A Group-theoretical Approach to Gabor Analysis

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ABSTRACT

We describe new methods to obtain non-orthogonal Gabor expansions of discrete and finite signals and reconstruction of signals from regularly sampled STFT-values by series expansions. By this we understand the expansion of a signal of a given length n, into a (finite) series of *coherent* building blocks, obtained from a *Gabor atom* through discrete time- and frequency shift operators. Although bump-type atoms are natural candidates the approach is not restricted to such building blocks. Also the set of time/frequency shift operators does not have to be a (product) lattice, but just an ordinary (additive) subgroup of the time/frequency-plane, which is naturally identified with the two-dimensional $n \times n$ cyclic group. In contrast, other, non-separable subgroups, turn out to be more interesting for our task: the efficient determination of a suitable set of coefficients for the coherent expansion. It is sufficient to determine the so-called *dual Gabor atom*. The existence and basic properties of this dual atom are well known in the case of lattice groups. As use here that this is true for general groups. But more importantly, we demonstrate that the conjugate gradient method reduces the computational complexity drastically. Given the dual atom the required Gabor coefficients are obtained as short time Fourier coefficients of the given signal with the dual atom being the moving window.

1 Gabor Expansions, Short Time Fourier Transforms

Although, some of the main assertions of this paper are just special cases of more general facts about atomic decompositions based on group theoretical methods as described in a series of joint papers with K. Gröchenig[1–4] or are contained in statements in the second author's thesis[22] about projective group representations, we shall give a down to earth description for these results here, and some direct calculations instead of indirect hints how to carry out those specializations. This paper is therefore a contribution in line with other recent publications about the finite – or equivalently, the discrete and periodic – Gabor transform, as described for example very recently in [5–7].

In this section we shall set up the notations and describe the basic problem: the expansion of a given signal into a so-called *Gabor series.* In order to give a detailed description of the problem we have to fix some notations.

A signal $x = (x_k)_{k=0}^{N-1}$ is considered as N-dimensional row vector in the complex space $\mathbb{C}^N$. The inner product of two signals is given as $\langle x, y\rangle = \sum_{j=0}^{N-1} x_j \overline{y}_k$. Here and throughout the paper we extend finite sequences in $\mathbb{C}^N$ to infinite sequences of period N by setting $x(n + kN) = x(N)$ for $n = 0, 1, \dots, N-1, k \in \mathbb{Z}$.

Although the usual Fast Fourier Transform (shortly FFT) will play an important role in our approach we shall not restrict our attention to the case $n = 2^j, j \in \mathbb{N}$. We will frequently use other natural numbers n as signal length, preferably with many prime factors. As we shall see, typical numbers of interest for us are $n = 144, 216, 240, 288, \ldots$. For many purposes it is even more natural to consider vectors of length n as functions on the cyclic group Z_n of order n which may be understood as the group of unit roots of order n in the complex domain. Consequently the time shift operator T_k – which denotes cyclic rotation – is defined by

$$T_k(x_0, x_1, \ldots, x_{n-1}) := (x_k, x_{k+1}, \ldots, x_{n-1}, x_0, \ldots, x_{k-1})\,. \tag{1}$$

The modulation operator – i.e. the frequency shift operator – is given as

$$M_l(x_0, \cdots, x_{n-1}) := (x_0, e^{-2\pi j/n}x_1, e^{-4\pi j/n}x_2, \cdots, e^{-2\pi j(n-1)/n}x_{n-1})\,. \tag{2}$$

The *Short Time Fourier Transform* (in short STFT) of a signal x with respect to some window w is therefore defined as $n \times n$-matrix, whose k-th column contains the Fourier transforms of the pointwise product $x \cdot T_k b$. In terms of inner products we may write this STFT (with $\overline{w}$ as complex conjugate of w):

$$(STFT_w x)(k, n) = \langle x, M_n T_k \overline{w} \rangle \qquad \text{for } 0 \leq k, n \leq n-1\,. \tag{3}$$

Most often, the STFT is only needed at certain subsets of the time/frequency-plane (shortly TF-plane), such as a 2-D lattice, with lattice gaps a in the (horizontal) time- and b in the (vertical) frequency-direction. It may be mentioned here that a subsampling of the Fourier transform corresponds – up to normalization – to take the FFT of a periodized version of the localized function. This is the reason why it is enough to determine "small" FFTs only of length n/a if the support of the window w is not longer than this number, i.e. if $w_j = 0$ for $j > n/a$.

Finally, we call a family of time-frequency shifted copies of a single function g a *coherent family*, if it is of the form $g_i := (M_{n_i} T_{k_i} g)_{i \in I}$, where I is any finite index family with l elements, say. Without loss of generality we assume that $\|g\|_2 = 1$.

