ON DISCRETE BAND-LIMITED SIGNAL EXTRAPOLATION

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Abstract

The aim of this paper is to present extrapolation methods which can be applied to real data problems. We introduce a method which dramatically improves upon the Papoulis-Gerchberg method. Further we propose the so-called **ACT**-algorithm [5] – originally designed for the nonuniform sampling problem – as an efficient extrapolation method. The proposed techniques have many advantages over known methods – which were especially designed for the extrapolation problem – in terms of computational savings and in accuracy of the results. We support this claim by some numerical examples.

1 Introduction

We begin by recalling what is meant by continuous signal extrapolation and discrete signal extrapolation. Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a band-limited signal of finite energy

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty \quad \text{and} \quad supp\hat{f} \subseteq [-\Omega, \Omega]$$

for some $\Omega > 0$, where $\hat{f}$ denotes the continuous Fourier transform of f. Now if we are given a piece of f only (i.e. for a certain interval I we know the function $g : g(t) = f(t), t \in I$), then the continuous extrapolation problem is, to find $f : \mathbb{R} \mapsto \mathbb{C}$ such that

a) f is band-limited to Ω,

b) $f(t) = g(t)$ for $t \in I$.

It is well known that f is theoretically uniquely determined because f is an analytic function. Several approaches for the theoretical solution of this extrapolation problem can be found in [18, 15, 2, 21, 22].

For the problem to be accessible for numerical solution, we first have to set up a finite-dimensional model. For the discrete theory used below we follow [11] and [5].

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A discrete signal x of length N is a finite sequence $x(0), x(1), \ldots, x(N-1) \in \mathbb{C}^N$. Its ℓ^2-norm is

$$\|x\| = \left(\sum_{n=0}^{N-1} |x(n)|^2 \right)^{1/2} .$$

The discrete Fourier transform of x is given by

$$\hat{x}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-2\pi i k n / N} \qquad k = 0, \ldots, N-1 \quad .$$

The space of band-limited discrete signals with bandwidth $M < N/2$ is defined by

$$\mathcal{B}_M = \left\{ x \in \mathbb{C}^N | \hat{x}(k) = 0 \ \text{ for } |k \bmod (N)| > M \right\} .$$

We extend finite sequences in $\mathbb{C}^N$ to infinite sequences of period N by $x(n + lN) = x(n)$ and $\hat{x}(k + lN) = \hat{x}(k)$ for $n, k = 0, \ldots, N-1, l \in \mathbb{Z}$. In other words we identify the index set $\{0, 1, \ldots, N-1\}$ with the cyclic group $\mathbb{Z}_N$. The **Nyquist interval** ν is defined by $\nu := \frac{N}{2M+1}$.

The discrete extrapolation problem for band-limited signals can now be stated as follows. Let x be in $\mathcal{B}$ and assume we know x only on the interval $I_r = [n_r, n_r + 1, \ldots, n_r + r - 1]$. The extrapolation problem considers two questions: The first is, when can x be reconstructed? The theoretical answer is trivial. We have to know at least $2M+1$ sampling values of x. The more important question is how can x be reconstructed stably and fast? Observe that from this point of view extrapolation and interval interpolation coincide.

There exists a vast literature on extrapolation algorithms [18, 20, 22, 21, 2, 30, 31]. However many of them have in common that they are not stable or efficient enough to be applied to real world extrapolation problems.

Before we discuss the new extrapolation methods and compare them to standard methods, we will interpret the extrapolation problem for discrete signals by the use of trigonometric polynomials. Since $x \in \mathcal{B}_M$ we have

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} \hat{x}(k) e^{2\pi i k n / N} \ . \tag{1}$$

A discrete signal of length N and bandwidth M can be interpreted as the restriction of a trigonometric polynomial of period 1 and degree M to an arithmetic sequence in $[0, 1)$. More precisely, if $x \in \mathcal{B}_M$ is given by (1) and

$$p(t) = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} \hat{x}(k) e^{2\pi i k t} \tag{2}$$

then $x(n) = p(\frac{n}{N}), n = 1, \ldots, N$. The original question turns now into the problem of how to reconstruct p from its samples $p(\frac{n_r + j}{N}), j = 0, \ldots, r-1$. For the remainder of the paper we consider therefore the $2M+1$-dimensional space

$$\mathcal{P}_M = \left\{ p \mid p(t) = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} a_k e^{2\pi i k t} \right\} \tag{3}$$

of all trigonometric polynomials of degree M and period 1. The problem discussed in this paper can now be formulated as follows: Given a sampling interval $[0 \leq t_r, t_r + \frac{1}{N}, \ldots, t_r + \frac{r-1}{N} < 1]$, we ask how to reconstruct $p \in \mathcal{P}_M$ from its sampling values $p(t_i)$.

