Recovery of missing segments and lines in images

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Abstract

The so-called irregular sampling problem is concerned with the problem of reconstruction of an image from irregularly located samples. In a series of papers (many of them together with K. Gröchenig) the authors have developed new algorithms for efficient reconstruction of signals from arbitrary sampling sets, which are guaranteed to converge as long as the image is band-limited and the sampling set does not have too big holes. It is however experimental evidence that for a typical real world image very satisfactory reconstructions can be obtained even if it is not band-limited in the strict sense.

In the present paper only a certain type of irregularities is considered, i.e. missing lines or missing rectangles in a given image. This problem occurs in practical situations where such parts may be lost during the transmission of an image or badly recorded by a camera. The basic idea is to solve the problem by iterative solution of one-dimensional subproblems, i.e. by interpolation along horizontal and then vertical lines or vice versa. The use of fast and efficient 1-D reconstruction methods is the basis for very fast and highly parallelizable algorithm. We demonstrate the efficiency of the product ACT algorithm by typical applications.

Subject terms: image recovery, nonuniform sampling, Fourier transforms.

1 Introduction

One of the principal problems in digital image processing is the reconstruction of missing pixels of an image. This may occur if the image is transmitted pixel per pixel, and noise in the channel or other disturbances distort the image. In a series of earlier papers K. Gröchenig and the authors [11, 10, 9] have developed methods for the reconstruction of the band-limited signals from irregular samples. From a practical point of view many images can be well approximated by band-limited images and in some instances – due to limitations of measurement instruments – one has truly band-limited images.

The article [11] gives a summary of the results obtained up to 1992. Compared to earlier methods (especially the reconstruction methods described by Marvasti) the so-called adaptive weights method compensates very well for irregularities, such as clusters, in the sampling set. The basic fact concerning this method is that one can guarantee convergence at a geometric rate if the maximal gap is smaller than the Nyquist rate. The combination of this approach with the conjugate gradients method, together with the use of a Toeplitz structure of the system matrix, have resulted in the so-far most efficient algorithm, called ACT-method, which has been obtained recently. It

is guaranteed to converge below the Nyquist rate and also that it will converge faster than the standard iterative methods.

In the present note special sampling sets which have a kind of product structure, i.e. missing lines or missing rectangular segments are considered. They allow the reduction of the problem to the iterated solution of one-dimensional problems. The fact that the many 1-D problems occuring have a lot in common (i.e. the basic features: the sampling geometry as well as the size of the spectrum) can be incorporated in the design of the algorithm. The combination of these features makes the algorithm presented in this paper a very efficient one.

The irregular sampling problem for product sampling sets has been considered previously by Butzer and Hinsen [4]. They solve the problem by means of product representations. This approach requires a large number of multiplications and appears to be feasible only for small band-width and a small number of sampling points. According to the authors, this method has not been implemented so far.

Concerning the general 2D reconstruction of band-limited images from irregularly spaced samples the theoretical background has not yet been developed in such detail. The best results known to us so far are given in section 8.5.2 of our report [11].

After the early, but more experimental paper by Sauer/Allebach [22] the suitable mathematical background and far reaching qualitative results have been obtained in [8]. In this paper also the so-called Adaptive Weights method (short ADPW) has been introduced (based on the operator D_Ψ^+). Numerical experiments based on the MATLAB-toolbox IRSATOL developed by the NUHAG (Numerical Harmonic Analysis Group at the Math. Dept. of the Univ. of Vienna) have been carried out meanwhile. They have shown [10, 26] that the so-called Voronoi method by Sauer/Allebach based on nearest neighborhood interpolation performs well in terms of convergence per iteration, but in terms of speed the ADPW method is much better. For this reason ADPW and its accelerated versions based on the conjugate gradient method (cf. [10] for a first presentation of results in this direction) can be recommended for the general irregular sampling problem. Furthermore the Adaptive Weights Conjugate Gradient method has turned out to be quite robust and flexible and can handle clusters and holes in the sampling set much better than other methods described in the literature. However, it does not take into account or use explicitly any possible structure of the underlying sampling set.

