RECONSTRUCTION OF SIGNALS FROM IRREGULAR SAMPLES OF ITS SHORT–TIME FOURIER TRANSFORM*

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ABSTRACT

The short–time Fourier transform (STFT) leads to a highly redundant linear time–frequency signal representation. In order to remove this redundancy it is usual to sample the STFT on a rectangular grid. For such regular sampling the basic features of the reconstruction problem are well understood. In this paper, we consider the problem of reconstructing a signal from *irregular* samples of its STFT. It may happen that certain samples of the STFT from a regular grid are lost or that the STFT has been purposely sampled in an irregular way. We investigate that problem using Weyl-Heisenberg frames, which are generated from a single *atom* by time–frequency–shifts (along the sampling set). We compare various iterative methods and present typical numerical experiments. Whereas standard frame iterations are doing not very well it turns out that for many reasons the conjugate gradient algorithm behaves best, most often even better than one might expect from the observations made for general frame operators.

Keywords: short time Fourier transform, irregular sampling, Weyl-Heisenberg frames, conjugate gradient, Gabor expansions

1 INTRODUCTION

The short–time Fourier transform (STFT) is a classical tool for the analysis and processing of natural signals. In the modern, signal decomposition point of view we interpret the STFT as an inner product representation based upon time–frequency–shifted versions of a prototype signal (analysis window). Such an expansion set is highly linear–dependent, thus the STFT is a highly redundant signal representation. Due to the existence of a reproducing formula one can easily reconstruct the signal from the full STFT (even without knowledge of the analysis window provided that it is normalized).

By sampling the STFT on a rectangular grid the redundancy can be conveniently reduced leading to the Gabor expansion. The reduction of the redundancy comes at the cost of a more sophisticated reconstruction problem.

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Note that the sampled STFT does no more satisfy a reproducing formula. Moreover, it is now well–known that a certain degree of redundancy is necessary in order to keep the concept of a time–frequency–localized analysis window for the Gabor expansion.

In many applications such as source coding or signal enhancement, it is desirable to have some freedom to match the signal transform to the signal at hand. The basic design parameter of the Gabor expansion are the sampling gaps of the rectangular grid and the choice of the analysis window. With regard to natural signals it is obvious that the restriction to regular grids restricts the signal adaptivity of the expansion set considerably compared to general nonregular time–frequency sampling sets. This gives the motivation to study the reconstruction problem of an irregularly sampled STFT. The basic ingredients we need to tackle this problem come from the analogy to the theory of irregular sampling of band-limited signals[1] and the theory of Weyl-Heisenberg frames.[2,3] In both cases one has *coherent* frames, i.e. frames which are generated from a single *atom* through some group action.[4] This leads to a reproducing kernel Hilbert space of transforms – here the STFT, in the band-limited case the signals themselves – which can be used to reconstruct it from irregular samples. After reconstructing the full STFT from its samples it is of course easy to obtain the signal. Practical algorithms will carry out the whole construction at the level of signals instead of their STFT.

There are a few mathematical papers describing this situation for the continuous setting (signals over R^d for some natural number d, mostly $d = 1$). It follows from the most general, but qualitative statements in the theory developed by Feichtinger and Gröchenig (cf. [5, 6, 7) that for any given window there is a pair of lattice constants, such that for *any* sampling set *not* allowing to squeeze any rectangle of that size between the sampling set (without including at least one point) the corresponding sampling set is enough to connstructively obtain the signal completely due to iterative algorithms. For so-called product sampling sets, i.e. in the case that at certain (irregularly spread) frequences irregular samples (but the same at all those frequencies) are obtained Benedetto and Heller have given some more explicit sufficient conditions for complete reconstruction in terms of frame theory (cf. [8, Thm. 4.1, which is also referred in [9], Thm. 30 and Cor. 31). The most general quantitative results are due to K. Gröchenig, who deals with the case of sampling sets concentrated at different frequencies, but with different sampling at each frequency level in his paper [10]. Acceleration methods in connection with frames are first suggested in [11,12].