2 Acceleration of Papoulis-Gerchberg method

In [18, 7] Papoulis and Gerchberg presented independently an iterative method for the extrapolation of band-limited functions. Although this method shows only a poor rate of convergence, it is still proposed in recent literature [19, 14, 8, 13, 23].

The discrete PG algorithm can be described by making use of the concept of frames [4]. Recall that a system of elements $\{f_j\}_{j\in I}$ in a Hilbert space $\mathcal{H}$ is called a frame, if there exist two constants $0 < A \leq B$, such that

$$A\|x\|^2 \leq \sum_{j\in I} |\langle x, f_j\rangle|^2 \leq B\|x\|^2 \qquad \forall x \in \mathcal{H}\,. \tag{4}$$

The largest lower and smallest upper bound, respectively are the **optimal frame bounds** A_{opt}, B_{opt}.

Let D_M denote the Dirichlet kernel

$$D_M(t) = \sum_{k=-M}^{M} e^{2\pi ikt} = \frac{\sin(2M+1)\pi t}{\sin \pi t}$$

and note that

$$p(t) = \int_0^1 p(u) D_M(t-u)\, du = \langle p, D_M(.-t)\rangle\ . \tag{5}$$

Assume that $r \leq 2M+1$, then $\{D_M(.-t_j)\}_{j=1}^{r}$ is a frame for $\mathcal{P}_M$ and we can define the *frame operator* S by

$$Sp(t) = \sum_{j=1}^{r} p(t_j) D_M(t-t_j)\ . \tag{6}$$

Since the frame operator is invertible [4] on $\mathcal{P}_M$, we can compute the original function p from its samples by the well-known **frame algorithm** [6, 1]:
Papoulis-Gerchberg method:

$$\begin{aligned} p_0 &= \lambda S p, \\ p_{m+1} &= p_m + \lambda S(p - p_m) \qquad \text{for } m = 1, 2, \ldots \end{aligned} \tag{7}$$

with relaxation parameter $\lambda < \frac{1}{B}$.

Unfortunately the PG method delivers only a poor rate of convergence, even with optimal relaxation (i.e., $\lambda = \frac{2}{A_{opt}+B_{opt}}$). Even in cases where the missing part of the signal is not larger than the Nyquist interval the rate of convergence of PG is unfit for practical use.

From the point of view of linear algebra PG is just the **Richardson algorithm**. Linear algebra provides several acceleration techniques for the Richardson iteration [12, 27, 10, 28]. The frame operator is self adjoint [4] and since $\langle Sp, p\rangle = \sum_{j=1}^{r} |p(t_j)|^2$, it is also positive. Therefore we can apply the **Conjugate Gradient method** for accelerating the Papoulis-Gerchberg algorithm.

Theorem 1 (Conjugate Gradient Acceleration of Papoulis-Gerchberg) *Let $p \in \mathcal{P}_M$ and $[0 \le t_r, t_r + \frac{1}{N}, \ldots, t_r + \frac{r-1}{N} < 1]$ be the sampling interval with $r \le 2M+1$. Let S denote the frame operator of (6) with associated optimal frame bounds A_{opt} and B_{opt}. Initialize $r_0 = Sp$ and $q_0 = r_0$. Then p can be reconstructed iteratively from the sampling values $p(t_j)$ in absence of rounding errors in at most $2M+1$ iterations by the following algorithm:*

$$
\begin{aligned}
\alpha_k &= \frac{\langle r_{k-1}, r_{k-1}\rangle}{\langle q_{k-1}, Sq_{k-1}\rangle} \qquad \text{for } k = 1, 2 \ldots \\
p_k &= p_{k-1} + \alpha_k q_{k-1} \\
r_k &= r_{k-1} - \alpha_k S q_{k-1} \\
\beta_k &= \frac{\langle r_k, r_k\rangle}{\langle r_{k-1}, r_{k-1}\rangle} \\
q_k &= r_k + \beta_k q_{k-1}
\end{aligned}
$$

