Computationally Attractive Reconstruction of Band-Limited Images from Irregular Samples

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Abstract— **An efficient method for the reconstruction of band-limited images and the approximation of arbitrary images from nonuniform sampling values is developed. The novel method is based on the observation that the reconstruction problem can be formulated as linear system of equations using 2-D trigonometric polynomials, where the matrix is of block-Toeplitz type with Toeplitz blocks. This system is solved iteratively by the conjugate gradient method. We show that the usage of so-called adaptive weights in the establishment of the block Toeplitz matrix can be seen as efficient preconditioning. The superiority of the new method over conventional approaches is demonstrated by numerical experiments.**

Keywords— **image reconstruction, nonuniform sampling, block Toeplitz matrix, conjugate gradients**

I. Introduction

Reconstruction of two-dimensional signals from a finite set of irregularly spaced samples is of importance in image processing, computer tomography, geology and oceanography. Since in many problems the number of samples is large, efficient reconstruction algorithms are required. While irregular sampling in one dimension is well-developed both in theory and its numerical exploitation – see [1], [2], [3] – there exist only few efficient reconstruction methods for the 2-D problem [4], [5]. Most of these methods are restricted to very special cases or need much computational effort and therefore their application is restricted to small scale image problems [6], [7], [8]. Our approach is based on a generalization of the 1-D method presented in [2] and on observations made by Gröchenig [9].

The paper is organized as follows. In Section II a discrete model for the 2-D nonuniform sampling problem is presented. In Sections III and IV it is shown that the reconstruction problem, formulated as least squares problem based on trigonometric polynomials, can be attacked by solving a linear system of equations where the matrix is of block Toeplitz type. The new algorithm is described in detail in Section V. Numerical experiments and comparisons to standard methods are presented in Section VI.

II. Statement of the Problem

The extension of the classical Shannon-Whittaker-Kotel'nikov sampling theorem to two dimensions is straightforward [10]. The extension of nonuniform sampling theorems to two dimensions is more delicate. Some theoretical results can be found in [11], [12], [1], [13].

For the problem to be accessible for numerical solution we have to create a *finite-dimensional model*. For clearity of presentation we consider square images, but all results hold for images of rectangular shape in a straightforward manner. A discrete 2-D signal of size $N \times N$ is an array of real or complex numbers, i.e. a rectangular matrix x of size $N \times N$ with entries $x(j,k) \in \mathbb{C}$ for $j,k = 0,1,\dots,N-1$.

We identify the index set $\{0,1,\dots,N-1\} \times \{0,1,\dots,N-1\}$ with the finite cyclic group $\mathbb{Z}_N \times \mathbb{Z}_N$. Hence the image x is understood as two-dimensional bi-periodic function with period N. That means $x(j,k+mN) = x(j,k)$ for $k = 0,1,\dots,N-1$, $m \in \mathbb{Z}$ and $x(j+nN,k) = x(j,k)$ for $j = 0,1,\dots,N-1$, $n \in \mathbb{Z}$.

The discrete two-dimensional Fourier transform on $\mathbb{Z}_N \times \mathbb{Z}_N$ is defined by

$$\hat{x}(k_1,k_2) = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} x(n,m) e^{-2\pi i (k_1 n + k_2 m)/N^2} \right) \quad (1)$$

for $k_1, k_2 = 0,\dots,N-1$. $\|x\|$ notates the ℓ^2-norm of x (the square-root of its energy) given by

$$\|x\| = \left(\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} |x(j,k)|^2 \right)^{1/2}$$

Band-limited functions on $\mathbb{Z}_N \times \mathbb{Z}_N$ are defined similar to band-limited functions on $\mathbb{R}^2$. Using the periodicity of x and $\hat{x}$ we define for $0 < M < \frac{N}{2}$ the space of discrete band-limited images of band-width M by

$$\mathcal{B}_M = \{x : \|x\| < \infty, \hat{x}(k_1,k_2) = 0 \text{ for } |k_1| \text{ or } |k_2| > M\} \quad (2)$$

A sampling set is a set of points $\{(n_j,m_j)\}_{j=1}^{r}$ with $0 \le n_1 < n_2 \dots < n_r < N, 0 \le m_1 < m_2 \dots < m_r < N$. We say that a sampling set satisfies the *Nyquist criterion*, if the whole image can be covered by rectangles of size $\frac{N}{2M+1} \times \frac{N}{2M+1}$, where the sampling points (n_j,m_j) are the midpoints of these rectangles. The *Nyquist interval* ν is given by the rectangle of size $\frac{N}{2M+1} \times \frac{N}{2M+1}$.

The problem 2-D reconstruction of discrete band-limited signals from a nonuniform set of samples can now be stated in the following way: Given a sampling set $(n_j,m_j), j = 1,\dots,r$ and the samples $x(n_j,m_j)$ of a signal $x \in \mathcal{B}_M$, is x uniquely determined by its samples? And what is an efficient method to reconstruct or at least approximate x?

Before we proceed to the deduction of the new reconstruction algorithm, we give a different interpretation of the irregular sampling problem of images.

For $x \in \mathbb{B}_M$ the inverse DFT of (1) is given by

$$x(n,m) = \frac{1}{N^2} \sum_{k_1=-M}^{M} \left(\sum_{k_2=-M}^{M} \hat{x}(k_1,k_2) e^{2\pi i (k_1 n + k_2 m)/N^2} \right) \quad (3)$$

and x can be interpreted as the restriction of a 2-D trigonometric polynomial p of period 1 and degree M to

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an arithmetic double-sequence in $[0,1)\times[0,1)$. More precisely, if $x \in B_M$ is given by (3) and

$$p(u,v) = \frac{1}{N^2}\sum_{k_1=-M}^{M}\left(\sum_{k_2=-M}^{M}\hat{x}(k_1,k_2)e^{2\pi i(k_1u+k_2v)}\right)$$

then $x(n,m) = p(\frac{n}{N},\frac{m}{N})$ for $n,m = 0,\ldots,N-1$. The original question now turns into the problem how to reconstruct p from its samples $p(\frac{n_j}{N},\frac{m_j}{N})$.