The task of determining a *Gabor expansion* for signals in $\mathbb{C}^n$ usually is understood as finding some linear mapping from the signal space $\mathbb{C}^n$ into the coefficient space $\mathbb{C}^l$ (where $l \geq n$ according to our assumptions), i.e. $x \mapsto (\lambda_i)_{i \in I}$ satisfies

$$x = \sum_{i \in I} \lambda_i g_i \qquad \text{for all } x \in \mathbb{C}^n. \tag{4}$$

The mapping $(\lambda_i)_{i \in I} \mapsto \sum_{i \in I} \lambda_i g_i$ from $\mathbb{C}^l$ to $\mathbb{C}^n$ is referred to as *Gabor synthesis.*

Note that in general such a sequence of coefficients is not uniquely determined , since the typical situation is a mostly redundant family $(g_i)_{i \in I}$ with more than n elements, and hence linear dependency of the (non-orthogonal) family g_i follows. The task of finding suitable coefficients is therefore non-trivial, especially, if n is too large to apply ordinary matrix inversion.

According to the general theory of *frames*[8,9] we may call such a family $(g_i)_{i \in I}$ a *frame* for $\mathbb{C}^n$ if the linear span of this family is all of $\mathbb{C}^n$, i.e. if any signal can be written as a linear combination $x = \sum_{\in I} \lambda_i g_i$ of such building blocks, with complex coefficients λ_i. A coherent family which is a frame in this sense will be called a *coherent frame* or *Gabor frame* for $\mathbb{C}^n$.

2 Standard Methods for Gabor Expansions

2.1 Frame Method

It is meanwhile common knowledge that for a given frame $(g_i)_{i \in I}$ in $\mathbb{C}^n$ it is possible to obtain the minimal ℓ^2-norm solution λ_i by means of the *inverse frame operator*. Recall, that the *frame operator* S is given by $Sx = \sum_{i \in I} < x, g_i > g_i$, for $x \in C^n$. This linear operator is surjective and injective. Therefore the minimal eigenvalue A is strictly positive, i.e. $A > 0$, and the maximal eigenvalue B is obviously finite. Usually these two values are also called the *frame bounds* of the frame operator S. The fact that $0 < A \leq B < \infty$ ensures the existence of the inverse frame operator S^{-1}, which can be written in the form of a Neumann series whenever $\gamma > 0$ is small enough (cf. [10]):

$$S^{-1} = \gamma \sum_{k=1}^{\infty} (I - \gamma S)^k \,. \tag{5}$$

$S^{-1}x$ can be used to obtain as the limit of a sequence which can be described iteratively as follows:

$$x_0 = \gamma x \qquad \text{and} \qquad x_{n+1} = (I - \gamma S) x_n + \gamma x. \tag{6}$$

This algorithm is known as the *frame algorithm.* For the case of Gabor frames some iterative methods to solve for the corresponding coefficients have been described in References [11,12].

The existence of the inverse frame operator implies a pair of very useful reconstruction or expansion formulas:

$$x = S(S^{-1}x) = \sum_{i \in I} \langle x, S^{-1} g_i \rangle g_i = S^{-1}(Sx) = \sum_{i \in I} \langle x, g_i \rangle S^{-1} g_i. \tag{7}$$

We call $(h_i) := (S^{-1} g_i)$ the *dual frame.* As can be seen from these formulas this dual frame also spans the linear space $\mathbb{C}^n$, but on the other hand it can be used to determine the minimal norm coefficients.

The speed of the frame algorithm to determine the dual frame depends very much on the use of some suitable relaxation parameter γ. Although theory tells us that the optimal choice is $\gamma = 2/(A + B)$ this does not help in many practical situations, because the true values of A and B may not be known and actually are hard to calculate in such cases. On the other hand insufficient knowledge about them may lead to poor (geometric) rate of convergence, often enough with a rate of convergence not much below 1. Although the application of the *steepest descent method* should improve this situation we cannot expect more than a geometric rate of convergence from this iterative approach.

2.2 Zak transform

We only mention here the method to obtain the dual window b by going to the Zak-transform domain. It requires the calculation of the Zak transform, followed by some pointwise division, and then taking the inverse Zak transform, see [5,13–16]. However, to our knowledge the application of the Zak transform is strictly confined to lattice subgroups and cannot be used for the more general subgroups of the TF-plane which are the main concern of this note.

2.3 Direct Methods

There are of course also "direct" methods, to "solve" any of the linear systems, e.g. the one describing the coefficients for a given signal x, or directly the values of the dual Gabor atom. We mention only few recent references in this direction, such as [17,6,18]. The possible redundancy and linear dependency of the building blocks will in general require to make use of the pseudo-inverse of suitable matrices. Of course, for small signal sizes (depending on the available computer) this might be an efficient way, at least in order to gain some general theoretical insight. However, the size of such matrices will be always at least the size of a square matrix of dimension n, which is prohibitive large for realistic signals. In some cases, however, with parallel machines and powerful computers such methods might be of interest.[19–21]

3 Gabor Frames and Subgroups of the TF-plane

The basic result of this note (described in the general context of projective group representation in Christensen's thesis[22]) is the following characterization of the dual frame of a Gabor frame.