If $\kappa = \frac{B_{opt}}{A_{opt}}$, the following error estimate holds in the S-norm (defined by $\|q\|_S = \sqrt{\langle Sq, q\rangle}$)

$$\|p - p_k\|_S \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|p - p_0\|_S \,. \tag{8}$$

Remarks:

a) Actually in general the rate of convergence of CG does not only depend on the quotient of the smallest and largest value of the spectrum of the underlying matrix, but mainly on the clustering of the spectrum of the matrix [32, 26, 16]. Especially for the extrapolation problem it turns out that the spectrum of S is always clustered, which often results in a high rate of convergence for the Papoulis-Gerchberg Conjugate Gradient method. See Section 4 for an explanation of this behavior.

b) The computational effort for one iteration for the method of Theorem 1 is $\mathcal{O}(N \log N)$, since the action of S on a vector v can be carried out via Fast Fourier Transform.

c) The conjugate gradient acceleration of PG allows to reconstruct large missing segments of a signal – e.g. ten times larger than the Nyquist interval – which was by no means possible with the standard PG method.

3 Method of Jain and Ranganath

In [15] Jain and Ranganath presented an extrapolation algorithm, where they make use of the fact that the so-called sinc matrix is of Toeplitz type if the sampling set is an arithmetic progression.

To be more precise, let us write the PG algorithm of (7) in terms of matrices. Let U be the $r \times r$ matrix with entries

$$U_{kl} = D_M(t_k - t_l) = \frac{\sin(2M+1)\pi(t_k - t_l)}{\sin \pi(t_k - t_l)} \tag{9}$$

and set $b_j = p(t_j), j = 1, \ldots, r$. Then

$$Uv = b$$

and

$$p(t) = \sum_{j=1}^{r} v_j D_M(t - t_j).$$

Unfortunately U is **not invertible** if $r > 2M+1$, which is the usual case in applications. Additionally the solution of $Uv = b$ is often very **unstable**, also Jain mentioned this problem in [15].

Jain et al. proposed to solve the system $Uv = b$ in one step (via pseudo inverse) or iteratively by CG. Since for all k, l the difference $|t_k - t_l| = const$, U is a Toeplitz matrix. Consequently the multiplication of U with a vector can be carried out by FFTs [25, 5]. Hence the main computational effort for solving the Toeplitz system $Uv = b$ via CG is of $\mathcal{O}(2r \log 2r)$ for one iteration.

4 ACT method for signal extrapolation

The ACT method (for Adaptive weights Conjugate gradient Toeplitz method) was originally proposed by Feichtinger, Gröchenig and the author for the nonuniform sampling problem. We will demonstrate that ACT is especially efficient for many extrapolation problems.

4.1 The ACT algorithm

In contrast to the original approach we omit the adaptive weights here, because they were introduced to compensate the irregularities in the sampling set, which is not necessary if the sampling set is an arithmetic progression as is the case in extrapolation.

To derive the ACT method let us look at the action of the frame operator S on the coefficients of trigonometric polynomials as in [11].

Let $p(t) = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} a_k e^{2\pi i k t} \in \mathcal{P}_M$ with coefficients $\mathbf{a} = (a_k)_{k=-M}^{M} \in \mathbb{C}^{2M+1}$. Then

$$\begin{aligned}
Sp(t) &= \sum_{j=1}^{r} p(t_j) D_M(t - t_j) \\
&= \sum_{j=1}^{r} \sum_{k=-M}^{M} \frac{1}{\sqrt{N}} \sum_{l=-M}^{M} a_k e^{2\pi i k t_j} e^{2\pi i l t} e^{-2\pi i l t_j} \\
&= \sum_{l=-M}^{M} \left[\sum_{k=-M}^{M} \left(\frac{1}{\sqrt{N}} \sum_{j=1}^{r} e^{-2\pi i (l-k) t_j} \right) a_k \right] e^{2\pi i l t}.
\end{aligned}$$