In the present paper we discuss a specialized and faster version for the special situation of product sampling, which occurs in many different applications.

2 A Discrete Model

In digital image processing an image is an array of real or complex numbers i.e. a rectangular matrix x of size $N_1 \times N_2$ with entries $x(j,k), i = 0,1,\ldots,N_1-1, j = 0,1,\ldots,N_2-1$.

We identify the index set $\{0,1,\ldots,N_1-1\} \times \{0,1,\ldots,N_2-1\}$ with the finite cyclic group $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, where $\mathbb{Z}_k$ means $\mathbb{Z}$ modulo k. Hence the image x is understood as two-dimensional bi-periodic function with period N_1 and N_2 respective. That means $x(j,k+mN_2) = x(j,k)$ for $k = 0,1,\ldots,N_2-1,\ m \in \mathbb{Z}$ and $x(j+nN-1,k) = x(j,k)$ for $j = 0,1,\ldots,N_1-1,\ n \in \mathbb{Z}$, so e.g. $x(N_1,N_2) = x(0,0)$.

Band-limited functions on $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ are defined similar as on $\mathbb{R}^2$. Using the periodicity of x and $\hat{x}$, we admit negative indices and define for $0 < M_1 < \frac{N_1}{2}, 0 < M_2 < \frac{N_2}{2}$ the space of discrete band-limited images of bandwidth $M_1 \times M_2$ by

$$\mathbb{B}^2_{\mathrm{M_1 M_2}} = \{x(t_1, t_2) \in \ell^2(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}) |\, \hat{x}(k_1, k_2) = 0 \text{ for } |k_1| > M_1,\, |k_2| > M_2\}.$$

A sampling set is a set of points $\{t_j\}_{j=1}^r$ where $t_j = (u_j, v_j), 0 \le u_1 < u_2 \ldots < u_r < N_1, 0 \le v_1 < v_2 \ldots < v_r < N_2$. A sampling set $\{t_j\}_{j=1}^r$ satisfies the Nyquist criterion, if the whole image can be covered by rectangles of size $\frac{N_1}{2M_1+1} \times \frac{N_2}{2M_2+1}$, where the sampling points t_j are the midpoints of these rectangles. Hence the *Nyquist interval* ν is given by the rectangle of size $\frac{N_1}{2M_1+1} \times \frac{N_2}{2M_2+1}$. The problem is now, how can the image $x \in \mathbb{B}^2_{\mathrm{M_1 M_2}}$ be recovered from its samples $\{x(t_j)\}_{j=1}^r$.

The case where $\{t_j\}_{j=1}^r$ is a uniform sampling set is well understood and the reconstruction of x can be carried out by the extended version of Shannon's sampling theorem [5, 3, 18]. However if the samples $\{t_j\}_{j=1}^r$ are nonuniform distributed the recovery of x poses much more problems. Several reconstruction methods can be found in [22, 23, 31, 19], see [26, 10] for more efficient methods.

If the sampling points are arbitrarily distributed, it is demonstrated in [26, 10] that the combination of the so-called Adaptive Weights conjugate gradient method is a very powerful and practicable method.

3 Product Sampling Sets

In many image processing applications one is confronted with the problem of recovering missing vertical and horizontal lines of an image. From the point of view of sampling theory the missing pixels of the image have to be recovered from a *product sampling set*. A two-dimensional product sampling set is created as follows: a nonuniform sampling set is established for each axis, let us call these one-dimensional sampling sets $\{u_i\}_{i=1}^{r_1}$ and $\{v_j\}_{j=1}^{r_2}$. Then all possible pairs are formed with the first coordinate from $\{u_i\}$ and the second coordinate from $\{v_j\}$. Thus the sampling coordinates are of the form $(u_i, v_j)_{1 \le i \le r_1, 1 \le j \le r_2}$.

The problem is now to recover the image x from the samples $x(u_i, v_j)$ and the question arises which method can take most profit out of this special sampling structure. The 2-D Adaptive Weights Conjugate Gradient method [10] – an efficient method for arbitrary nonuniform sampling sets – will perform very good in such a situation. We do not go more into detail here about the adaptive weights conjugate gradient method, because in the sequel we will derive a reconstruction method which is especially designed for the recovery of images from product sampling sets and works even faster for this situation.