THEOREM 3.1. *For any Gabor frame $(g_i)_{i\in I} := (M_{n_i}T_{k_i}g)_{i\in I}$, where $(k_i, n_i)_{i\in I}$ forms an additive subgroup of the 2-D cyclic TF-plane $Z_n \times Z_n$, the dual Gabor frame is of the form $h_i := (M_{n_i}T_{k_i}h)_{i\in I}$. Consequently the following Gabor representation formula holds true:*

$$x := \sum_{(k,n)\in H} (STFT_{\bar{h}}x)(k,n)(M_nT_kg) \qquad \textit{for all } x \in \mathbb{C}^n. \tag{8}$$

The dual atom h is the unique solution of the equation $Sh = g$.

REMARK. Note that "the" dual atom is uniquely determined (in the way described above) and that the corresponding coefficient sequence $\lambda_i = \langle x, h_i\rangle$ is exactly the minimal norm solution to the representation problem. There are not just other different sequences of coefficients which also yield a representation of a given signal, but there are even other coefficient sequences which arise as STFT-coefficients with respect to some window different from "the" dual one. Indeed, there is a whole linear space of such "trivial" windows m, which satisfy

$$0 = \sum_{i\in I} \langle x, m_i\rangle g_i \qquad \text{for } x \in \mathbb{C}^n. \tag{9}$$

The existence of non-zero "trivial" Gabor windows can be derived from the so-called Wexler-Raz biorthogonality relation.[23] It makes sense to call any signal of the form $h + m$ a dual one, because any such atom can be used for as a replacement of h in equation (8).

For the proof of Theorem 3.1 the following general observation is basic:

THEOREM 3.2. *Under the assumptions of Theorem 3.1 the frame operator S corresponding to the Gabor frame $(g_i)_{i\in H} := (M_{n_i}T_{k_i}g)_{i\in H}$, satisfies*

$$S \circ M_{n_i}T_{k_i} = M_{n_i}T_{k_i} \circ S \qquad \textit{for all } i \in I. \tag{10}$$

Proof. For $i \in I$ we have

$$M_{n_i}T_{k_i}Sx = M_{n_i}T_{k_i} \sum_{(k,n)\in H} \langle x, M_nT_kg\rangle M_nT_kg = \sum_{(k,n)\in H} \langle M_{n_i}T_{k_i}x, M_{n_i}T_{k_i}M_nT_kg\rangle M_{n_i}T_{k_i}M_nT_kg.$$

But $M_{n_i}T_{k_i}M_nT_k$ is equal to $M_{n+n_i}T_{k+k_i}$ times a complex number of absolute value 1, so

$$M_{n_i}T_{k_i}Sx \;=\; \sum_{(k,n)\in H} \langle M_{n_i}T_{k_i}x, M_{n+n_i}T_{k+k_i}g\rangle M_{n+n_i}T_{k+k_i}g \tag{11}$$

$$= \sum_{(k,n)\in H} \langle M_{n_i}T_{k_i}x, M_nT_kg\rangle M_nT_kg \tag{12}$$

$$= SM_{n_i}T_{k_i}x\,. \tag{13}$$

□

Proof. (of Theorem 3.1) The statement in Theorem 3.1 is an easy consequence of Theorem 3.2 and the representation of S^{-1} by Neumann's series (5). In fact, showing by induction that all the powers of the frame-operator commute with all the TF-shift operators associated with the elements of H we see that any polynomial in S commutes with them, and therefore S^{-1} as a (norm) limit of such polynomials. □

The combination of standard frame theory together with the observation described in Theorem 3.2 already gives some constructive and iterative way of calculating the dual Gabor atom:

THEOREM 3.3. *In the situation described in Theorem 3.1 the dual Gabor atom can be obtained iteratively through the following iteration*

$$h_0 = \gamma g \qquad \textit{and} \qquad h_{n+1} = (I - \gamma S)h_n + \gamma g \quad \textit{for } n \geq 1 \textit{ and } \gamma \leq \tfrac{1}{B}. \tag{14}$$

Proof. The iteration given in (14) can easily be seen to describe exactly the partial sums of the Neumann series for S^{-1}, applied to g. □

Observe that choosing $\gamma \leq 1/B$ usually results in a very slow rate of convergence. The optimal relaxation parameter is given by $\gamma_{opt} = 2/(A+B)$, as mentioned in section 2.1. For practical purposes it takes too much computational effort to calculate γ_{opt}. Numerical experiments have shown that a good estimate for the optimal relaxation parameter is given by the relative sampling density, i. e. $\gamma_{est} = \frac{ab}{N}$ if g is well concentrated to some extend. It can be shown that in this case S is a diagonal dominant operator.[24] This fact suggests another plausibility explanation for the choice of γ_{est}. It is obvious that in this case the inverse of the mean value of the diagonal entries of S gives a good estimate for $2/(A+B)$. The k-th diagonal entry of S equals $\frac{N}{b}\sum_{j=0}^{N/a-1}|T_{ja}g_k|^2$ (see Reference [25]), hence an easy calculation shows that the inverse of this mean value is exactly $\frac{ab}{N}$, compare also Figure 4.