Therefore we consider the $2M+1\times 2M+1$ dimensional space $\mathcal{P}_M$ of 2-D trigonometric polynomials

$$\mathcal{P}_M = \left\{p \,|\, p(u,v) = \sum_{k_1=-M}^{M}\sum_{k_2=-M}^{M} y_{k_1k_2}e^{2\pi i(k_1u+k_2v)}\right\}.$$

In order to make the following considerations more easy to read we change to vector-notation. In this form p is described by

$$p(\vec{t}) = \sum_{\vec{k}=-M}^{M} y_{\vec{k}}e^{2\pi i\vec{k}\vec{t}/N^2} \tag{4}$$

with $\vec{k} = (k_1,k_2), \vec{t} = (u,v)$ and $\vec{t} \in [0,N)\times[0,N)$.

To avoid unnecessary introduction of new variables we slightly abuse our notation and will treat the $(2M+1)\times(2M+1)$ array y also as vector by stacking the columns of y.

In practice images are rarely band-limited in the strict sense of definition (2), they are e.g. usually not periodic and often contaminated by noise. But images are often *essentially M-band-limited*, i.e. $\hat{x}(n,m)$ is negligible for $|n| > M$ or $|m| > M$. This fact suggests to approximate such images by trigonometric polynomials of degree M. In Corollary 1 we will derive an estimate for the resulting approximation error.

To reconstruct or approximate f from its samples $f(\vec{t}_j)$ by a trigonometric polynomial of degree M we have to minimize $\sum_{j=1}^{r}|p(\vec{t}_j)-f(\vec{t}_j)|^2$, where the minimum is taken over all $p \in \mathcal{P}_M$.

In the 1-D irregular sampling problem a successful concept for signal reconstruction is based on the *adaptive weights method*, see [1] and [5] for its application in two dimensions. We borrow from this concept the strategy of using adaptive weights $\{w_j\}$ to compensate for irregularities in the sampling set $\{\vec{t}_j\}$, see section VI for details.

The problem we have to investigate can now be stated as follows. Given the samples $f(\vec{t}_j)$ we have to solve

$$\sum_{j=1}^{r}|p(\vec{t}_j) - f(\vec{t}_j)|^2 w_j = minimum\,, \tag{5}$$

for $p \in \mathcal{P}_M$. Throughout the paper we denote by $\tilde{p}$ the trigonometric polynomial $\in \mathcal{P}_M$ which solves the least squares problem (5).

III. Irregular Sampling and 2-D Trigonometric Polynomials

Many methods for signal reconstruction are based on the solution of a least squares problem. These methods often involve the (pseudo-)inversion of a large-scaled matrix, which usually results in a high computational cost and extensive memory usage for matrices without specific structure (such as circulant, Hankel or Toeplitz structure). Therefore these methods are only of theoretical interest. In the sequel we demonstrate that the least squares problem formulated in (5) can be solved numerically efficient and with low memory effort.

Before we proceed with the derivation of a fast algorithm, we first analyze the least squares problem (5) and start with two useful observations, following the work of Gröchenig [14].

Observation 1: Given $\tilde{p}(\vec{t}) = \sum_{\vec{k}=-M}^{M}\tilde{y}_{\vec{k}}e^{2\pi i\vec{k}\vec{t}/N^2} \in \mathcal{P}_M$ and weights $\{w_j\}_{j=1}^{r}, w_j > 0\,\forall j$, then

$$\sum_{j=1}^{r}|\tilde{p}(\vec{t}_j)|^2 w_j = \sum_{j=1}^{r} w_j\left(\sum_{\vec{k}=-M}^{M}\tilde{y}_{\vec{k}}e^{2\pi i\vec{k}\vec{t}_j/N^2}\right)\overline{\left(\sum_{\vec{l}=-M}^{M}\tilde{y}_{\vec{l}}\,e^{2\pi i\vec{l}\vec{t}_j/N^2}\right)}$$

$$= \sum_{\vec{k}=-M}^{M}\sum_{\vec{l}=-M}^{M}\tilde{y}_{\vec{k}}\overline{\tilde{y}_{\vec{l}}}\left(\sum_{j=1}^{r}w_je^{2\pi i(\vec{k}-\vec{l})\vec{t}_j/N^2}\right).$$

Let T denote the $(2M+1)^2\times(2M+1)^2$ matrix with entries

$$T_{\vec{k}\vec{l}} = \sum_{j=1}^{r}w_je^{2\pi i(\vec{k}-\vec{l})\vec{t}_j/N^2} \text{ for } |k_1|,|k_2|,|l_1|,|l_2| \le M \tag{6}$$

and let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{C}^{(2M+1)^2}$. Then

$$\sum_{j=1}^{r}|\tilde{p}(\vec{t}_j)|^2w_j = \langle Ty,y\rangle \tag{7}$$

and T is positive semi-definite.

Observation 2: Given $\tilde{p} \in \mathcal{P}_M$ with coefficients $\tilde{y}_{\vec{k}}$, the vector $T\tilde{y}$ can be computed using only the sampling coordinates $\vec{t}_j$ and the samples $\tilde{p}(\vec{t}_j)$, since

$$\begin{aligned}(T\tilde{y})(\vec{k}) &= \sum_{\vec{l}=-M}^{M}\sum_{j=1}^{r}\tilde{y}_{\vec{l}}w_je^{2\pi i(\vec{k}-\vec{l})\vec{t}_j/N^2} &(8)\\ &= \sum_{j=1}^{r}\tilde{p}(\vec{t}_j)w_je^{2\pi i\vec{k}\vec{t}_j/N^2}\,. &(9)\end{aligned}$$

Let us assume for the moment that T is invertible, we discuss the question of invertibility of T later. The observations above lead to following formulation for a solution of the least squares problem (5).

Theorem 1: Given the nodes $\{\vec{t}_j\}$, the samples $\{f(\vec{t}_j)\}$ and weights $\{w_j\}, j = 1,\ldots r$ and suppose that $r \ge (2M+1)^2$. The polynomial $\tilde{p}$ that solves (5) is given by

$$\tilde{p}(\vec{t}) = \sum_{\vec{k}=-M}^{M}\tilde{y}_{\vec{k}}e^{2\pi i\vec{k}\vec{t}/N^2} \in \mathcal{P}_M\,. \tag{10}$$

where its coefficients $\tilde{y}_k$ satisfy

$$\tilde{y} = T^{-1}b \in \mathbb{C}^{(2M+1)^2} \tag{11}$$

with

$$b_{\vec{k}} = \sum_{j=1}^{r}f(\vec{t}_j)w_je^{2\pi i\vec{k}\vec{t}_j/N^2} \qquad \text{for } |k_1|,|k_2| \le M\,. \tag{12}$$