2 NOTATION AND PROBLEM STATEMENT

For the problem to be accessible for numerical solution we have to create a finite-dimensional model. We consider discrete, complex valued, N-periodic (or finite length N) signals $x \in \mathbb{C}^N$. The STFT is understood as a function over the so-called discrete time–frequency plane, respectively as the following $N \times N$ matrix:

$$(STFT_g x)(k,l) := \langle x, M_l T_k g \rangle, \qquad \text{for } k,l = 0, \dots N-1$$

where g is the analysis window. M_l denotes the modulation operator (frequency shift)

$$M_l(x_0, x_1, \dots, x_{N-1}) := \left(x_0, e^{2\pi j l/N} x_1, e^{2\pi j 2l/N} x_2, \dots, e^{2\pi j (N-1)l/N} x_{N-1} \right),$$

T_k is the translation operator (time shift)

$$T_k(x_0, x_1, \dots, x_{N-1}) := (x_k, x_{k+1}, x_{k+2}, \dots, x_{N-1}, x_0, \dots, x_{k-1}),$$

and the inner product is defined as usual ($\bar{y}$ is the conjugate of y)

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x_n \bar{y}_n .$$

Irregular sampling of the STFT can be seen as a map from $\mathbb{C}^N$ into $\mathbb{C}^K$ (K = number of samples):

$$x(t) \mapsto (STFT_g x)(k_i, l_i), \tag{1}$$

where $(k_i, l_i), i = 1, 2, \ldots, K$ denotes the sampling set. The question of existence and numerical stability of the inverse mapping of (1) can be studied by the properties of the nonorthogonal expansion set:

$$g_i := M_{l_i} T_{k_i} g$$

using frame theory.

3 FRAMES AND FRAME ITERATION

Recall the definition of a frame:

DEFINITION 3.1. *A sequence $\{g_i, i \in \mathbb{Z}\}$, with $g_i \in \mathbb{C}^N$, is called a frame for $\mathbb{C}^N$, if there exists constants $0 < A \leq B < \infty$ such that for all $x \in \mathbb{C}^N$*

$$A\|x\|^2 \leq \sum_{i \in \mathbb{Z}} \left|\langle x, g_i \rangle\right|^2 \leq B\|x\|^2 \tag{2}$$

The frame operator S corresponding to the irregularly sampled STFT expansion set is given by

$$Sx = \sum_i \langle x, g_i \rangle g_i = \sum_i (STFT_g x)(k_i, l_i) M_{l_i} T_{k_i} g,$$

in our specific discrete context the frame operator is a $N \times N$ matrix

$$S(m, n) = \sum_i \overline{(M_{l_i} T_{k_i} g)_m} (M_{l_i} T_{k_i} g)_n,$$

which acts on signals defined as row vectors.

Existence of the lower frame bound $A > 0$ assures that the signal can be recovered from the inner products $\langle x, g_i \rangle$ in terms of a so-called *dual frame* $\{\tilde{g}_i\}$:

$$x = \sum_{i \in \mathbb{Z}} \langle x, g_i \rangle \tilde{g}_i,$$

where $\tilde{g}_i = S^{-1} g_i$. Based on these definitions we can state that the following theorem which gives the precise connection between irregular sampling of the STFT and frame theory:

THEOREM 3.2. *Given a signal $x \in \mathbb{C}^N$, an STFT analysis window $g \in \mathbb{C}^N$ and a time–frequency sampling set $\{(k_i, l_i)\}_{i=1}^M$, then x can be exactly reconstructed from the irregular STFT sampling values $\{\langle x, M_{l_i} T_{k_i} g \rangle\}_{i=1}^M$ if and only if the set $\{M_{l_i} T_{k_i} g\}_{i=1}^M$ constitutes a frame for $\mathbb{C}^N$.*

If the signal length is small, the dual frame can be obtained by direct matrix inversion. Recall that the elements of the dual frame are (up to conjugation) just the columns of the pseudo-inverse matrix M^+ (also known as Moore–Penrose inverse) if the rectangular matrix M carrying the elements of the original frame as rows. Since a frame for $\mathbb{C}^N$ typically contains much more than N elements the following well known identity[13] may help to reduce the dimensionality of the problem:

$$M^+ = M^* (M M^*)^+ ,$$

where M^* denotes the conjugate transpose of M. However, for comparatively large N matrix inversion is getting impossible or at least very slow such that iterative methods have to be taken into consideration.