Figure 1 shows the Gabor atom, which has been chosen invariant with respect to the (unitary) FFT. The chess-type subgroup of the 108×108 frequency plane has critical density, i.e. $n = 108$ sampling points. The atom is real-valued, but the dual Gabor atom is complex-valued (the complex part is shown as dotted line), since the sampling set is not invariant under the involution $(k,n) \mapsto (k,-n)$ of the TF-plane, see also Lemma 5.1. The Fourier invariance of the atom also implies that the dual Gabor atom has the same property, as can be seen in Figure 1c) and 1d).

Figure 2 displays 4 different subgroups of the TF-plane, let us call them $H_1, \ldots, H_4$, with $n = 108$. All these subgroups have the same degree of redundancy, i.e. they have $162 = 3n/2$ sampling points. H_1, shown in Figure 2a) is a chess-type subgroup, which can also be seen as a union of two lattice subgroups, with equal gaps in both directions. It gives the most equilibrated picture in this case. The second and the third are lattices with gaps 18 and 9 respectively. Thus they can be obtained from each other by interchanging the role of time and frequency variable. H_4, illustrated in Figure 2d) arises from sampling the TF-plane with gapsize 18 and 4 respectively.

Figure 3 shows the four dual Gabor wavelets Starting with the Gabor atom shown in Figure 1a), we calculate the dual Gabor atom for each of these 4 subgroups, let us call them $h_1, \ldots, h_4$. Due to the Fourier invarince of the Gabor atom, we also have have that h_2 coincides (up to normalization) with the FFT of h_3, shown in Figure 3b) and 3c). All dual Gabor atoms in this example are real, because the corresponding subgroups $H_1, \ldots H_4$ are invariant under the involution $(k,n) \mapsto (k,-n)$.