Proof: We define the subspace $\mathcal{R} = \{\{p(\vec{t}_j)\}_{j=1}^{r} : p \in \mathcal{P}_M\} \subseteq \mathbb{C}^r$ with the weighted inner product $\langle a,b\rangle_{\mathcal{R}} = \sum_{j=1}^{r}a_j\bar{b}_jw_j$ for $a,b \in \mathbb{C}^r$. The solution of the least squares problem (5) is the orthogonal projection of the vector $\{f(\vec{t}_j)\}_{j=1}^{r} \in \mathbb{C}^r$ onto $\mathcal{R}$ and therefore must satisfy

$$\langle\{\tilde{p}(\vec{t}_j)\}-\{f(\vec{t}_j)\},\{\tilde{p}(\vec{t}_j)\}\rangle_{\mathcal{R}} = \sum_{j=1}^{r}(\tilde{p}(\vec{t}_j)-f(\vec{t}_j))\overline{\tilde{p}(\vec{t}_j)}w_j = 0$$

On the one hand we get by (7) and (11)

$$\sum_{j=1}^{r} |\tilde{p}(\vec{t}_j)|^2 w_j = \langle T\tilde{y}, \tilde{y}\rangle = \langle TT^{-1}b, \tilde{y}\rangle = \langle b, \tilde{y}\rangle \ .$$

On the other hand by equation (12) we have

$$\begin{aligned}\langle\{f(\vec{t}_j)\}, \{\tilde{p}(\vec{t}_j)\}\rangle_{\mathcal{R}} &= \sum_{j=1}^{r} f(\vec{t}_j)\overline{\tilde{p}(\vec{t}_j)} w_j \\ &= \sum_{\vec{k}=-M}^{M} \overline{\tilde{y}_{\vec{k}}} \sum_{j=1}^{r} f(\vec{t}_j) w_j e^{2\pi i \vec{k}\vec{t}_j/N^2} \\ &= \langle b, \tilde{y}\rangle \ .\end{aligned}$$

■

In other words, the coefficient vector $\tilde{y}$ for the polynomial $\tilde{p}(\vec{t}) = \sum_{\vec{k}=-M}^{M} \tilde{y}_{\vec{k}} e^{2\pi i \vec{k}\vec{t}/N^2}$ that minimizes (5) is the least squares solution of $Ty = b$. In the following corollary we derive an estimate for the deviation of an essentially band-limited image f from its approximation $\tilde{p}$ computed via Theorem 1.

Corollary 1: Let us assume that the function f given by $f(\vec{t}) = \sum_{\vec{k}=-N/2}^{N/2} a_{\vec{k}} e^{-2\pi i \vec{k}\vec{t}/N^2}$ is an essentially M-band-limited image, such that

$$\sum_{|\vec{k}|=M+1}^{N/2} |a_{\vec{k}}|^2 \le \delta \sum_{\vec{k}=-N/2}^{N/2} |a_{\vec{k}}|^2 \ . \tag{13}$$

If we approximate f from its samples $\{f(\vec{t})_j)\}_{j=1}^{r}$ via Theorem 1, then

$$\|f - \tilde{p}\| \le \sqrt{\delta(1 + A^{-1})}\ \|f\|$$

where $A = \|T^{-1}\|$ (i.e. the minimal eigenvalue of T).

Proof: Let f_M be the orthogonal projection of f onto $\mathcal{P}_M$ and $f_\delta = f - f_M$, then $f_\delta(t) = \sum_{|\vec{k}|=M+1}^{N/2} a_{\vec{k}} e^{-2\pi i \vec{k}\vec{t}/N^2}$ and $\|f - \tilde{p}\|^2 = \|f_M - \tilde{p}\|^2 + \|f_\delta\|^2$. From [15, Corollary 2.2] it follows that

$$\|f_M - \tilde{p}\|^2 \le A^{-1} \sum_{j=1}^{r} |f_\delta(\vec{t}_j)|^2$$

From condition (13) we get $\sum_{j=1}^{r} |f_\delta(\vec{t}_j)|^2 \le \|f_\delta\|^2 = \frac{1}{N^2}\|\hat{f}_\delta\|^2 \le \delta\frac{1}{N^2}\|\hat{f}\|^2 = \delta\|f\|^2$ and $\|f_M - \tilde{p}\|^2 \le A^{-1}\delta\|f\|^2$, thus $\|f - \tilde{p}\| \le \sqrt{\delta(1 + A^{-1})}\|f\|$. ■

IV. Irregular Sampling and Block-Toeplitz matrices

In the sequel we demonstrate that the linear system of equations $Ty = b$ can be solved in an efficient manner, due to the *block-Toeplitz structure* of T. Before we proceed with a detailed analysis of the properties of T, we briefly recall the structure of a block Toeplitz matrix.

Assume that A is an $N \times N$ block-matrix, each block B_i being a $n \times n$ Toeplitz matrix. Specifically, A is of the form

$$A = \begin{bmatrix} B_0 & B_1 & \cdots & B_{N-1} \\ B_{-1} & B_0 & \cdots & B_{N-2} \\ \vdots & \vdots & & \vdots \\ B_{-N+1} & B_{-N+2} & \cdots & B_0 \end{bmatrix} \tag{14}$$

where the entries of B_i satisfy $b_{jk} = b_{j-k}$. Then we call A a *block Toeplitz matrix* with *Toeplitz blocks* (BTTB for short).

The following lemma is substantially based on an observation due to Gröchenig [9].

Lemma 1: Let T be the matrix defined in (6). Then T is a Hermitian block-Toeplitz matrix with non-Hermitian Toeplitz blocks.

Proof: Since

$$(\vec{k} - \vec{l})\vec{t}_j = (k_1 - l_1)u_j \times (k_2 - l_2)v_j,$$

the entries of T are

$$\begin{aligned}(T_{k_1 l_1})_{k_2 l_2} &= T_{k_1(2M+1)+k_2, l_1(2M+1)+l_2} \\ &= \sum_{j=1}^{r} w_j e^{2\pi i(k_1 - l_1)u_j/N} e^{2\pi i(k_2 - l_2)v_j/N}\end{aligned} \tag{15}$$

for $|k_1|, |l_1|, |k_2|, |l_2| \le M$.

