Trigonometric approximation

The task of this project is to compare various numerical algorithms for trigonometric approximation.

Let

$$P_n = \{p : p(x) = \sum_{k=0}^{n-1} c_k \cos(\pi k x) \quad \text{for } x \in [0,1) \text{ and } c = \{c_k\}_{k=0}^{n-1} \in \mathbb{R}\}$$

be the space of cosine polynomials of degree $n-1$. Let $q \in P_n$ and assume that you are given the sampling points $x_k, k = 0, \ldots, m-1$ with $m \geq n$ and the sampling values $\{q(x_k)\}_{k=0}^{m-1}$. The problem is to find the coefficient vector c of the polynomial q from the data $\{x_k\}$ and $\{q(x_k)\}$ and then approximate q on a uniform grid $\{y_l\}_{l=0,N-1}$ with $y_l = l/N$.

In this project we consider several variations of this problem. For all these variations we fix our test polynomial to be

$$q(x) = \sum_{k=0}^{n-1} c_k \cos(\pi k x), \quad x \in [0,1], \tag{1}$$

with

$$c_k = \begin{cases} 1 & \text{for } 0 \leq k \leq 9, \\ \frac{1}{k^2} & \text{for } 10 \leq k \leq 19. \end{cases}$$

Let $b = [b_0, \ldots, b_{m-1}] \in \mathbb{C}^m$ be given by $b_l = q(x_k)$ where q is as desribed in (1) and let A be the Vandermonde-type matrix that arises when you compute c, i.e., when you solve $\min \|Ac - b\|$ or $\min \|Ac - b^\delta\|$, respectively (see below for the definition of b^δ). Let $\tilde{c}$ denote the solution.

(A) Assume the sampling points $\{x_k\}_{k=0}^{m-1}$,$m = 30$, are randomly spaced points in $[0,1]$, generated via $x = sort(rand(m,1))$ (Note: use rand and not randn to generate the x_k). What condition numbers to you observe on average for A if you use, say, 10 different random sampling sets?

For one such random sampling set use SVD, QR, and normal equations to compute $\tilde{c}$ and compare their relative errors $\|c - \tilde{c}\|_2 / \|c\|_2$. Compute an approximation $\tilde{q} = \sum_{k=0}^{n-1} \tilde{c}_k \cos(\pi k x)$ to q at the grid points $\{y_l\}_{l=0,N-1}$ with $y_l = l/N$ and $N = 100$. Plot the various approximations and the original polynomial and comment on the results.

Now add about 5% random noise to b, i.e. $b^\delta = b + \delta$ where $\delta = 0.05\|b\|_2 \cdot \text{randn}(m,1)/\sqrt{m}$ and repeat the experiments.

(B) Same setup as in (A), but now choose the random sampling points such that all sampling points are in the interval $[0.1, 1]$ (i.e., there is no sampling point in the interval $[0, 0.1]$). How does the condition number of the matrix A change compared to the random sampling case in (A)? How do the singular values behave? Again use SVD, QR, normal equations, but also truncated SVD (TSVD) to compute approximations (relative errors, plots) as in (A). Does TSVD help in this case? What kind of truncation level would you choose?