Thus let T be the $(2M+1) \times (2M+1)$ Hermitian Toeplitz matrix with entries

$$T_{lk} = T_{l-k} = \frac{1}{\sqrt{N}} \sum_{j=1}^{r} e^{-2\pi i(l-k)t_j} \quad \text{for } |l|, |k| \leq M \ . \tag{10}$$

Hence we get

$$T\mathbf{a} = \mathbf{y}$$

with

$$y_k = \sum_{j=1}^{r} p(t_j) e^{-2\pi i k t_j} \ \text{ for } |k| \leq M \ .$$

In contrast to Jain's approach T is Toeplitz for **any arbitrary** nonuniform sampling set. Since $\langle Sp, p \rangle_{L^2[0,1)} = \langle T\mathbf{a}, \mathbf{a} \rangle_{\mathbb{C}^{2M+1}}$ the matrix T is **invertible**. We solve $T\mathbf{a} = \mathbf{y}$ by CG carrying out the matrix-vector multiplication by FFT's. Summing up we have the following result (see also [5] for a more comprehensive statement):

Theorem 2 *Let M be the size of the spectrum and let $0 \leq t_1 < \ldots < t_r < 1$ be an* arbitrary *sequence of sampling points with $r \geq 2M+1$. Compute*

$$\gamma_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{r} e^{-2\pi i k t_j} \qquad \text{for } k = 0, 1, \ldots, 2M \ .$$

The associated Toeplitz matrix has $(T)_{lk} = \gamma_{l-k}$ for $|l|, |k| \leq M$.

To reconstruct a trigonometric polynomial $p \in \mathcal{P}_M$ from its samples $p(t_j)$, compute first

$$y_k = \sum_{j=1}^{r} p(t_j) e^{-2\pi i k t_j} \quad \text{for } |k| \leq M \ ,$$

and set $r_0 = q_0 = \mathbf{y} \in \mathbb{C}^{2M+1}$, $\mathbf{a}_0 = 0$. Compute iteratively for $m \geq 1$

$$\mathbf{a}_m = \mathbf{a}_{m-1} + \frac{\langle r_{m-1}, q_{m-1} \rangle}{\langle T_w q_{m-1}, q_{m-1} \rangle} \ q_{m-1}$$

$$r_m = r_{m-1} - \frac{\langle r_{m-1}, q_{m-1} \rangle}{\langle T_w q_{m-1}, q_{m-1} \rangle} \ T_w q_{m-1}$$

and

$$q_m = r_m - \frac{\langle r_m, T_w q_{m-1} \rangle}{\langle T_w q_{m-1}, q_{m-1} \rangle} \ q_{m-1} \ .$$

Then $\mathbf{a}_m$ converges in at most $2M+1$ steps to a vector $a \in \mathbb{C}^{2M+1}$ solving $T\mathbf{a} = \mathbf{y}$. The reconstruction $p \in \mathcal{P}_M$ is then given by $p(t) = \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} a_k e^{2\pi i k t}$.

It was shown in [5] that the entries of the Toeplitz matrix T can be computed in $\mathcal{O}(N \log N)$ via Fourier Transform, if the sampling sequence $0 \leq t_1 < \ldots < t_r < 1$ is a subsequence of the arithmetic progression $0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}$. In the extrapolation problem the computational effort to establish the matrix can once more be reduced.

Corollary 1 *If the sampling set is of the form* $0 \leq t_r < t_r + \frac{1}{N} < \ldots < t_r + \frac{r-1}{N} < 1$ *then the entries of the matrix can be computed in* $\mathcal{O}(2M+1)$ *by using the relations* [34]

$$\begin{aligned} \gamma_0 &= \frac{r}{\sqrt{N}} \qquad \textit{and for } k = 1, \ldots, 2M \\ \gamma_k &= \frac{1}{\sqrt{N}} \sum_{j=1}^{r} e^{-2\pi i k t_j} = \frac{1}{\sqrt{N}} \sum_{j=n_r}^{n_r+r-1} e^{-2\pi i k j/N} = \frac{e^{-2\pi i k r/N} - 1}{\sqrt{N}(e^{-2\pi i k/N} - 1)} e^{-2\pi i k n_r/N} . \end{aligned}$$

Remark: Note, that the PG-CG method of Theorem 1 and the ACT method of Theorem 2 deliver the same rate of convergence, assuming ecaxt arithmetics. However the computational effort for PG-CG depends on the signal length, whereas the computational effort for the ACT method essentially depends on the bandwidth of the signal, which is usually much smaller, compare Figure 5.