To show that T is BTTB, we consider the entries of T for fixed k_1, l_1 and denote $(B_{k_1 l_1})_{k_2 l_2} = (T_{k_1 l_1})_{k_2 l_2}$. Define the complex constant $c_j := e^{2\pi i(k_1 - l_1)u_j/N}$. Now it is easy to see that the submatrix $B_{k_1 l_1}$ is of Toeplitz type, since the entries

$$(B_{k_1 l_1})_{k_2 l_2} = \sum_{j=1}^{r} w_j c_j e^{2\pi i(k_2 - l_2)u_j/N^2}$$

satisfy $(B_{k_1 l_1})_{k_2 l_2} = (B_{k_1 l_1})_{k_2 - l_2}$. The global Hermitian block structure, i.e. $B_{k_1 l_1} = B^H_{l_1 k_1}$, follows immediately from

$$e^{2\pi i(k_1 - l_1)u_j/N} e^{2\pi i(k_2 - l_2)v_j/N} = \overline{e^{2\pi i(l_1 - k_1)u_j/N} e^{2\pi i(k_2 - l_2)v_j/N}}$$

hence

$$(B_{k_1 l_1})_{k_2 l_2} = \overline{(B_{l_1 k_1})_{l_2 k_2}}. \tag{16}$$

The submatrix $B_{k_1 l_1}$ itself is in general not Hermitian, since

$$\sum_{j=1}^{r} w_j c_j e^{-2\pi i(k_2 - l_2)u_j/N^2} = \overline{\sum_{j=1}^{r} w_j c_j e^{2\pi i(l_2 - k_2)u_j/N^2}}$$

is true if c_j is real, which is only valid for $k_1 - l_1 = 0$. Hence merely the blocks B_0 along the main diagonal of T are Hermitian. ■

When is T invertible?

The fundamental theorem of algebra states that a polynomial of degree m may always be uniquely factored into a product of m first-order polynomials. An immediate consequence of the fundamental theorem of algebra is that a one-dimensional trigonometric polynomial of degree M is uniquely defined by $2M + 1$ samples. An analogous result in dimensions greater than one does not hold, since the fundamental theorem of algebra is not true in higher dimensions. In other words it is necessary to have at least $2M + 1$ samples to uniquely define a 2-D trigonometric polynomial $p \in \mathcal{P}_M$, but it is not a sufficient condition. Assume that $p \in \mathcal{P}_M$ is uniquely defined by its samples $\{p(u_j, v_j)\}_{j=1}^{r}$, then $\sum_{j=1}^{r} |p(\vec{t}_j)|^2 w_j > 0$ which implies that T is positive definite by (7) and thus invertible.

Many uniqueness conditions for two-dimensional polynomials are based on *Eisenstein's irreducibility criterion* [16], [17], [18]. Almost all random selections of sampling points result in a unique solution [19]. And although in many applications the sampling sets are likely to be chosen on well-defined geometric objects such as lines or circles, and therefore are not randomly distributed, there are only few examples arising in practice where the special form of the sampling set results in singularity of the BTTB matrix T. However from a numerical analysis point of view, the computation of the coefficients of the polynomial can be very unstable, since T may be invertible but ill-conditioned. In [1], [20] it is shown that T is invertible (and well-conditioned) if the sampling set is the Kronecker product of two 1-D sequences, where each sequence satisfies the 1-D Nyquist criterion.

However we are interested in more applicable and conveniently-to-verify criterions which guarantee invertibility and moreover provide a small condition number of T.

Conjecture: In [21] Gröchenig has derived explicit estimates for the condition number of the Toeplitz matrix for the 1-D case, if the sampling set satisfies a Nyquist criterion. We have the conjecture that a similar condition holds for two dimensions. Whenever the maximal gap γ in a random sampling set $\{(u_j, v_j)\}_{j=1}^r$ is smaller than the Nyquist interval as defined in section II, the BTTB matrix T turns out to be invertible with a condition number $\kappa(T) < (\frac{N^2+2\gamma M^2}{N^2-2\gamma M^2})^2$. Extensive numerical experiments support this conjecture. We hope to find a formal proof for this observation.

Remark: *Restoration of blurred images* can be formulated as least squares problem where the system matrix is of block Toeplitz type, cf. [22]. But observe that the task of restoring blurred images is different from our purpose and the arising block-Toeplitz matrices have different origin, each has its own special properties. Clearly, some techniques developed for image restoration may be successfully applicable to image reconstruction (e.g. preconditioning techniques).

V. A New Image Reconstruction Method

We have seen that the solution of the least squares problem (5) can be obtained by solving $Ty = b$, where T is BTTB. An important advantage of block Toeplitz matrices is that they can be inverted with considerably less computational effort than matrices without special structure. Linear algebra provides a large repertory of Toeplitz- and block-Toeplitz solvers[23], [24], [25], [26]. Depending on the size of T and the required precision one may choose either *direct* methods for the complete inversion of T or *iterative* methods for an approximation of T^{-1}. The stability of the fast direct solvers is still a problem [27]. Although some new ideas seem to overcome these stability problems and still preserve numerical efficiency [28].

The *conjugate gradient method* (CG) is known as an efficient iterative method to solve linear systems of equations for positive definite matrices [29]. Moreover it is particularly well suited for Toeplitz-like systems [24], [25], [22]: (i) the matrix-vector multiplication in the CG iteration can be carried out very efficient (see below); (ii) the rate of convergence of CG essentially depends on the distribution of the singular values of the matrix, see Axelsson and Lindskog [30]. If the singular values are clustered, CG will converge fast, if they are not clustered, one can apply one of the many existing Toeplitz-preconditioners to achieve clustering of the singular values around a fixed point, see also VI.

If T is invertible, T is obviously positive definite due to equation (7) thus CG can be applied to solve $Ty = b$. Since the conjugate gradient algorithm can be found in many numeric books [29], we omit a presentation of its pure formulas. Instead we present the CG algorithm in connection with our least squares problem. For Algorithm 1 we assume T to be invertible. Combining the properties of CG and Theorem 1 we obtain following algorithm.

Algorithm 1: Given a sampling set $\{(u_j, v_j)\}_{j=1}^r$ with $0 \le u_1 < u_2 < \dots u_r \le N-1, 0 \le v_1 < v_2 < \dots v_r \le N-1$, the samples $\{f(u_j, v_j)\}_{j=1}^r$ and weights $\{w_j\}_{j=1}^r$. To obtain the trigonometric polynomial $\tilde{p} \in \mathcal{P}_M$ with coefficient vector $\tilde{y}$, that solves

$$\sum_{j=1}^r |p(\vec{t}_j) - f(\vec{t}_j)|^2 w_j = minimum$$

where $r \ge (2M+1)^2$, compute:

Step 1:

$$(T_{k_1 l_1})_{k_2 l_2} = \sum_{j=1}^r w_j e^{-2\pi i (k_1 - l_1) u_j / N} e^{-2\pi i (k_2 - l_2) v_j / N} \quad (17)$$

for $|k_1|, |l_1|, |k_2|, |l_2| \le M$.

