Selected solutions to 2. Homework

Problem 2: We compute

$$\|A - Q\|_F^2 = \|A\|_F^2 - 2\text{Re tr}(AQ^*) + \|Q\|_F^2.$$

Since $\|Q\|_F^2 = n$ and $\|A\|_F^2$ is fixed, we must find a matrix Q that maximizes Re tr(AQ^*). Let $A = U\Sigma V^*$ denote the singular value decomposition of A with singular values $\sigma_k, k = 1, \ldots, n$. Then there holds

$$\text{Re}(\text{tr } AQ^*) = \text{Re}(\text{tr } U\Sigma V^* Q^*) = \text{Re}(\text{tr } \Sigma V^* Q^* U) = \sum_{i=1}^{m} \sigma_i t_{ii},$$

where $T = [t_{ij}] = V^* Q^* U$ is unitary matrix. The sum is maximized when $t_{ii} = 1$ for all i, that is when $Q = UV^*$.

Problem 6:

$$(I - 2P)^*(I - 2P) = (I - 2P)^2 - I - 4P + 4P^2 = I.$$

$I - 2P$ is a reflection, a kind of generalization of the Householder reflection.

Problem 7: Let $A = \hat{U}\hat{\Sigma}V^*$ be the reduced SVD of A. Then $A^*A = V\hat{\Sigma}V^*$. Hence the square roots of eigenvalues of A^*A are singular values of A. The rest follows from the usual properties of rank and eigenvalues.