Corollary 1. *Let S be a frame operator (i.e., strictly positive and bounded) then S is invertible and x can be recovered from Sx by the following iteration:*

$$\begin{aligned} x^{(0)} &= \gamma S x \\ x^{(n+1)} &= x^{(n)} + \gamma S(x - x^{(n)}), \end{aligned}$$

where γ is a sufficiently small constant, the so-called relaxation parameter.

Applicability of the frame algorithm hinges on the choice of the relaxation parameter. Too small γ leads to slow convergence while too large γ leads to divergence. One can show that the optimal convergence rate is $\frac{B-A}{B+A}$, and can only be achieved for:

$$\gamma_{opt} = \frac{2}{A+B}.$$

while convergence (slower than possible) will take place if one chooses a relaxation parameter which is too small or a little bit too large.

In practice one does not have exact knowledge of the frame bounds and their computation requires substantial numerical effort, similar to the reconstruction problem itself.

One possible way to obtain some partial information about the upper bound of the frame operator is to run a few steps of the vector iteration method for the frame operator. This may give a reasonable approximation of the constant B. It is then a question of good guess to find out about the constant A. If the frame is supposed to be a *bad* one, which means that the sampling set has relatively large holes, then it is natural to expect that A is rather small, and a value close to $2/B$ will be a good choice. The closer the frame comes to a tight frame (which means a good one, where A is close to B), the more reasonable it is to choose the relaxation parameter in the order of $1/B$. See Figure 1 for details.

4 STEEPEST DESCENT AND CONJUGATE GRADIENT

It is typically not easy to obtain a reliable estimate for the two (optimal) frame constants A and B , in particular for rather irregular sampling sets. Since however any wrong choice of the relaxation parameter – other than $2/(A+B)$ – deteriorates the usually not very good rate of convergence even more, it makes sense to replace the frame iteration which coincides with the Richardson iteration[12] by the *steepest descent method*[14] (SD). It has the advantage of adjusting the relaxation parameter automatically during the iterative process. But also the rate of convergence of SD is still not sufficient. One way to overcome this problem is to combine SD and under-relaxation, see [15] for details.

The most efficient acceleration technique of the frame iteration is given by the *conjugate gradient algorithm* (CG), cf. [14]. Since the frame operator is a positive definite operator, experts in numerical linear algebra will not find it very surprising that CG does best in all the cases we have investigated. However, it is remarkable how much better it works in some interesting cases.

This fact is due to the following well-known feature of the CG-method: The correction terms throughout the iterative process are mutually orthogonal with respect to a new scalar product (which can be derived by means of the positive definite frame operator). This has the big advantage that the iterative process will be finished (for the case of ideal numerical accuracy) if the mathematical problem can be reduced to a lower dimensional equivalent problem. This is indeed the case, whenever the sampling set is a complete subgroup, let us say with lattice constants a and b, in the time and frequency direction respectively. In this case polynomials in S (obtained by replacing the frame operator S in the algebraic polynomial) form an algebra of dimension at most ab. This implies that this is the maximal number of steps required for complete reconstruction. Of course the method of dual Gabor windows[3] is the method of choice in this case. However, even if the sampling situation is in some sense *close* to such a group situation, CG will still show a similar (good) behaviour, since the eigenvalues of the frame operator are still highly clustered, which results in a high rate of convergence.[16] This is the case for sampling sets obtained by jitter errors from a strict lattice, but also for the case of lattices – even jittered ones – with missing samples, as long as the holes are not too large. Figure 2 demonstrates such a behavior.

Here the sampling set should match the characteristics of the window g. If it has an ambiguity function $STFT_g(g)$ which is flat in the time domain and well concentrated in the frequency domain, gaps which are slim in the time direction, but extended in the frequency direction will pose more problems for good reconstruction than similar gaps having their orientation in the opposite directions, see also Figure 5.