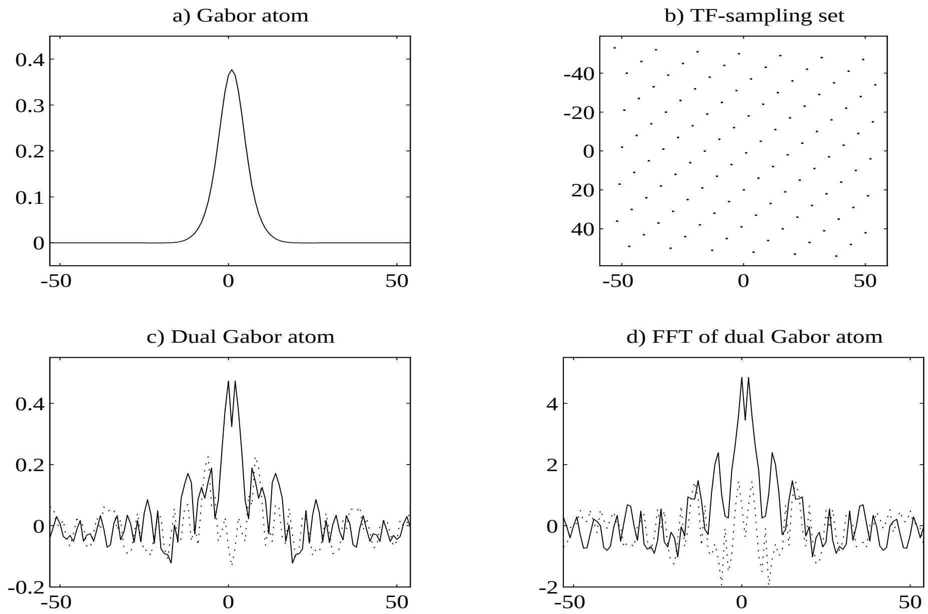

Figure 1: Gabor atom g and dual Gabor atom h

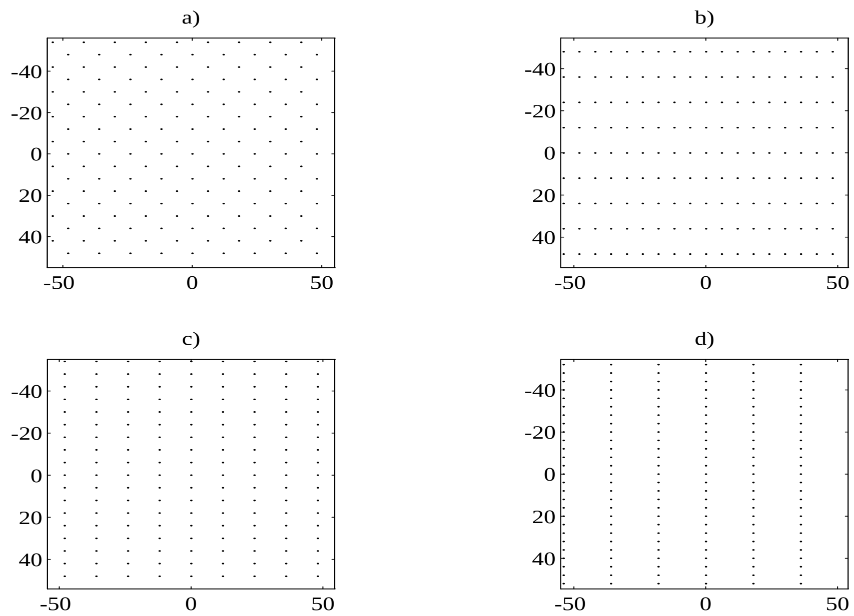

Figure 2: 4 different subgroups of the 108×108 TF-plane

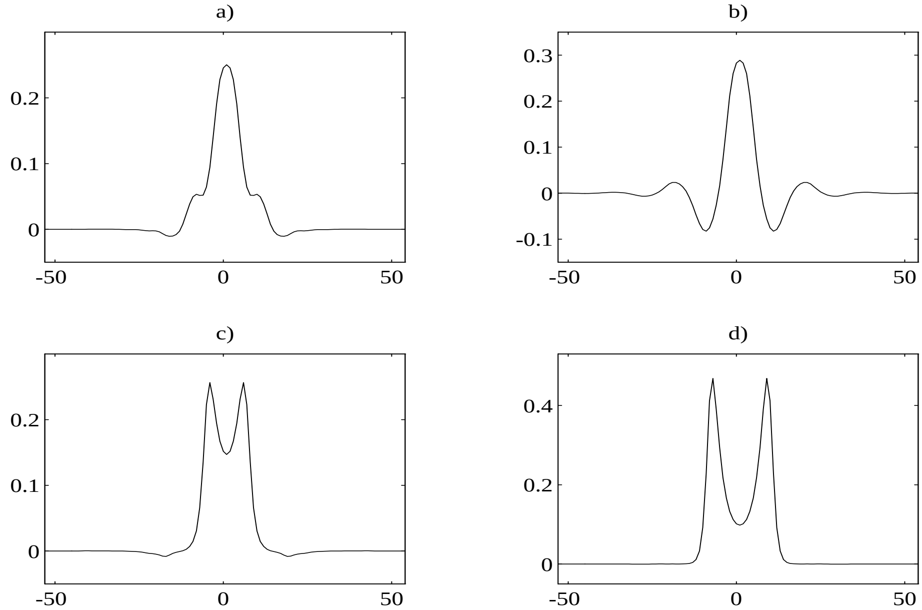

Figure 3: The dual Gabor atoms associated with the sampling patterns H_1, H_2, H_3, H_4

4 The Conjugate Gradient Method

Since the frame operator S as described in Theorem 3.1 is not just a general linear operator, but represented by a positive definite matrix, we may expect to have efficient "solvers". Indeed, as we have verified through numerical experiments the equation can be efficiently solved applying the conjugate gradient method.[26] The pseudo-code (almost identical with the MATLAB code that has been tested in our algorithms), reads as follows:

```
% GABDCG.M     HGFei   Jan. 93
% dual Gabor atom via Conjugate Gradients
% using the function GABFR.M
%
% input:   atom  = Gabor atom (building block)
%          H     = 0/1 matrix (n * n) describing H
%          iter  = maximal number of iterations
%          tresh = treshold for termination

function   gd  =   gabdcg(atom,H,iter,tresh);

n = length(atom);
gd = zeros(1,n);         %initialization
r = atom - gabfr(gd,atom,H);
% gabfr performs the action of the
% Gabor frame operator S (cf. sect.2.1)
```

```
% on the vector  gd in this case
p = r;
rr_new = r*r';

%****************** I T E R A T I O N S
for j = 1:iter;

  Sp = gabfr(p,atom,H);
  alpha = rr_new/(p*Sp');
  gd = gd + alpha * p;
  % update of the current approximation of gd
  r = r - alpha*Sp;
  rr_old = rr_new;
  rr_new = r*r';
  if rr_new < tresh; return; end;
  beta = rr_new/rr_old;
  p = r + beta*p;
end;
```

The frame operator can also be written as a matrix (operating from the right and which we like to denote by $\mathbf{S}$), which splits into a product of some rectangular matrix G with its conjugate and transpose G^*:

$$Sx = x \cdot \mathbf{S} = x \cdot (G^* \; G), \tag{15}$$

where the matrix G contains the signals g_i of the Gabor frame as row vectors. We usually take the following order: Starting with the atom in the first row, we write a block of all the modulated versions of the original atom one after the other, from the lowest frequency, in steps of b (for gap in frequency), up to the highest arising frequency. Then follows a second and third block, which arises in the same way, but starting from a version of the atom which is cyclically rotated by a or a multiple of it.