Let us consider a band-limited image x of size $N_1 \times N_2$ with rectangular spectrum of the form $[-M_1, M_1] \times [-M_2, M_2]$. The restriction of x to a single row or column is a band-limited 1-D signal with spectrum $[-M_1, M_1]$ or $[-M_2, M_2]$ respectively, since the 1-D spectrum is just the projected rectangle on the corresponding axis. Hence we can reconstruct e.g. the rows $(v_j)_{1 \le j \le r_2}$ from the corresponding sampling values. More precisely, let x_{v_j} be the v_j-th row of the image x. Then for fixed j we can reconstruct x_{v_j} from the sampling values $\{x(u_i, v_k)\}_{i=1}^{r_1}$. As mentioned x_{v_j} is a one-dimensional band-limited signal, thus we can apply any 1-D reconstruction method. Obviously all

rows x_{v_j} have the same bandwidth $[-M_1, M_1]$ and have to be reconstructed from the same sampling geometry.

Having recovered the rows $x_{v_j}, j = 1, \ldots, r_2$ we can reconstruct all columns of x from the values at the sampling coordinates $(v_j)_{1 \le j \le r_2}$. Observe that we reconstruct not only the columns corresponding to the sequence $\{u_i\}_{i=1} r_1$ but *all* columns. Again all columns of x have the same bandwidth and are sampled at the same sampling pattern.

Summing up, the original 2-D reconstruction problem has been transformed into a total of $r_2 \cdot N_2$ very similar 1-D reconstruction problems. In principle we can choose any 1-D reconstruction method for the successive reconstruction of the image x. However from the facts described above it is clear that we will look for a 1-D reconstruction method which is very efficient and takes advantage of the constant geometry.

4 The product version of the ACT method

The most efficient method for the reconstruction of band-limited nonuniform sampled 1-D signals is an iterative one. It arises by a combination of the Adaptive Weights method [11] and conjugate gradients [15, 14] applied to a small Toeplitz system of equations and is called *ACT method* [9, 26]. Actually the performance of ACT compared to other methods [21, 30, 29, 13, 7] is very convincing. It reduces the computational effort drastically and improves the rate of convergence by several orders of magnitude [26, 9]. It is also robust in the following sense: Many standard methods allow only the reconstruction of segments which are only slightly larger than the Nyquist interval. In contrast ACT can recover segments which are 8 –10 times larger than the Nyquist interval. Furthermore for problems which are rather ill-conditioned, one can easily apply one out of the large variety of preconditioning methods for Toeplitz systems and still obtains a satisfactory solution.

For a detailed discussion of the ACT method we refer to [9]. Here we state only the main theorem. See [9] for a proof of this theorem.

Theorem 1 *Let M be the size of the spectrum and let $0 \le t_1 < \ldots < t_r < 1$ be an* arbitrary *sequence of sampling points with $r \ge 2M + 1$. Set $t_0 = t_r - 1, t_{r+1} = t_1 + 1$ and $w_j = \frac{t_{j+1} - t_{j-1}}{2}$ and compute the sequence*

$$\gamma_k = \sum_{j=1}^{r} e^{-2\pi i k t_j} w_j \quad \textit{for } k = 0, 1, \ldots, 2M$$

iwhich generates the associated hermitean Toeplitz matrix $T, (T)_{lk} = \gamma_{l-k}, \quad |l|, |k| \le M$.