In connection with rather redundant sampling sets the question arises if one should make a subselection of "relevant" points. Since the CG-algorithm turned out to be not only performing very well in all the cases that have been considered it was also the most stable with respect to clusters and irregularities in the sampling set. Such clusters may force small relaxation parameters, hence poor convergence rates in the case of the ordinary frame iteration method. That making a subselection of points may improve the speed of reconstruction even in the case of CG is plausible: While more points lead to better rates of convergence per iterations, the computational cost *per iteration* will increase. If there are already sufficiently many points involved this extra effort will *not* be compensated by the gain in terms of better approximation per iteration. This statement, however, only applies to noise-free data, while in the presence of noise it might be better to include as many sampling points as possible in order to obtain a better SNR.

Finally we would like to point to the following observation made during our numerical experiments: Having more points does not necessarily mean that the CG-method – nor any other of the methods – will perform better. Also this is true "in the statistical sense" it may very well happen that (e.g. for a fixed number of iterations, or even throughout the iterative process) the larger sampling set is leading to a slightly slower convergence of the approximate reconstructions towards the true signal.

5 POCS BASED METHODS

For a general description of the method of projections onto convex sets (POCS) see the book of H. Stark,[17] but also the book of N. Hurt,[18] for a Hilbert space description of such methods one may consult the work of D.Youla, see e.g. [19].

POCS is based on the idea that each sampling value of the STFT (for fixed window) corresponds to the knowledge that the signal has to satisfy one specific equation. The STFT being a linear mapping from the signal space into the complex numbers it is clear that the family of all signals x satisfying $STFT(x, g, n, k) = \alpha_{(n,k)}$ forms a convex (in fact affine) manifold $A_{(n,k)}$. If the family of TF-shifted windows forms a frame, the intersection of all those affine subspaces consists of a single element, which is precisely the original signal. In order to identify this unique element in the intersection one chooses an iterated projection onto those convex sets. One *cycle* consists of a sequence of projections onto all those affine subspaces (there are as many as there are sampling points in the sampling set). Thus one has to keep in mind that enlarging the sampling set makes one cycle longer, and of course the pure computational load per cycle increases linearly with the number of sampling points. The POCS algorithm as such is performing a number of cycles, until the relative error is below a certain threshold.

The POCS method is used often straightforward, choosing the affine conditions in their "natural" order, which means in our case that one would make a list of the points in the TF-plane, ordered according to increasing time and/or frequency. However, especially for relatively high sampling density it turns out to be a disadvantage to

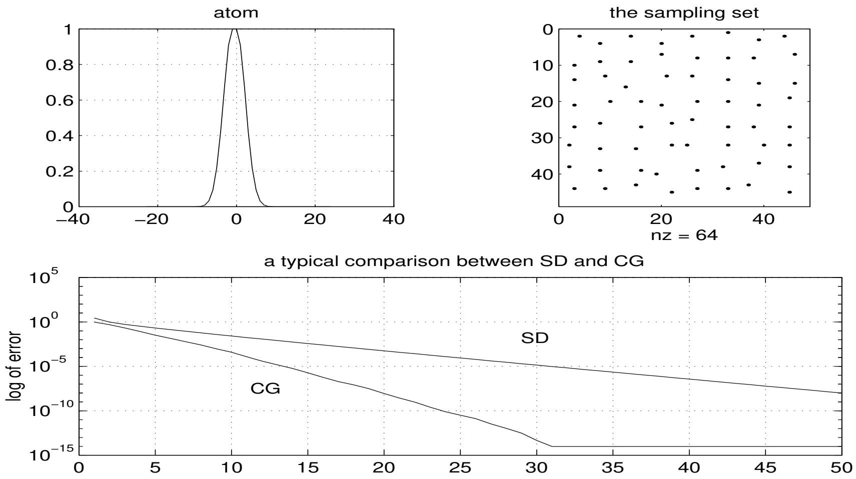

Figure 1: CG outperforms SD

choose this particular order, because neighboring points correspond to TF-shifted versions of the window which are relatively close to each other. Thus a correction at one point, following too closely another correction based on a sampling point nearby will not require a significant change, while taking the same computational load as a correction at a point "far away". For this reason "random order" arrangements appear to be quite efficient.[20]

We found it most difficult to make any a priori judgement on the expected approximation behaviour of POCS. There is by no means a guarantee that a larger sampling set will lead to faster reconstruction. Indeed, it may happen that enlarging the sampling set just increases the length of one cycle and therefore the computational effort, without really giving the appropriate improvement in terms of higher gain per cycle.