We have compared the standard frame algorithm as described in formula (14) with the the performance of the proposed CG-method. As typical test example we start with the Gabor atom of Figure 1 and the sampling pattern shown in Figure 2b).

Figure 4 shows a comparison of the rate of convergence of the standard frame algorithm with the CG acceleration. We measure the error between the the solution h and the approximation h_n after each iteration by $\|h - h_n\|_2/\|h\|_2$. The dotted line corresponds to the standard frame method using $\frac{ab}{n}$ as estimate for the optimal relaxation parameter – which is usually not known. One can see that even with optimal relaxation the frame algorithm needs about 70 iterations to reach full precision compared to 16 iterations of the CG method.

Furthermore, observe that it makes sense to compare only the number of iterations, since the most expensive step for the frame method and CG is the application of the frame operator, which is done in our code by the routine GABFR.M.

It is also clear from the last remark that whatever implementation of the Gabor frame operator (i.e. the content of GABFR.M) one uses, it can be applied for both methods. Since it is called in each iteration step the actual execution time for any algorithm will be proportional to the execution time of GABFR.M. Detailed discussions of this point will be the content of further communications, but let us mention at least two natural choices used in our numerical experiments. Both of them do not put any serious limitations on the size of the signal in terms of memory of the computer used (in other words both do not use huge matrices):

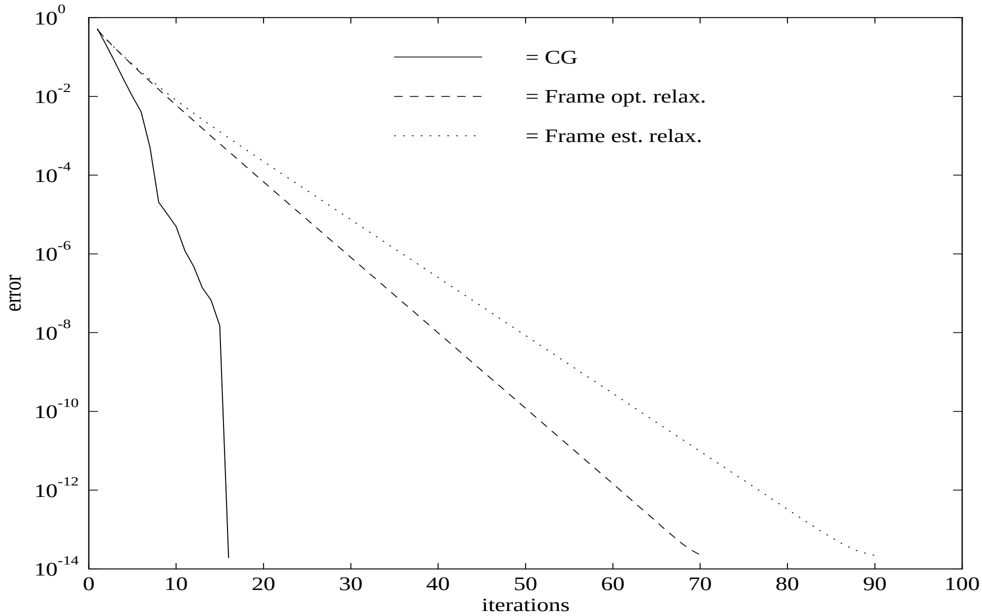

Figure 4: Comparison of rate of convergence for the frame method and CG

a) Following the definition one has first to take the coefficients with respect to the Gabor family. It is well known that this corresponds to taking the samples of the STFT (over the subgroup H) with respect to the conjugate window $\bar{g}$, in suitable order. Thinking now of those coefficients as a matrix (in the same ordering as the corresponding points are placed in the TF-plane) we have to do Gabor synthesis. It seems to us that collecting terms column-wise is the natural form. In fact, each shifted copy (arising in the Gabor family) has to be multiplied by some suitable trigonometric polynomial which can be assembled using the coefficients of one column of the STFT applying a simple FFT (of short length).

b) The alternative is to make use of the fact that by direct computation it can be shown that the Gabor frame operator can be written as a superposition of multiplication operators, in combination with shift operators, at least if the group happens to be an ordinary (product) lattice, see Lemma 2.1 in [27]. In this case one has to precompute the multipliers in terms of the atom g and the lattice constants a, b. Consequently the actual action of GABFR.M only requires to use those multipliers, see also [28], Appendix A. In this form – which is usually preferable – the computational effort is in the order of Gabor synthesis, which is the second part of the previous method. Detailed explanations will be given elsewhere .