To reconstruct a signal x with bandwidth $[-M, M]$ from its samples $x(t_j), j = 1, \ldots, r.$, compute first

$$b_k = \sum_{j=1}^{r} x(t_j) w_j e^{-2\pi i k t_j} \quad \textit{for } |k| \le M\,,$$

Setting $r_0 = p_0 = b, a_0 = 0$ compute iteratively for $n \ge 1$

$$a_n \;=\; a_{n-1} + \frac{\langle r_{n-1}, p_{n-1} \rangle}{\langle T p_{n-1}, p_{n-1} \rangle}\, p_{n-1}$$

$$
\begin{aligned}
r_n &= r_{n-1} - \frac{\langle r_{n-1}, p_{n-1}\rangle}{\langle Tp_{n-1}, p_{n-1}\rangle}\, Tp_{n-1} \\
p_n &= r_n - \frac{\langle r_n, Tp_{n-1}\rangle}{\langle Tp_{n-1}, p_{n-1}\rangle}\, p_{n-1} \ .
\end{aligned}
$$

Then a_n converges in at most $2M+1$ steps to a vector $\mathbf{a} \in \mathbb{C}^{2M+1}$ solving $T\mathbf{a} = \mathbf{b}$. The reconstruction x is then given by $x(t) = \sum_{k=-M}^{M} a_k e^{2\pi ikt}$.

If in addition $\delta < \frac{1}{2M}$ and $x_n(t) = \sum_{k=-M}^{M} a_{n,k} e^{2\pi ikt}$ denotes the approximation after n iterations, then

$$
\left(\sum_{j=1}^{r} |x(t_j) - x_n(t_j)|^2 w_j \right)^{1/2} \leq 2(2\delta M)^n \left(\sum_{j=1}^{r} |x(t_j)|^2 w_j \right)^{1/2} . \tag{1}
$$

Since a Toeplitz matrix is constant along its diagonals, a hermitean Toeplitz matrix is completely defined by its first row. It will be crucial for our discussion that one requires only the sampling points and the bandwidth to establish T. Another important fact from a numerical point of view is that the multiplication of a hermitean Toeplitz matrix T with a vector can be carried out by FFT's, since T can be embedded in a circulant matrix. The entries of a circulant matrix C of size $n \times n$ satisfy $c_{ij} = c_{i-j} = c_{i-j+n}$ [24, 25]. It is a well known fact that the multiplication of a circulant matrix C by a vector is identical with discrete convolution of the first row of C with the vector and hence can be carried out by FFT. Note that the inverse of a Toeplitz matrix is not Toeplitz.

Since the main calculation in the ACT algorithm of Theorem 1 is a matrix-vector multiplication as described above, each iteration requires roughly $\mathcal{O}(MlogM)$ operations. Furthermore according to our model, b and γ can be computed via FFT [9].

Observe that in our case the Toeplitz matrices corresponding to the rows x_{v_j} required for reconstruction of the image coincide. This observation can be used to reduce the overall computational costs drastically by carrying out the most expensive step (inversion of the Toeplitz matrix) only once and just applying it successively to the data arising from the different rows. Unfortunately these facts alone do not provide a fast algorithm, if one has to deal with large sized images with a large bandwidth. The drawbacks are that one has to store the matrix T^{-1} and – more important – one has to carry out a large number of matrix-vector multiplications, which cannot be replaced directly by FFT-methods.

The best way to circumvent these problems is to calculate and establish the inverse matrix using the following fact. The inverse T^{-1} is completely determined by the solution of the equation

$$
Ta = e_1 \qquad \text{where} \quad e_1 = (1, 0, \ldots, 0)^T .
$$

Specifically

$$
T^{-1} = \frac{1}{a_1}(L^*L - GG^*) \tag{2}
$$

where

$$
L = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ a_2 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ a_N & a_{N-1} & \ldots & a_1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & a_N & \ldots & a_2 \\ 0 & 0 & \ldots & a_3 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{bmatrix} .
$$

This fact is known as one of the Gohberg-Semencul formulations [12] of the Trench formula [28].

Several of the $\mathcal{O}(n \log n)$ Toeplitz inversion algorithms are based on these formulas [6, 1, 2]. The advantage associated with relation (2) is fairly clear. First, it allows in our present application a representation of the inverse that requires only $\mathcal{O}(M)$ storage. Second, as pointed out by Jain [17] and others, the factors L and G are Toeplitz, and hence the solution $T^{-1}b$ can be calculated in only $\mathcal{O}(M \log M)$ operations using the FFT technique.

4.1 Description of the product ACT algorithm

The reconstruction method can now be stated as follows:

1. Establish the Toeplitz matrix T_1 corresponding to the horizontal 1-D sampling set $\{u_i\}_{i=1}^{r_1}$ and to M_1 as described in Theorem 1.

