# Selected solutions to 1. Homework

**Problem 1.4:** This is a problem of interpolation of 8 data points by a linear combination of 8 prescribed basis functions. The set-up is just as in Example 1.1 in the book, except that instead of the monomials $1, x, x^2, \ldots$ we now have arbitrary functions $f_1, f_2, \ldots, f_8$.

Consider the mapping from coefficients $c_1, \ldots, c-8$ to data $d_1, \ldots, d_8$. This mapping is linear, and thus can be represented by an $8 \times 8$ matrix. Let's call it $B$, with entries

$$B_{i,j} = f_j(i),$$

a kind of generalized Vandermonde matrix. By assumption, the mapping is *onto*: for every data $\{d_i\}$ there is a set of coefficients $\{c_i\}$. In other words $B$ has full rank.

By Theorem 1.2, since $B$ has full rank, the mapping it defines is one-to-one. This completes part (a).

by Theorem 1.3, since $B$ has full rank, it is nonsingular. In fact, the inverse is just $B^{-1} = A$ as defined in the statement of the problem. Thus $A^{-1} = B$, and so the $i, j$-th entry of $A^{-1}$ is $F_j(i)$. This completes part (b).

**2.3:** (a) Take $Ax = \lambda x$ with $\|x\| = 1$ and consider the scalar $x^*Ax$. Since $Ax = \lambda x$, it is equal to $\lambda$. On the other hand since $x^*A = x^*A^* = (Ax)^* = (\lambda x)^* = \overline{\lambda} x^*$, it must also be equal to $\overline{\lambda}$. This implies $\lambda = \overline{\lambda}$, that is, $\lambda$ is real.

(b) Suppose $Ax = \lambda x$ and $Ay = \nu y$ with $\lambda \neq \nu$ and $x, y \neq 0$, and consider the scalar $y^*Ax$. Since $Ax = \lambda x$, it is equal to $\lambda y^*x$. On the other hand since $y^*A = y^*A^* = (Ay)^* = (\nu y)^* = \nu y^*$ (using our knowledge that $\nu$ is real), it must also be equal to $\nu y^*x$. Thus $(\lambda - \nu)y^*x = 0$, and since $\lambda \neq \nu$ this implies $y^*x = 0$.

**Problem 2.5:** (a) If $S$ is skew-hermitian, then $i$ is hermitian, so by Problem 2, $i$ has real eigenvalues, therefore $iS$ has imaginary eigenvalues. (Or prove it directly as in Problem 2).

(b) If $S$ has imaginary eigenvalues, then $I - S$ has eigenvalues on the line $\text{Re}\, z = 1$ in the complex plane. In particular, none of the eigenvalues of $I - S$ are zero, so by Theorem 1.3 $I - S$ is non-singular.

(c) Let $Q = (I - S)^{-1}(I - S)$. We check if

$$(I - S)^{-1}(I - S)((I - S)^{-1}(I - S))^* = I.$$

Multiplying on the left by $(I - S)$ and on the right by $(I - S)^*$ converts this to

$$(I + S)(I + S)^* = (I - S)(I - S)^*,$$

that is

$$I + S + S^* + SS^* = I - S - S^* + SS^*.$$

Since $S = -S^*$, this equality certainly holds, and we are done.

**Problem 3.2** Pick a vector $x \neq 0$ and a scalar $\lambda$ such that $|\lambda| = \rho(A)$ and $Ax = \lambda x$. Then $\|Ax\| = |\lambda|\|x\|$, or in other word, $\|Ax\|/\|x\| = \rho(A)$. Since $\|A\|$ is the supremum of all quotients $\|Ax\|/\|x\|$, this implies $\|A\| \geq \rho(A)$.

**Problem 3.3:** Proof of 3.3(d), (3.3(c) is similar):

Let $A \in \mathbb{C}^{n \times n}$. We prove that

$$(1) \qquad \|A\|_2 \leq \sqrt{n}\|A\|_\infty$$

We first show that

$$(2) \qquad \|A\|_2 \leq \|A\|_F$$

and

$$(3) \qquad \|A\|_F \leq \sqrt{n}\|A\|_\infty$$

Proof of (2):

$$\|A\|_2^2 = \rho(A^*A) \leq \sum_{k=1}^{n} \lambda_i(A^*A) = \text{trace}(A^*A) = \|A\|_F^2,$$

where $\rho(A^*A)$ denotes the spectral radius of $A^*A$ and $\lambda_i(A^*A)$ are the eigenvalues of $A$.

Proof of (3)

$$\|A\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{i,j}|^2 \leq \sum_{i=1}^{n} \Big( \sum_{j=1}^{n} |a_{i,j}|^2 \Big) \leq n \Big( \max_i \sum_{j=1}^{n} |a_{i,j}|^2 \Big) = n\|A\|_\infty^2.$$

Combining (2) and (3) proves (1)

**Problem 4.4:** No. Here is a counterexample: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then $A$ and $B$ have the same singular values, but are not unitarily equivalent, since $QAQ^* = I \neq B$ for any unitary matrix $Q$.