Step 2: Compute the vector b by

$$b_{(2M+1)k_1+k_2} = \sum_{j=1}^r w_j f(u_j, v_j) e^{2\pi i (k_1 u_j/N + k_2 v_j/N)}$$

for $|k_1|, |k_2| \le M$ and set $r^{(0)} = q^{(0)} = b, y^{(0)} = 0$.

Step 3: For $n \ge 1$ compute iteratively

$$\begin{aligned}
\alpha^{(n)} &= \frac{\langle r^{(n-1)}, r^{(n-1)} \rangle}{\langle q^{(n-1)}, Tq^{(n-1)} \rangle} \\
y^{(n)} &= y^{(n-1)} + \alpha^{(n)} q^{(n-1)} \\
r^{(n)} &= r^{(n-1)} - \alpha^{(n)} T q^{(n-1)} \\
\beta^{(n)} &= \frac{\langle r^{(n)}, r^{(n)} \rangle}{\langle r^{(n-1)}, r^{(n-1)} \rangle} \\
q^{(n)} &= r^{(n)} + \beta^{(n)} q^{(n-1)}.
\end{aligned}$$

Assuming exact arithmetics, $y^{(n)}$ converges in at most $(2M+1)^2$ steps to $\tilde{y}$. The solution $\tilde{p}$ is then given by

$$p(u, v) = \sum_{k_1=-M}^{M} \sum_{k_2=-M}^{M} \tilde{y}_{k_1 k_2} e^{2\pi i (k_1 u/N + k_2 v/N)} . \quad (18)$$

Remarks and Further Discussion:

• The main effort of CG is one matrix-vector multiplication in each iteration. Multiplication of a vector by a Toeplitz matrix can be performed using the FFT [24]. And analogous to Toeplitz matrices multiplication of a vector by the BTTB matrix T can be computed by embedding the $(2M+1)^2 \times (2M+1)^2$ matrix T into a $(4M)^2 \times (4M)^2$ block circulant matrix C with circulant blocks. We remark that C can be diagonalized with a 2-D FFT in the same way as a circulant with a 1 -D FFT [31]. Thus the matrix vector multiplies in the algorithm above can be performed in $\mathcal{O}(4M^2 \log 2M)$ operations.

• Due to BTTB structure it can be easily checked that the whole information of T is stored in the first and M-th column of T. Hence for general sampling sets the computational effort for the calculation of the relevant entries of T is $\mathcal{O}(r(2M+1)^2)$ operations.

• There is an important special case of irregular sampling sets, where in a regular grid – like an image stored as matrix – some of the pixels are randomly lost. This case is investigated e.g. in [4], [5], [32], [20], [33]. In such a situation the computational effort to compute the entries of T can be drastically reduced because the first and M-th column of T and the vector b can be calculated via 2-D FFT. In the following corollary we want to make this statement more precise.

Corollary 2: Assume that the sequences $0 \leq u_1 < u_2 \ldots < u_r \leq N-1, 0 \leq v_1 < v_2 \ldots < v_r \leq N-1$ are subsequences of $\{0, 1, \ldots, N-1\}$ and $\{0, 1, \ldots, N-1\}$ respectively. Let $\{\vec{t}_j\}_{j=1}^r$ be a sampling set with $\vec{t}_j = (u_j, v_j)$. Then the first column of T can be computed in $\mathcal{O}(n^2 \log n)$ operations by one 2-D FFT.

Proof: From (15) we know that the first column of T is given by

$$T_{k_1(2M+1)+k_2,0} = \sum_{j=1}^{r} w_j e^{2\pi i(k_1 u_j + k_2 v_j)/N^2}$$

for $|k_1|, |k_2| \leq M$. Define $g(n_1, n_2) = w_j/N^2$ if $(n_1, n_2) \in \{(u_j, v_j)\}_{j=1}^r$ and $g(n_1, n_2) = 0$ if $(n_1, n_2) \notin \{(u_j, v_j)\}_{j=1}^r$ for $n_1, n_2 = 0, \ldots, N-1$. Compute

$$h(m_1, m_2) = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} g(n_1, n_2) e^{2\pi i(m_1 n_1 + m_2 n_2/N^2)}$$

for $m_1, m_2 = 0, 1, \ldots, N-1$ which is just the 2-D inverse DFT of g. The entries of the first column of T are then given by $T_{k_1(2M+1)+k_2,0} = h(k_1, k_2)$ for $k_1, k_2 = 0, \ldots, 2M$. ∎

Obviously a similar statement holds for the computation of b and the M-th column of T. Since the proposed algorithm is a combination of **A**daptively chosen weights, **B**lock Toeplitz matrices and the **C**onjugate gradient method, we refer to it as **ABC** method.

How to determine the weights?

If the sampling set is very irregular in the sense that it has many clusters or variable sampling density, there is a simple way to compensate for those irregularities. We use a similar strategy as in the so-called adaptive weights method developed by Feichtinger and Gröchenig [1]. Simply spoken samples in sparse sampled regions get a larger weight (because we consider them as more important) than samples in dense sampling regions. More precisely, the value of w_j is given by the size of the area of the Voronoi polygon for the point (u_j, v_j). Recall that the vertices of this Voronoi polygon are the midpoints of the corresponding Delaunay triangles of (u_j, v_j), see [34]. An efficient FFT-based method for computing these adaptive weights for 2-D has been developed in [5].

From the viewpoint of solving the linear system $Ty = b$, the role of the weights w_j can be interpreted as preconditioning the matrix T. Thus choosing the weights in equation (17) "properly", should provide better clustered singular values and thus a higher rate of convergence, compare also Figure 5. Since the sampling geometry does not change during the iterations, the weights have to be computed only once at the beginning of the algorithm.

VI. Numerical Results

In this section we will present some numerical results and experiments. We will demonstrate the efficiency of the proposed algorithm by comparing it to standard methods found in the literature. Further we discuss the effect of preconditioning the system $Ty = b$. We will apply T. Chan's preconditioner to the unweighted BTTB matrix and compare it to the effect of "preconditioning" by using adaptive weights.