2. Solve the system $T_1 a = e$ for $e := (1, 0, \ldots, 0)$ by the conjugate gradient method using fast matrix-vector multiplication by FFT's. Then establish T_1^{-1} in the Gohberg-Semencul formula.

3. The reconstruction of each of the r_2 rows corresponding to the indices v_j of the vertical sampling set can now be done by FFT. 7 FFT's of $\mathcal{O}(M \log M)$ are required for the multiplication of T^{-1} in the Gohberg-Semencul formula by b and two FFT's of $\mathcal{O}(N_1 \log N_1)$ are required to establish b according to $T_1 a = b$ and to compute the associated row from its Fourier coefficients.

4. Repeat the first two steps for the vertical sampling set (replace M_1 by M_2, $\{u_i\}$ by $\{v_j\}$ and T_1 by T_2) and reconstruct all columns of the image.

At this point the question of the qualtiy of the reconstruction arises. The following Corollary gives simple and rather general conditions which guarantee complete reconstruction.

Corollary 1 *Assume that the image x in $\mathbb{B}^2_{M_1 M_2}$ of size $N_1 \times N_2$ is sampled on the product sampling set $(u_i, v_j)_{1 \le i \le r_1, 1 \le j \le r_2}$. Let δ_u and δ_v denote the maximal gap of the 1-D sampling sets $\{u_i\}$ and $\{v_j\}$. If both 1-D sampling sets satisfy the corresponding Nyquist criterion, i.e. $\delta_u < \frac{N_1}{2M_1+1}$ and $\delta_v < \frac{N_2}{2M_2+1}$, then x can be completely reconstructed by the method above.*

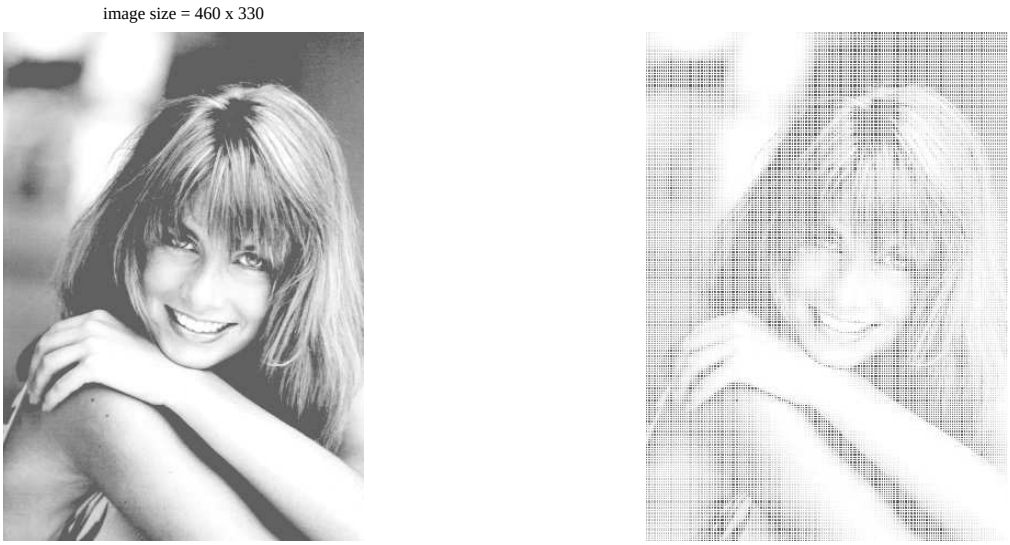
Proof: The assertion follows immediately by applying condition 1 of Theorem 1 successively in each variable. ■

Remark:

a) As outlined in [26] in practice the method converges most often for much larger gaps in the sampling set, since convergence of the ACT method depends mainly on the condition number of the Toeplitz matrix (see also Figure 3 and Figure 4).

b) Clearly this algorithm can also be used as an efficient tool for 2-D signal extrapolation. It turns out that the computational effort for this approach is much smaller than the effort for the 2-D Papoulis-Gerchberg method [20] or for the algorithm of Jain-Ranganath [16]. See [27] for details.

c) The algorithm of Theorem 1 is highly parallelizable. Hence real time applications are also possible for large images.