If the Gabor atom is well concentrated in the time domain, the frame operator S is diagonal dominant.[24] Therefore instead of the CG method we can apply the preconditioned CG method[26] (PCG for short), with the inverse of the diagonal of $\mathbf{S}$ as preconditioner. Figure 5 shows a comparison of the rate of convergence of CG with PCG for the same Gabor atom and sampling pattern as in the experiment above.

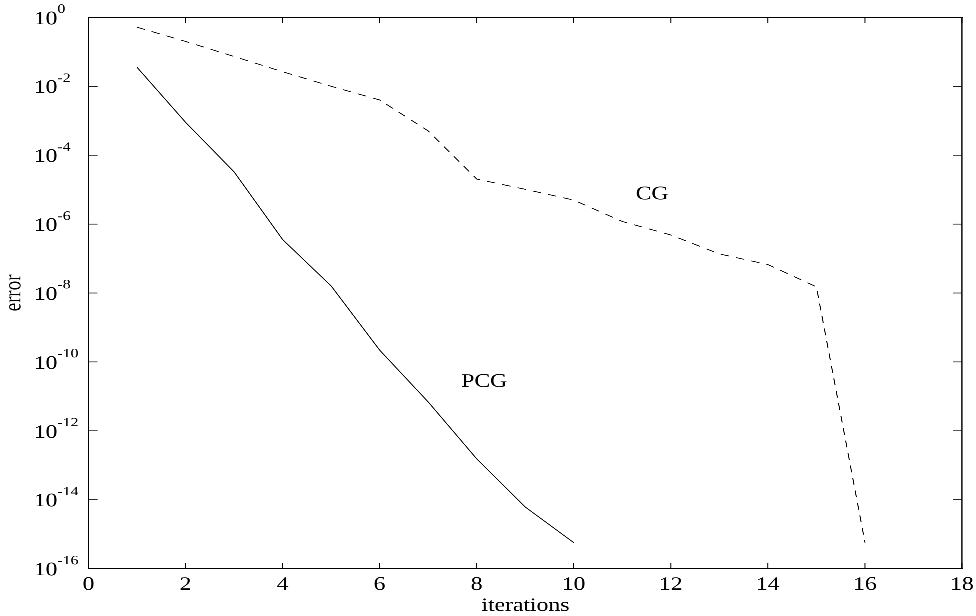

Figure 5: Comparison of rate of convergence for CG and PCG

5 Special Structure

In many cases it is of interest to require additional invariance properties of the Gabor atom and/or its dual. The properties given in this section do not depend on the method by which we calculate the dual Gabor atom, but just on the group theoretical background.

Starting with a real-valued Gabor window, it turns out that the dual Gabor window is also real-valued. That this is in fact true for most (but not all subgroups, see Figure 1) is described in the following Lemma.

LEMMA 5.1. *Suppose that g is real. If the subgroup H is invariant under the involution $(k,n) \mapsto (k,-n)$ of the TF-plane, then the frame operator takes real-valued signals into real-valued signals. In other words, the matrix corresponding to the frame operator is real-valued. In particular, the dual Gabor window to some real Gabor atom g will be real.*

Proof.

$$\overline{Sx} = \overline{\sum_{(k,n)\in H} \langle x, M_n T_k g \rangle M_n T_k g} = \sum_{(k,n)\in H} \langle \overline{x}, M_{-n} T_k g \rangle M_{-n} T_k g = \sum_{(k,n)\in H} \langle \overline{x}, M_n T_k g \rangle M_n T_k g = S\overline{x}.$$

In particular Sg is real: by induction $S^l g$ is real for any natural number l, and therefore $S^{-1}g$ is real. □

REMARK. It can be seen through simple examples, that this invariance property cannot be omited. In fact, it suffices to consider the subgroup H generated by the point $(2,1)$ in $Z_5 \times Z_5$. The group H thus evidently consists of the pairs $(0,0),(1,3),(2,1),(3,4)$ and $(4,2)$. Using any non-trivial atom, such as the vector $x = [1,1,0,0,0]$ one can show by direct calculation that the corresponding matrix describing the frame-operator is not real-valued, and that even the frame operator, applied to g is not a real-valued sequence anymore. In view of the above lemma this is not surprising, since the group is not invariant under a change of sign in the second component.

REMARK. In a similar way symmetry of the atom (with respect to the origin) is preserved by passage to the dual Gabor atom in case the subgroup H is invariant with respect to the involution $(k, n) \mapsto (-k, n)$ of the TF-plane.

6 Reconstruction from Sampled STFT

So far we have been considering the aspect of expanding a given signal in terms of a given family of building blocks, the coherent family $(g_i)_{i \in I}$. The second part of formula (7) however contains another aspect of this circle of question: It shows that there is an easy way of reconstructing an unknown signal x from the inner products of x with the elements of the coherent Gabor frame, i.e. from the Gabor coefficients $\langle x, g_i \rangle$. However, in view of the definition (3) of the STFT, these are (up to conjugation of the atom) the values of the STFT of x with respect to the given window at the points of a subgroup. We thus have the following Shannon-type reconstruction formula for *general* subgroups of the TF-plane.