4.2 Distribution of Eigenvalues

In section 2 we mentioned that the spectrum of S is always clustered for the extrapolation problem. By construction it is clear that T and S have the same spectrum. Let G denote the $r \times (2M+1)$-matrix with entries $g_{jk} = e^{2\pi i k t_j}$. Then $T = G^*G$ and $U = GG^*$, thus T and U have the same non-zero singular values.

To support the conjecture that the spectrum of T – hence also of S and U – is always clustered at 1 and 0, let us have a look at the corresponding continuous extrapolation problem [18, 15]. Instead of the matrix U we have to consider finite segments $\tilde{U}_n$ of the singly infinite symmetric Toeplitz matrix $\tilde{U}_\infty$ with entries $\tilde{u}_0 = 2\omega, \tilde{u}_k = (\sin 2\pi\omega k)/\pi k$ for $k = 1, 2, \ldots,$ ...and $0 < \omega < \frac{1}{2}$.

Applying Szegö's Theorem [9] about the distribution of the eigenvalues of Toeplitz matrices one can show that the eigenvalues of $\tilde{U}_n$ are always clustered at zero and one [33, 24]. Furthermore the eigenvalues are clustered exponentially to the limits zero and one, the transition region consists of only $\mathcal{O}(n \log n)$ eigenvalues.

We have the conjecture that similar results hold for the discrete extrapolation problem, i.e. for the matrix U and therefore for T. We do not have a formal proof, but we were able to verify this conjecture numerically. The clustering behavior of the spectrum of T has on the one hand advantages but on the other hand poses some problems. Roughly spoken as long as the small eigenvalues of T do not cluster "too close" to zero, the rate of CG is very high. But if the eigenvalues are very close to zero, CG delivers a slow rate of convergence or stagnates. Hence it is important to have estimates for the smallest eigenvalue of T. In [24] Slepian gives estimates for the eigenvalues of $\tilde{U}_n$, especially for the minimal eigenvalue.

We can adapt these formulas for the discrete setting. It is clear that the behavior of the eigenvalues of T depends on the size of the missing part of the signal. Let γ be the size of the gap in the sampling set, i.e. $\gamma = N - r$, and let ν denote the Nyquist interval (see section 1). By modifying Slepian's formula we

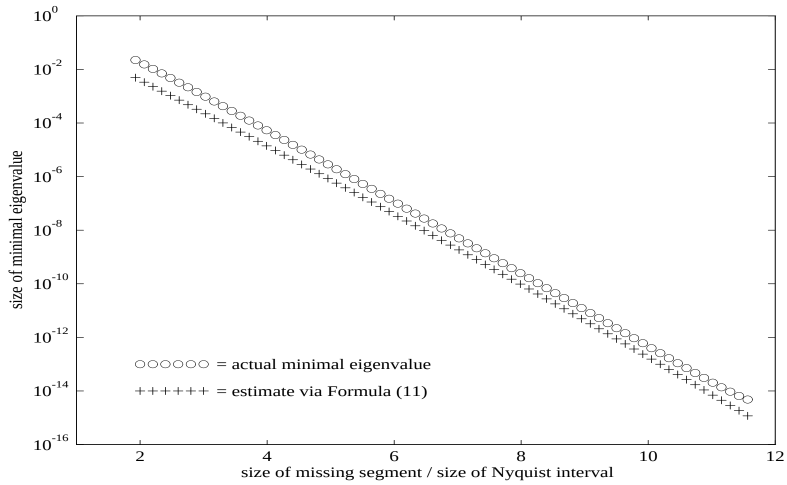

Figure 1: Comparison of minimal eigenvalue λ_{min} and estimate

obtain the following estimate for the minimal eigenvalue λ_{min} of T:

$$\lambda_{min} \cong \exp(c\gamma/\nu) \qquad \text{with } c = N/r\,. \tag{11}$$

Figure 1 demonstrates the quality of this estimate for a typical example. In this experiment we have chosen $N = 1024, M = 50$ and the choices for r rise from 940 to 1010. We establish T for each r and compute the exact minimal eigenvalue and its estimate via Formula (11). For $r < 940$ one cannot expect a reasonable comparison, because the values in the calculations are close to the roundoff level 10^{-16}.