(a) original non-bandlimited image

(b) distorted image, 51724 samples

Figure 1:

5 Experimental Results

In the first example we want to demonstrate that the proposed method can also be applied to non-band-limited images. Figure 5(a) shows the original non-band-limited image x of size 330×460, 5(b) the distorted image, where 137 vertical and 192 horizontal lines (but not more than two successive lines) of the original image have been removed. This means that more than 70 % of the information is missing. We apply the product ACT method of Theorem 1. The number of non-zero coefficients of the rectangular low-pass filter is 151 for the horizontal direction and 251 for the vertical one. For the choice of the bandwidth of the filter, one has to make sure that the number of non-zero coefficients in each direction is smaller than the corresponding number of samples, to obtain a stable solution. But choosing the bandwidth of the filter too small would result in a not satisfying approximation.

The reconstruction is shown in Figure 5. According to the algorithm the reconstructed image x_n is band-limited, the error $\|x - x_n\|/\|x\|$, measured in the ℓ^2 - norm – i.e. the mean square error – is smaller than 4%. Note that applying the low-pass filter to the original non-distorted image results in an aliasing error of about 3%.

A special form of a product sampling problem is the situation where we have to recover a missing segment in an image. Such an example is shown in Figure 3. The image is of size 640×480. A large segment (the region around the eyes) of the original band-limited image has been removed. The size of the missing segment is 16 times 230 pixels. Note that the height of this distorted segment is about 10 times larger than the Nyquist interval for the vertical direction.

In this example the method of Theorem 1 is applied only along the columns of the image. The

reconstructed image

relative mean square error = 0.039

Figure 2:

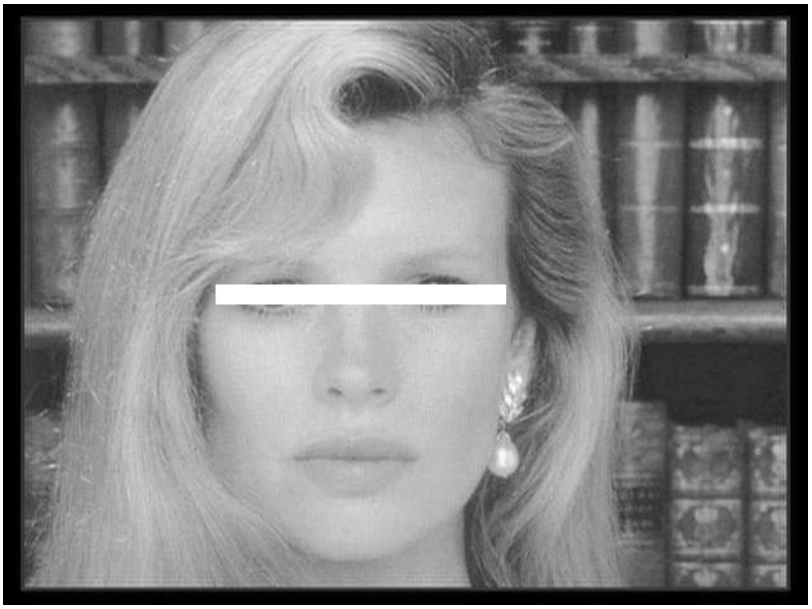


Figure 3: band-limited image with missing segment

Figure 4: reconstruction by product ACT method

reconstructed image is shown in Figure 4. This surprising effect is possible by the band-limitedness of the image shown in Figure 3, which has been obtained in our example by digital filtering with some rectangular low-pass filter (the number of non-zero filter coefficients is 321 in the horizontal and 241 in the vertical direction. For a sufficiently wide filter there is almost no visible effect on the image. However the coherence of the data set is largely increased, in a non-trivial and non-local manner. This makes at least plausible that "somehow" all the information about the missing part is "in principle" available from the rest of the image, similar to holography. The described method shows how to retrieve this information constructively.

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