In terms of floating point operations (flops) and gain of approximation POCS was sometimes doing rather well. No general conclusion about its efficiency in the general situation considered in this paper can be made. See Figure 4 for a comparison including POCS, with a fairly typical behaviour (the sampling set was a uniformly distributed random set in the TF-plane).

6 NUMERICAL EXAMPLES

We have carried out a number of numerical experiments concerning the efficiency of various iterative methods suitable to reconstruct a signal from irregular samples of its STFT with respect to a known window. The emphasis in this note is on the efficiency in terms of iterations for the case of general sampling sets. In other words, we do not look out for sampling sets allowing for special tricks in the implementation of these algorithms, mostly because these improvements (although interesting by themselves) usually apply to all the competing algorithms in a very similar way by making the iterative steps cheaper. This – however – usually does not influence the relative ranking of the methods discussed in this paper.

Figure 1 shows the behaviour typical for a rather low-redundancy sampling set in the TF-plane, obtained through jitter movement from a regular lattice. It shows the Gabor atom used, the TF-sampling set and the rates

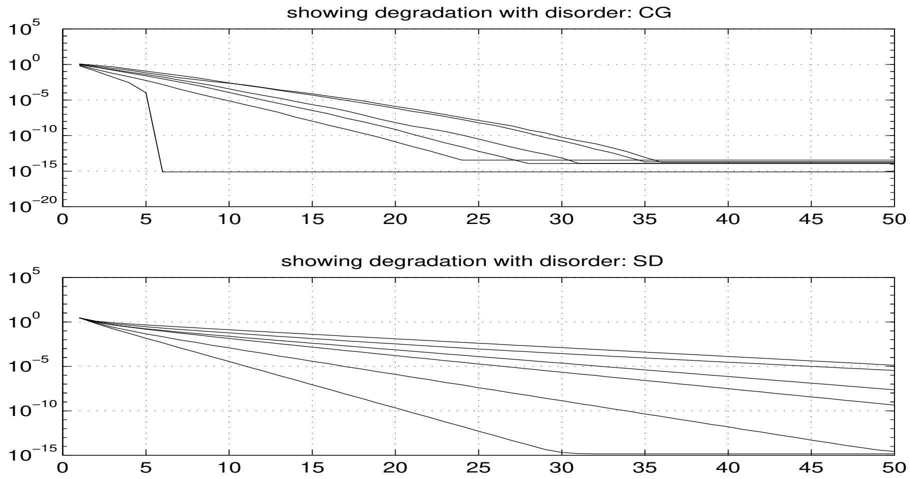

Figure 2: Degradation of method with increasing disorder

of convergence (log of mean square error versus number of iterations) for CG and SD.

Usually *stability* of algorithms is of great practical concern. In the present case we have tested the steepest descent and CG for the resistence towards perturbations on the sampling set. Starting from a slightly oversampled grid we have moved the lattice points randomly in both directions (by one unit per step). For each new and increasingly irregular sampling set we tried to recover the signal from the STFT values at those points. Figure 2 indicates that of course both methods show a degradation in terms of speed of convergence. However, it is clear from those plots that SD was suffering much more than CG. Continuing the experiment we found situations where SD simply was unable to recover the signal completely, while CG was still able to achieve this.

The third example (Figure 3) also indicates the good performance of the CG-reconstruction method. We have choosen a sampling set which looks like a hexagonal type lattice in the TF-plane as discussed in the paper of P.Prinz.[21] The original full lattice had 288 points, for a signal length of $N = 144$, thus a redundancy of 2. Then we took away 5 columns (each containing 12 sampling points). Thus the remaining lattice has 228 points, or a redundancy of 1.583. The idea is to show that even if during a transmission of Gabor coefficients full columns are lost the information contained in the remaining sampling values is still good enough to perform efficient reconstruction of the signal on the receivers side.