5 Numerical Results

We want to compare the new extrapolation methods of Theorem 1 and Theorem 2 to the standard PG method and to the method of Jain-Ranganath. The test signal for our experiment – a spectroscopy signal – has been kindly placed to our disposal by S. Spielman from the Lawrence Berkeley Lab.

The original signal of length 1020 and the observed signal – 60 samples in the middle part of the signal are missing – are both shown in Figure 2. We choose $M = 35$ and apply the PG method and the PG Conjugate Gradient method (PG-CG for short) to reconstruct the missing interval. First we want to demonstrate the magnitude of improvement of PG-CG upon the standard Papoulis-Gerchberg method. Figure 4 illustrates the rate of convergence of PG and PG-CG. Since we know the original signal x, we can compute the relative ℓ^2-error between x and the approximation after each iteration. The approximation

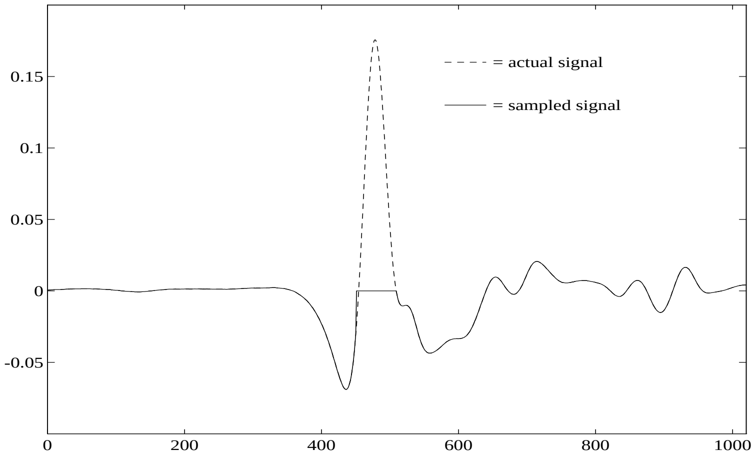

Figure 2: Original signal and sampled signal

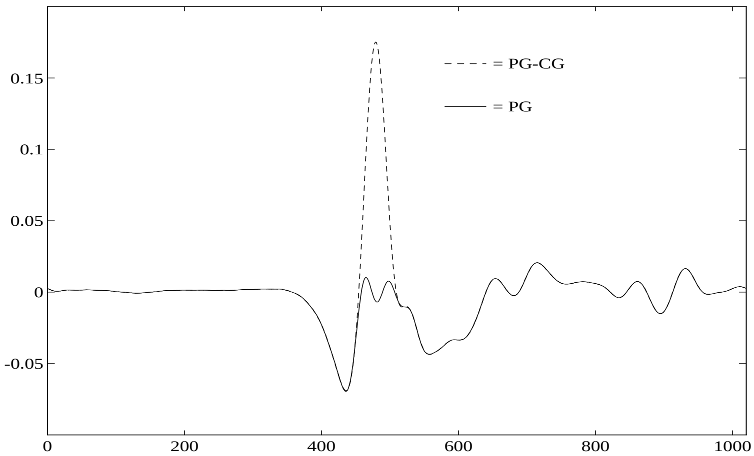

Figure 3: Approximation by PG and PG-CG

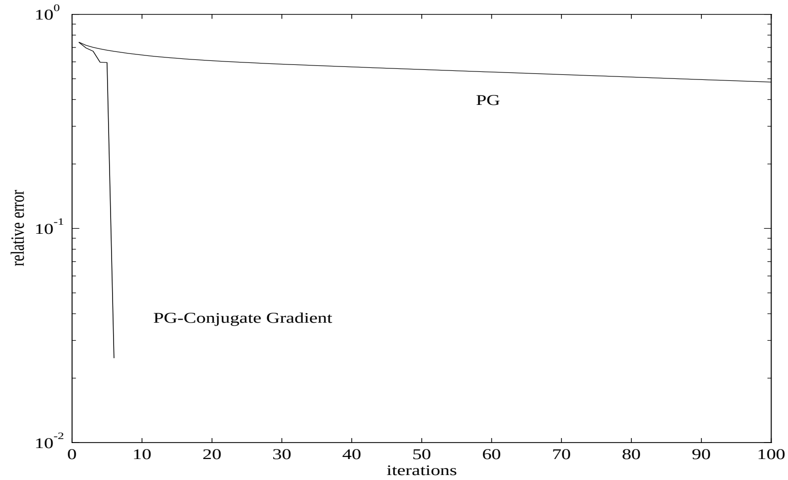

Figure 4: Comparison of rate of convergence

obtained by these methods are shown in Figure 3. PG-CG gives a very good reconstruction of the original signal after few iterations. Furthermore one can see that the non-circularity of the original signal has only negligible effects on the reconstruction of the signal.

For ACT and the Jain-Ranganath method it is easy – and does not change the solution – to make the FFT efficient by a proper enlargement of the Toeplitz matrix (see [5, 28]). Note that this enlargement is different to simple zero-padding (e.g. to a length 1024) which would affect the solution.