Following reconstruction methods are compared to ABC in the first experiment:

1. The so-called Voronoi method [1], which is a modified version of the well-known iterative algorithm proposed by Sauer and Allebach in [4]. We use a (FFT-based) nearest neighbor interpolation, based on Voronoi regions [34].
2. A method proposed by Marvasti in [33]. This method is also known as frame method [1] and depends essentially on the proper choice of the relaxation parameter. We use the optimal relaxation parameter since otherwise Marvasti's method would give poor results. However in practice the optimal relaxation parameter is not known a priori and is hard to compute.
3. The adaptive weights method [1], [5], without additional relaxation.
4. The CG acceleration of the adaptive weights method. Observe that the so-called *frame operator* which appears in the adaptive weights method has the same non-zero eigenvalues as the BTTB matrix T. This can be seen by investigating the action of the frame operator in the frequency domain, analogous to the 1-D case. For more details we refer to [2]. Hence the adaptive weights CG method should deliver the same rate of convergence as ABC (assuming exact arithmetics), but they differ in the required computational costs.

Since the computational cost of the POCS method [35] for each iteration is several orders of magnitude higher than other methods, and it provides only poor rate of convergence, it is not included in our comparison.

Experiment 1: Comparison of ABC to above four standard methods.

For the first experiment we consider an image of size 512×750, band-limited to the interval $[-127, 127] \times [-127, 127]$. The image is randomly sampled at 200266 sampling points, thus about 48% of the samples are missing, see Figure 1. Missing samples are displayed as black pixels. The reconstruction provided by ABC after 10 iterations is shown in Figure 2.

First we compare the rate of convergence of these methods. Since we know the original image x, we can compute the relative mean square error MSE $= \|x - x^{(n)}\|_2 / \|x\|_2$ of the approximation $x^{(n)}$ after the n-th iteration. The often used SNR for measuring the quality of reconstruction is related to the relative mean square error by SNR $=$

$10\log_{10}\text{MSE}^{-2}$. The results are shown in Figure 3. As mentioned before, ABC and the adaptive weights conjugate gradient method deliver the same rate of convergence, thus only the convergence curve of ABC is illustrated. ABC gives a significantly better rate of convergence than Marvasti's method and the adaptive weights method. As pointed out in [2] both the adaptive weights method and Marvasti's method are based on the Richardson iteration and therefore they come up with all well-known disadvantages of this algorithm.

Only the Voronoi version of the Sauer-Allebach method (labeled as "SA-Voronoi" in the plot) provides a better approximation during the first iterations than ABC. The reason is that the Voronoi method spends much computational effort in computing a good first approximation before the iterations start, whereas ABC starts the iterations with a more simple approximation, which requires significantly less computational cost.

More important for practical purposes is the number of floating point operations required to reach a certain precision, say, a mean square error of 10^{-4}. It is no surprise, that the Voronoi method needs much more floating point operations (flops) than the others – see Figure 4 – because at each iteration step it has to establish a nearest neighborhood tessellation of the sampled image, which is then low-pass filtered. In our example ABC needs only half the number of flops as the adaptive weights conjugate gradient acceleration. As a rule of thumb, ABC is preferable over adaptive weights CG, if $M < N/4$ or if N is not a power of 2, which is often the case.

The rate of convergence of ABC justifies the preference of the iterative CG algorithm over direct solvers. In this example, ABC provides full reconstruction (up to a precision of 10^{-15}) after 80 iterations, which is significantly less than the theoretical upper bound $(2M+1)^2 = 65025$.

Experiment 2: Comparison of preconditioners.
If an image is sampled near the Nyquist density or if there are many large gaps in the sampling set, it may happen that even the conjugate gradient method converges rather slowly. To remedy this problem we may turn to the preconditioned system $C^{-1}Ty = C^{-1}b$ [29], [24]. If the preconditioner C is chosen appropriately, such that the singular values of the resulting matrix are strongly clustered, the preconditioned conjugate gradient method (PCG) will converge much faster than CG. Various efficient preconditioners for BTTB matrices have been developed in the last ten years, cf. [22], [36], [37] for detailed discussions of preconditioners. We use the optimal circulant Frobenius norm approximation C_F introduced by T. Chan [38].

We have mentioned that the weights can be interpreted as preconditioning the BTTB matrix. We support this claim by numerical experiments. We investigate the rate of convergence of four variants of ABC: (i) without using weights in the calculation of T; (ii) using adaptive weights in the calculation of T; (iii) applying Chan's preconditioner C_F to the "unweighted T ; and (iv) a "double-preconditioned" ABC version by using weights *and* applying C_F. We choose the same image and sampling set as in Experiment 1.

First we compare the rate of convergence, the results are displayed in Figure 5. All four variants of the ABC method show a good convergence behavior. Nevertheless the rate of convergence of ABC without using weights is somewhat worse than that of the others. This behavior is due to the fact, that CG benefits from the better clustered singular values of the block-Toeplitz matrix, achieved by Chan's preconditioner and preconditioning by adaptive weights respectively. More remarkable is that in terms of convergence rate preconditioning by the proposed weights and Chan's preconditioner is about equally attractive. It is no surprise that a combination of both preconditioning techniques gives the best results.

However from the viewpoint of floating point operations required to reach a precision of 10^{-4}, the assessment of the efficiency of the various ABC modifications is quite different. Here the ABC variants with and without weights are much more attractive, see Figure 6. Clearly ABC without using weights needs more iterations to provide a solution with required precision, but it needs less flops per iterations than e.g. ABC with Chan's preconditioner. But additional preconditioning is more important if $r \sim (2M+1)$ or if the sampling set has large gaps, in other words, if T is ill-conditioned.

Experiment 3: Approximation of non-band-limited images.
In the next example we focus on approximation of non-band-limited images. Our test-image is the well-known "Lena"-image of size 512×512, see Figure 7(a). The image is sampled at 150320 randomly spaced locations, about 43% of pixels are lost – see Figure 7(b) – the the signal-to-noise ratio is 3.7dB. The approximations after 10 iterations are shown in Figure 7(c)–(f).

Marvasti's method provides a modest approximation of most of the lost samples, but there are still many black dots visible. The approximation error is MSE = 0.1 and SNR = 20.3dB respectively. The adaptive weights method returns a satisfactory approximation, with SNR = 27.2dB and MSE = 0.04. ABC provides a reconstruction with significantly less black dots compared to the other reconstructions. In fact there is nearly no visible difference between the approximation provided by ABC and the original image. The MSE is 0.032 and SNR = 29.9dB. The Voronoi modification of the Sauer-Allebach method provides a reconstruction with approximately similar quality as ABC, (MSE = 0.34, SNR = 29.5dB). All experiments have been carried out on a DEC Alpha workstation using MATLAB.