We have to point out the difference between lost samples of the STFT (the case we study) and sampling values which are not stored because they are *known to be small* and are eliminated during transmission or storage. In such cases it is the aim to obtain a rather good numerical approximation of the signal as "cheap" as possible. A typical case is the use of thresholds in the Gabor coefficient domain. Assuming that a signal does not cover the full TF-plane (e.g. some chirp type signal) one may preserve only those Gabor-coefficients over a sufficiently dense TF-lattice which surpass a certain threshold. Then the use of a *dual Gabor atom*, applied to the full coefficient scheme containing also the zeros obtained by thresholding, quite often achieves a rather satisfactory reconstruction. For the calculation of the dual Gabor window cf. a number of papers, such as [22,23].

The final experiment explained here indicates the connection between the shape of the Gabor atom, the geometry of the sampling set and the speed of reconstruction (for the CG-method). The sampling set is a grid with missing points, with a lattice constant in the time-direction smaller than the lattice constant in the frequence

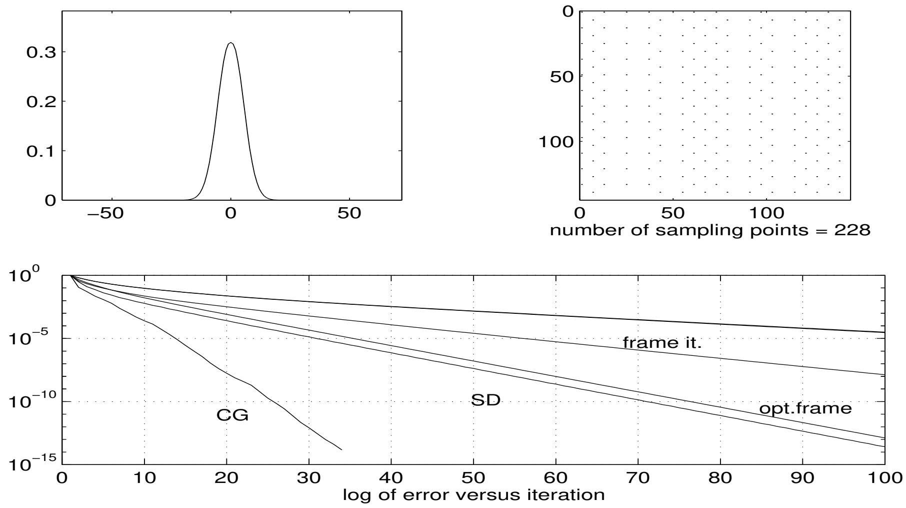

Figure 3: Comparison of rates of convergence

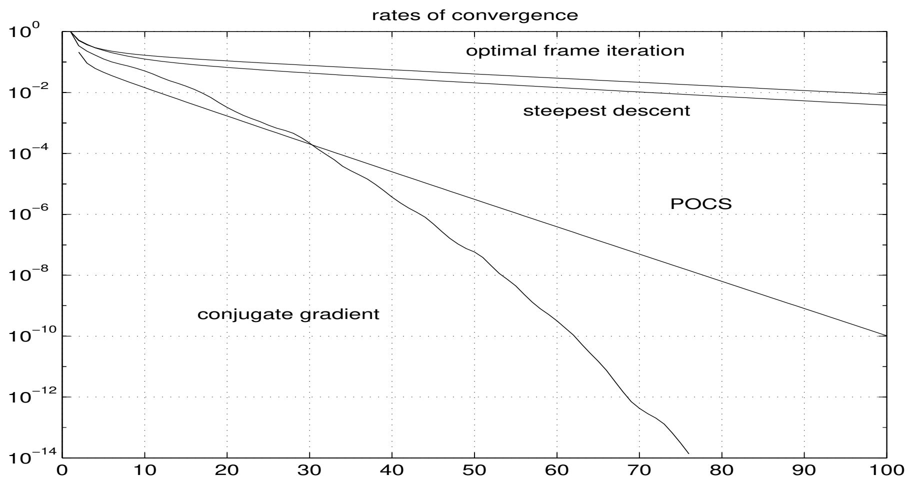

Figure 4: A comparison including POCS

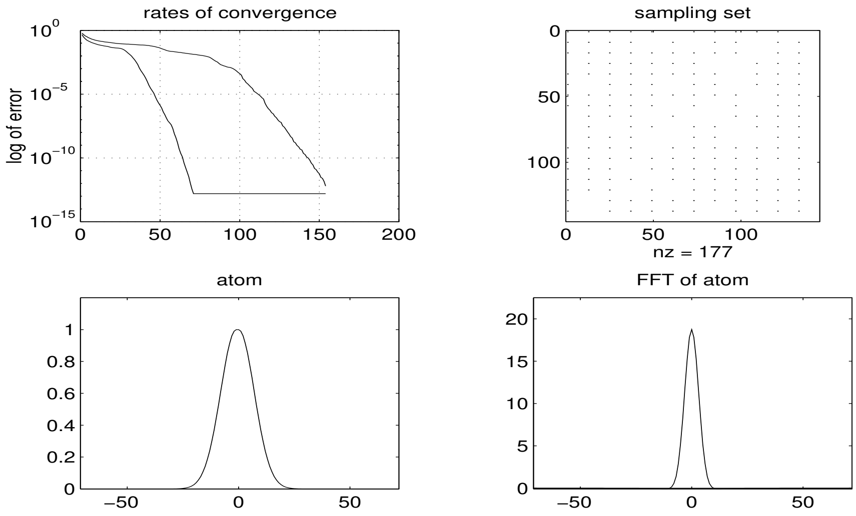

Figure 5: Comparison of rates of convergence

direction. We have tested the behaviour of the algorithm, using the atom shown (a rather broad Gaussian bump function, having a well concentrated FFT), and then the same setup, just turning around the sampling set by 90 degrees. It is not difficult to show that we would obtain the same decay by replacing the signal and the atom by its FFT while keeping the sampling set the same. In any case, the better the atom is adapted to the average type holes in the sampling set the better the performance of reconstruction algorithms will be.

7 CONCLUSION

The reader familiar with the theory of frames, positive operators and iterative methods for the inversion of such operators will not be surprised to hear that CG-methods are doing best on the problem of recovering the signal from the irregular samples of its STFT, given the window. It is always doing much better than ordinary frame iteration and avoids the problem of finding the appropriate relaxation parameters, since in practice the frame bounds are not