Theoretically PG-CG, Jain-Ranganath and ACT should deliver the same rate of convergence and the same reconstruction, however with different computational effort. But applying all these methods to the problem above, it turns out that the method of Jain-Ranganath is not stable enough to provide a reasonable result (whereas the results of PG-CG and ACT are nearly identical). Apart from convergence let us compare the number of floating point operations for these three methods for, say, 10 iterations. We use the same signal, but a smaller gap, so that also the Jain-Ranganath method converges. Figure 5 demonstrates that ACT needs significantly less floating point operations than the other extrapolation methods.

6 Extrapolation of 2-D signals

Consider a 2-D signal x of size $N \times N$ band-limited in $[-M, M] \times [-M, M]$, which is known over the observation window $[T, T_r] \times [T, T_r]$

Jain proposed a generalization of his method to two dimensions based on the properties of Kronecker products of matrices, see [15]. Consequently in Jain's

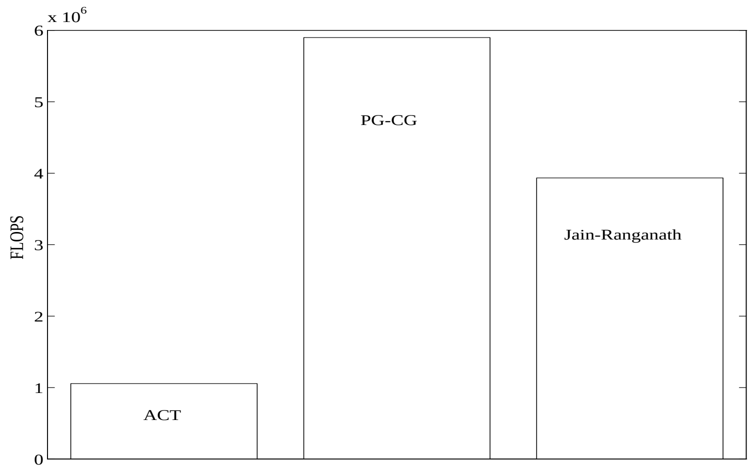

Figure 5: Comparison of *flops*

2-D extrapolation method the Toeplitz matrix is replaced by a block Toeplitz matrix. It is well known that the multiplication of a block Toeplitz matrix with a vector can be done via 2-D FFT's, see [3, 25].

Also the Papoulis Gerchberg method [17] and ACT can be generalized to 2 dimensions. Especially the 2-D ACT method gives a very powerful method, which was successfully applied to many geophysical and digital image problems[28, 29]. But for the situation above a much simpler generalization can be used.