VII. Discussion

The proposed algorithm has been successfully applied in geophysics to smooth approximation of potential fields from scattered data [3], [39]. Further a 3-D version of the ABC algorithm is currently under development. In [18] Zakhor criticizes the computational complexity of known methods for computing the coefficients of 2-D FIR filters from arbitrarily located nonuniform frequency samples. Algorithm 1 provides an efficient method to find these coefficients.

References

[1] H. Feichtinger and K. Gröchenig, "Theory and practice of irregular sampling," in *Wavelets: Mathematics and Applications* (J. Benedetto and M. Frazier, eds.), pp. 305–363, CRC Press, 1994.

[2] H. G. Feichtinger, K. Gröchenig, and T. Strohmer, "Efficient numerical methods in non-uniform sampling theory," *Numerische Mathematik*, vol. 69, pp. 423–440, 1995.

[3] T. Strohmer, *Efficient Methods for Digital Signal and Image Reconstruction from Nonuniform Samples.* PhD thesis, University of Vienna, 1994.

[4] K. D. Sauer and J. P. Allebach, "Iterative reconstruction of band-limited images from nonuniformly spaced samples," *IEEE Trans. CAS*, vol. 34, no. 12, pp. 1497–1506, 1987.

[5] H. G. Feichtinger and T. Strohmer, "Fast iterative reconstruction of band-limited images from irregular sampling values," in *Proc. on Computer Analysis of Images and Patterns* (D. Chetverikov and W. Kropatsch, eds.), (Budapest), pp. 82–91, Conf. CAIP, 1993.

[6] K. Sezan and H. Stark, "Applications of convex projection theory to image recovery in tomography and related areas," in *Image Recovery: Theory and application* (H. Stark, ed.), pp. 415–462, Acad. Press, 1987.

[7] D. C. Youla, "Generalized image restoration by the method of alternating projections," *IEEE Trans. CAS*, vol. 25, pp. 694–702, 1978.

[8] S. Kim and N. Bose, "Reconstruction of 2-D band-limited discrete signals from nonuniform samples," *IEE Proc.*, vol. 137, no. 3, pp. 197–204, 1990.

[9] K. Gröchenig, "personal communication."

[10] A. J. Jerri, "The Shannon sampling theorem - its various extensions and applications, a tutorial review," *Proc. IEEE*, vol. 65, pp. 1565–1596, 1977.

[11] H. G. Feichtinger and K. Gröchenig, "Iterative reconstruction of multivariate band-limited functions from irregular sampling values," *SIAM J. Math. Anal. 231*, pp. 244–261, 1992.

[12] K. Gröchenig, "Reconstruction algorithms in irregular sampling," *Math. Comp.*, vol. 59, pp. 181–194, 1992.

[13] P. L. Butzer and L. Hinsen, "Two-dimensional nonuniform sampling expansions - an iterative approach," *I, II. Appl. Anal. 32*, pp. 53–68 and 69–85, 1989.

[14] K. Gröchenig, "Irregular sampling, Toeplitz matrices, and the approximation of entire functions of exponential type," 1994. submitted.

[15] O. Christensen, "Moment problems and stability results for frames with applications to irregular sampling and Gabor frames," *J.Appl.Comp.Harm.Anal.*, vol. 3, no. 1, pp. 82–86, 1996.

[16] M. Fiddy, "The role of analyticity in image recovery," in *Image Recovery: Theory and application* (H. Stark, ed.), pp. 499–530, Acad. Press, 1987.

[17] M. Hayes, "The unique reconstruction of multidimensional sequences from Fourier transform magnitude or phase," in *Image Recovery: Theory and application* (H. Stark, ed.), pp. 195–230, Acad. Press, 1987.

[18] A. Zakhor and G. Alvstad, "Two-dimensional polynomial interpolation from nonuniform samples," *IEEE Trans. Sig. Proc.*, vol. 40, pp. 169–180, 1992.

[19] W. Rozwood, C. Therrien, and J. Lim, "Design of 2-D FIR filters by nonuniform frequency sampling," *IEEE Trans. Signal Proc.*, vol. 39, no. 11, pp. 2508–2514, 1991.

[20] H. Feichtinger and T. Strohmer, "Recovery of missing segments and lines in images," *Optical Engineering: special issue on "Digital Image Recovery and Synthesis"*, vol. 33, no. 10, pp. 3283–3289, 1994.

[21] K. Gröchenig, "A discrete theory of irregular sampling," *Lin. Alg. and Appl.*, vol. 193, pp. 129–150, 1993.

[22] R. Chan, J. Nagy, and R. Plemmons, "FFT-based preconditioners for Toeplitz-block least squares problems," *SIAM J. Numer. Anal.*, vol. 30, pp. 1740–1768, 1993.

[23] G. Ammar and W. Gragg, "Superfast solution of real positive definite Toeplitz systems," *SIAM J. Matrix Anal. Appl.*, vol. 9, pp. 61–76, 1988.

[24] G. Strang, "A proposal for Toeplitz matrix calculations," *Stud. Appl. Math.*, vol. 74, pp. 171–176, 1986.

[25] G. Strang and R. Chan, "Toeplitz equations by conjugate gradients with circulant preconditioner," *SIAM J. Sci. Stat. Comput.*, vol. 10, pp. 104–119, 1989.

[26] R. Bitmead and B. Anderson, "Asymptotically fast solution of Toeplitz and related systems of linear equations," *Lin. Alg. Appl.*, vol. 34, pp. 103–117, 1980.

[27] J. Bunch, "Stability of methods for solving Toeplitz systems of equations," *SIAM J. Sci. Statis. Comput.*, vol. 6, pp. 349–364, 1985.

[28] T. Huckle, "Symmetric Gaussian elimination for Cauchy-type matrices with application to positive definite Toeplitz matrices." submitted to Numerische Mathematik.

[29] L. Hageman and D. Young, *Applied Iterative Methods.* Academic Press, 1981.

[30] O. Axelsson and G. Lagendijk, "On the rate of convergence of the preconditioned conjugate gradient algorithm," *Numer. Math.*, vol. 48, pp. 499–523, 1986.

[31] P. Davis, *Circulant Matrices.* John Wiley, 1979.