easily available in the numerical sense. Steepest descent methods can be applied and allow to get rid of this problem, but do not behave much better than ordinary frame iterations (cf. Fig.4). For quite irregular sampling sets the method of putting weights on the elements of the Gabor frames, similar to the 2D-method of adaptive weights developed by the authors for the irregular sampling problem for band-limited images gives some improvement, in other words can be used as a preconditioner, improving the frame bounds. There is no space in this note to discuss this question in detail.

For all those methods (designed for relatively large signal length) one has to apply the frame operator iteratively (using different strategies) to various intermediate signals. In terms of implementations this means that any methods used can be improved in terms of computational load per iteration by making use of smart implementations of the Gabor frame operator. In practice, drastic improvements can be achieved for situations where the sampling grid has some regularities, or where a relatively small number of rows or colums in the indicator matrix (describing the sampling set) are carrying points. In such cases it is typically suggested to split the frame operator in two parts: Calculating the sampled STFT over the points in the TF-plane which are use, and performing Gabor synthesis afterwards. This can be done most efficiently by means of FFT-based methods and will be discussed in detail elsewhere.

Among the different methods tested on this problem the POCS-based methods cannot be compared directly. Whereas the other methods require a comparable computational load per iteration, POCS has to be counted in terms of cycles. It was interesting to see in our numerical experiments that in certain situations POCS did quite well (in terms of time required to obtain a fair approximate reconstruction), usually due to the fact that it did not require much of an initial setup. Certainly one can claim that the choice of the so-called "natural" order of a frame is about the least favourable order in which the elements of a Gabor frame should be used. Alternative (and improved variants) of the POCS-method which use more than one sampling point per iteration are to be discussed in a subsequent paper. It is also true that there is a very good chance of parallel implementation of all the methods discussed in this note.

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