Observe that the restriction of x to a single row or column is a band-limited 1-D signal with spectrum $[-M, M]$, since the 1-D spectrum is just the projected rectangle on the corresponding axis. Hence we can reconstruct e.g. the rows $(v_j)_{1 \leq j \leq r}$ from the corresponding sampling values. More precisely, let x_{v_j} be the v_j-th row of the image x. Then for fixed j we can reconstruct x_{v_j} from the sampling values $\{x(u_i, v_k)\}_{i=1}^{r}$. As mentioned x_{v_j} is a one-dimensional band-limited signal, thus we can apply any 1-D reconstruction method. Obviously all rows x_{v_j} have the same bandwidth $[-M, M]$ and have to be reconstructed from the same sampling geometry. Consequently the Toeplitz matrices T_j of (10) coincide for each row x_j. That means we have to establish and invert T only once. Observe that the inverse of a Toeplitz matrix is **not** Toeplitz. To make the multiplication of T^{-1} with a vector fast, we establish T^{-1} in the following way:

Note that T^{-1} is completely determined by solving

$$T\mathbf{a} = e_1 \qquad \text{where} \quad e_1 = (1, 0, \ldots, 0)^T.$$

Specifically

$$T^{-1} = \frac{1}{a_1}(L^* L - GG^*) \tag{12}$$

where

$$L = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ a_2 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_N & a_{N-1} & \dots & a_1 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & a_N & \dots & a_2 \\ 0 & 0 & \dots & a_3 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

is the Gohberg-Semencul formulation of the Trench formula. Since G and L are Toeplitz matrices, we can carry out the multiplication of T^{-1} via formula (12) by FFT's.

Summing up we have following 2-D extrapolation algorithm:

1. Establish the Toeplitz matrix corresponding to the horizontal 1-D sampling set.
2. Solve the system $T\mathbf{a} = e_1$ for $e_1 := (1, 0, \dots, 0)$ by the ACT method. Then establish T^{-1} in the Gohberg-Semencul formula.
3. Reconstruct the rows x_j, i.e. establish $\mathbf{y}_j$, multiply it by $T^{-1} = \frac{1}{a_1}(L^*L - GG^*)$ and apply the inverse FFT to obtain x_j.

We test this method for an image of size 640×480. The image was band-limited to the interval $[-160, 160] \times [-120, 120]$. Figure 6 shows the distorted image with a missing segment of size 230×16. Note that the height of this missing segment is about 10 times larger than the Nyquist interval for the vertical direction. We apply the described method along the columns of the image. The reconstruction is shown in Figure 7.

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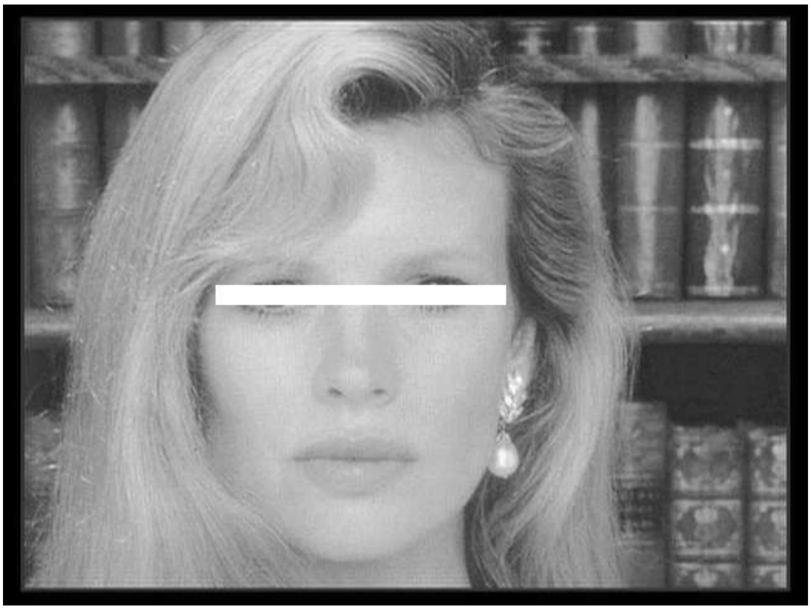

Figure 6: band-limited image with missing segment

Figure 7: reconstruction by 2-D ACT method

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