THEOREM 6.1. *Let H be a subgroup of the TF-plane such that the coherent family $(M_n T_k)_{(k,n) \in H} g$ spans $\mathbb{C}^k$. Then there exists some synthesis atom h such that*

$$x := \sum_{(k,n) \in H} (STFT_g x)(k, n)(M_n T_k h) \qquad \text{for all } x \in \mathbb{C}^n. \tag{16}$$

Formula (16) is a generalization of the reconstruction formula for lattices, as described for the "critical case" by Bastiaans[29,30] with the advantage that we do not have to assume that the Gabor family is linear independent. Figure 1 shows that the critical case (number of sampling points in the TF-plane equals the signal length, or redundancy = 1) also produces sharp edges in the graph of the dual atom, which is quite similar to the singularities of Bastiaan's γ-function.

Let us point out once more that the sampling set does not have to be a lattice group (a product of lattices in each variable), but it can be a general, possibly non-separable subgroup of the TF-plane.

REMARK. The fact that a slight perturbation of a subgroup H results in a corresponding frame operator which comes close to the frame operator of the related subgroup can be used to obtain reconstruction methods for irregular sampling sets which are close to some subgroup. The idea is similar to many methods in irregular sampling of band-limited signals. Formula (16) gives an approximate reconstruction operator, and by iteration on the remainder term it is possible to get the exact reconstruction applying it over and over again.

7 Questions of Redundancy, Examples

In general we are interested in modestly redundant Gabor expansion, for two reasons. The main reason is the bad stability of the Gabor frame expansion in the critical case, i.e. if the frame consists of exactly n elements. In fact, all the examples (even the use of nice subgroups, such as a chess-type subgroup, cf. Figure 2) indicate, that we cannot expect to have "nice" dual Gabor atoms.

On the other hand we do like too much redundancy, i.e. the number of elements in the Gabor frame should be not much larger than that of the signal space. In other words, if we define the *redundancy* of a Gabor frame as the quotient of the number of frame elements over the dimension n of the space to be spanned, we would like to have redundancy factors between 1 and 1.5 ideally. Note that we have redundancy $n/(ab)$ for lattice subgroups with lattice constants a, b.

At this point the choice of the signal length plays an important role. In fact, most authors dealing with finite Gabor expansion suggest signals of length 2^r , for some $r \geq 2$, because in that case the FFT provides an efficient way to determine the STFT, i.e. the Gabor coefficients.

From the point of view taken in this paper, however, it is not possible to have “good” subgroups with “decent” redundancy. In fact, in the case mentioned any subgroup of the TF-plane has to have the order (= number of elements) which is another power of two (these are the only divisors of the group order of the TF-plane, which is 2^{2r}. On the other hand, if we take groups of order n for some n having many (small) divisors we have a rich family of possible subgroups. Thus it is possible to find nice subgroups with low *redundancy* (number of elements of $H \subseteq Z_n \times Z_n$ divided by n). Thus we found e.g. the numbers $n = 48, 56, 72, 90, 108, 144, 240, 288$ very useful for our numerical experiments. Among them there are many of order $k(k+1)$, where it is actually possible to find subgroups of the TF-plane with redundancy $(k+1)/k$. The easiest of this type is of course a uniform lattice with lattice constant k in each direction.

8 Summary

It has been explained that for a Gabor frame, associated with *any additive subgroup* of the finite TF-plane, there is a dual Gabor frame, generated by a *single* element, called the *dual Gabor atom*. In general, non-separable subgroups (e.g. of chess-board type) are more useful than lattice subgroups with the same number of elements in the sense that the dual Gabor atom has better time-frequency localization. On the other hand, there are still efficient methods to determine the dual Gabor wavelet efficiently by means of the conjugate gradient method in for any such subgroup. The rich structure of the Gabor frame operator – small number of eigen-values in many cases – implies that often the final result is obtained within a rather small number of steps, in contrast to the standard iterative inversion of the frame operator which only decays at an exact geometric rate. The dual Gabor atom tends to be better concentrated compared to the lattice case, while the computational effort to obtain the coefficients (through calculation of some STFT) and subsequent Gabor synthesis is the same as for lattice subgroups. Further extensions to sampling patterns which are finite unions of subgroups or even irregular samples of the STFT are to be discussed in more details in a subsequent paper.[31] Any pair of dual Gabor windows also allows to derive TF-localization operators based on such Gabor expansions. The algorithm described here also does not put serious restrictions on the signal length that can be used, since it does *not* require the (pseudo)inversion or singular value decompositions of a corresponding large matrix. Finally we remind that by their very nature – being based on the repeated applications of an STFT algorithm – the algorithm is readily transferable to parallel machines without any extra programming effort.

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