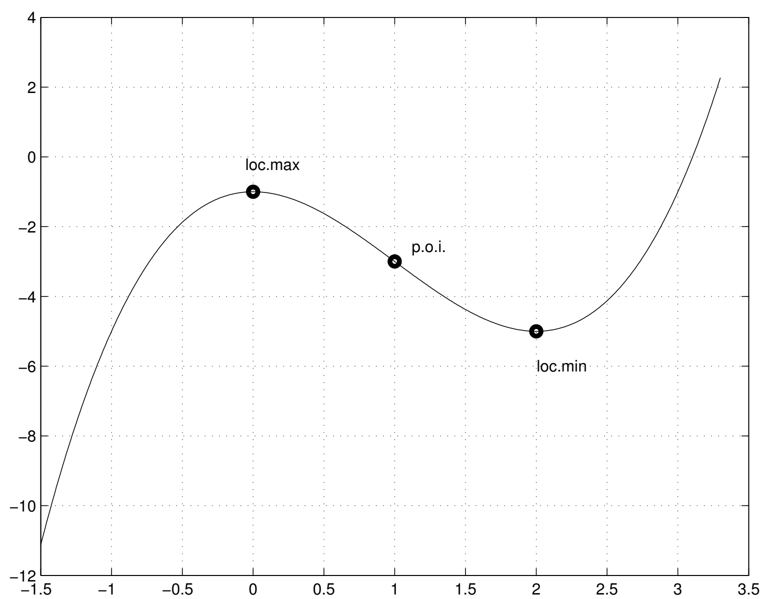
(b) Find points of inflection first:
$f''(x) = 6x - 6 = 0$, hence $x = 1$.

We get the intervals $(-\infty, 1)$ and $(1, \infty)$.

check concavity of f on those intervals:
$(-\infty, 1)$: Let $x = 0$, $f''(0) = -6$, hence f is concave down on that interval
$(1, \infty)$: Let $x = 3$, $f''(3) = 12$, hence f is concave up on that interval

(c) Since f changes concavity when crossing the point $x = 1$, f has an inflection point at $(1, -3)$.

Graph of the function:



4. Problem:
(a) Suppose that the equation $-7x^2 + 48xy + 7y^2 = 28$ holds. Find $\frac{dy}{dx}$ at the point $(0, -2)$.

Take derivatives on both sides:

$$-14x + 48y + 48x\frac{dy}{dx} + 14y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{14x - 48y}{48x + 14y}$$

hence $\frac{dy}{dx}$ at $(0, -2)$ is $-\frac{96}{28}$.
(b) Find the horizontal and vertical asymptotes of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

no horizontal asymptotest, vertical asympote at $x = 2$.

5. Problem:
(a) Use the definition of the derivative to compute the derivative of $f(x) = 4x - 2x^2$. (Note: give yourself zero points if you compute $f'(x)$ without using the definition of the derivative!)

$$f'(x) = \lim_{h \to 0} \frac{4(x+h) - 2(x+h)^2 - (4x - 2x^2)}{h} =$$

$$= \lim_{h \to 0} \frac{4h - 4xh - 2h^2}{h} = \lim_{h \to 0} 4 - 4x - 2h = 4 - 4x.$$

(b) Use linearization to estimate the value of $e^{0.1}$.
Use $a = 0$ and note that $(e^x)' = e^x$ and $(e^x)' = 1$ at $x = a = 0$. Hence

$$e^{0.1} \approx e^0 + 1(0.1 - 0) = 1.1$$

6. Problem: Find the derivative $f'(x)$ for the function $f(x)$ given below:
(a) $f(x) = 2^x x^2$

$$f'(x) = (\ln 2) 2^x x^2 + 2x 2^x$$

(b) $f(x) = (\sin \sqrt{x})^9$

$$f'(x) = 9(\sin \sqrt{x})^8 \cdot (\cos \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

(c) $f(x) = \sqrt{\frac{\cos x}{\ln x}}$

$$f'(x) = \frac{1}{2\sqrt{\frac{\cos x}{\ln x}}} \frac{-\sin x \ln x - \frac{1}{x} \cos x}{(\ln x)^2}$$

7. Problem: Let $f(x) = 2x^2 - 3x$.
(a) Find the tangent to the graph of $f(x)$ at the point $(2, 2)$.
$f(x)' = 4x - 3$, hence slope at $x = 2$ is 5. Tangent line is $y = 5x - 8$.
(b) Prove that $f(x)$ is equal to 0 somewhere in the interval $[-1, 1]$.

Since $f(-1) < 0$ and $f(1) > 0$ and f is continuous the Intermediate value theorem implies that there is a point $c \in [-1, 1]$ such that $f(c) = 0$.

8. Problem: What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of $1000cm^3$?

Let h be the height of the can, and r be the radius. The volume is V and the surface without top is S. The lightest can is the one that uses the least amount of material. $V = \pi r^2 h = 1000$, $S = \pi r^2 + 2\pi rh = \pi r^2 = 2000/r$. Hence $S' = 2\pi r - 2000/r^2 = 0$ from which we get $r = \frac{10}{\pi^{1/3}}$ hence $h = \frac{1000}{\pi r^2} = r$.

The answer is $h = r = \frac{10}{\pi^{1/3}}$ cm.

9. Problem: You are standing on top of a 32 ft tall tower. Assume the acceleration of the rock is constantly $-32ft/sec^2$. (a) You throw a rock straight *down* with velocity 16 ft/sec. When does the rock hit the ground? (b) What is the speed of the rock when it hits the ground?

(a) $f(t) = -16t^2 - 16t + 32 = 0$, hence $t = 1$ (since $t = -2 < 0$). The rock hits the ground after 1 second.

(b) $t = 1 : f'(1) = -32t - 16 = -48$ ft/sec. The speed is 48 ft/sec.

10. Problem: (a) Yes. The fact that $f'(x) = 0$ for all x means that the tangent to the graph of $f(x)$ has always slope 0, therefore $f(x) = C$ for some constant. The property $f(-1) = 4$ implies therefore that $C = 4$ and thus $f(x) = 4$ for all x.

(b) Assume that $f(4) - f(1) > 3$ and show that this leads to a contradiction: Since f' exists, f must be continuous. Thus we can apply the Mean Value Theorem, which says that there should exist a $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - f(1)}{3}.$$

But if $f(4) - f(1) > 3$ then

$$\frac{f(4) - f(1)}{3} > 1 = f'(c),$$

for some $c \in (1, 4)$, which cannot be since it violates the condition $f'(x) \leq 1$ for all $x \in [1, 4]$. [Part (b) was a bit harder than the other problems]

11. Problem: Answer will be presented in class (review session) on Monday.