[32] H. Stark, "Polar, spiral and generalized sampling and interpolation," in *Advanced Topics in Shannon Sampling and Interpolation Theory* (R. J. Marks II, ed.), pp. 185–218, Springer Verlag, 1993.

[33] F. A. Marvasti, C. Liu, and G. Adams, "Analysis and recovery of multidimensional signals from irregular samples using nonlinear and iterative techniques," *Signal Processing*, vol. 36, pp. 13–30, 1994.

[34] A. Okabe, B. Boots, and K. Sugihara, *Spatial Tessellation. Concepts and Applications of Voronoi Diagrams.* John Wiley & Sons, 1992.

[35] S. Yeh and H. Stark, "Iterative and one-step reconstruction from nonuniform samples by convex projections," *J. Opt. Soc. Amer. A*, vol. 7, no. 3, pp. 491–499, 1990.

[36] T. Ku and C. Kuo, "On the spectrum of a family of preconditioned block Toeplitz matrices," *Tech.Rep., USC, Signal and Image Proc. Inst.*, vol. 164, 1990.

[37] T. Chan and J. Olkin, "Preconditioners for Toeplitz-block systems," *Numer. Algorithms*, vol. 6, pp. 89–101, 1994.

[38] T. Chan, "An optimal circulant preconditioner for Toeplitz systems," *SIAM J. Sci. Stat. Comput.*, vol. 9, pp. 766–771, 1989.

[39] M. Rauth, "Application of 2-D methods for scattered data approximation to geophysical data sets," in *SampTA - Sampling Theory and Applications*, (Riga/Latvia), pp. 38–43, 1995.

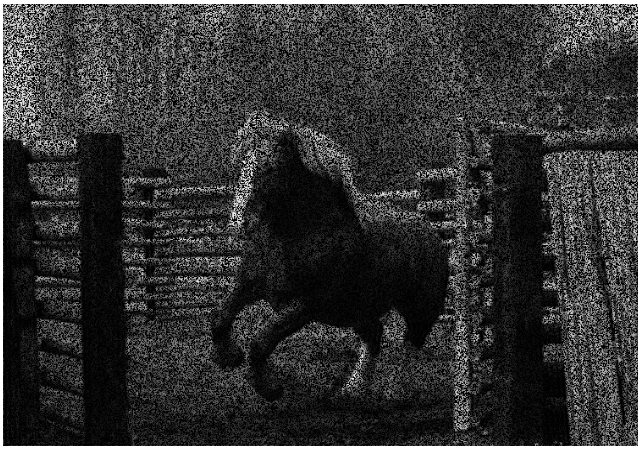

Fig. 1. Sampled image, about 48% of the samples are missing

Fig. 2. Reconstruction by ABC after 10 iterations

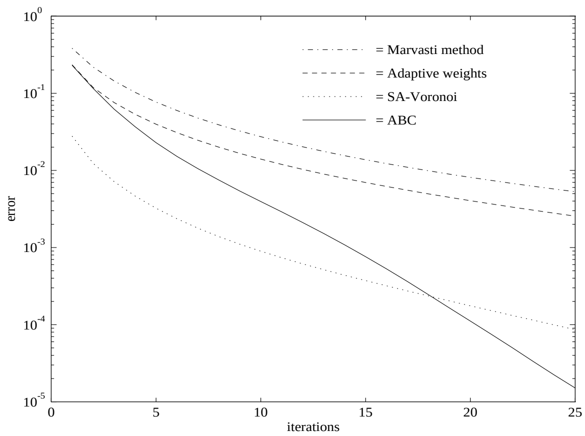

Fig. 3. Comparison of rate of convergence

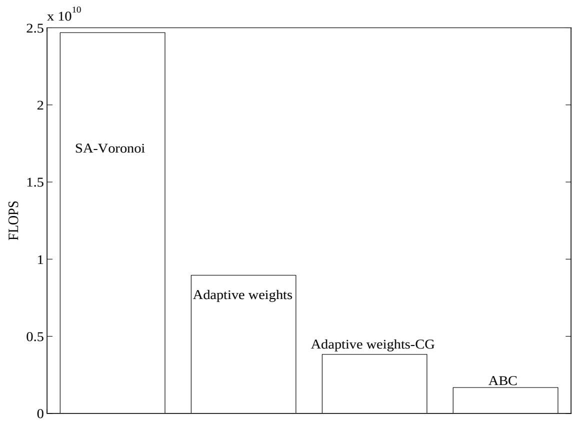

Fig. 4. Comparison of flops to reach precision 10^{-4}

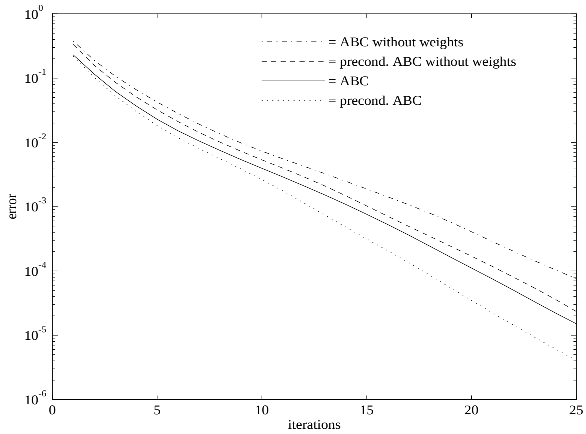

Fig. 5. Comparison of preconditioners: 25 iterations

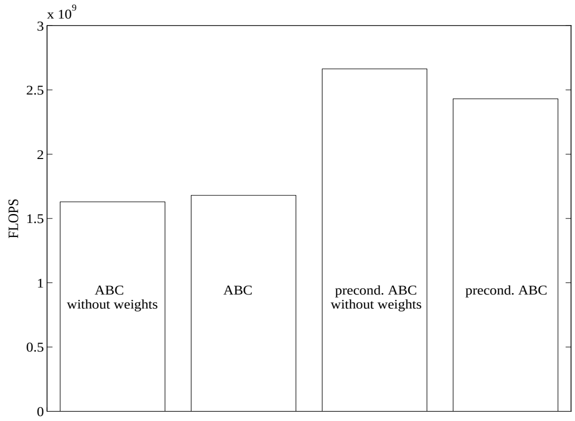

Fig. 6. Comparison of flops to reach precision 10^{-4}

(a) Lena

(b) sampled Lena: SNR = 3.7dB

(c) Marvasti: SNR = 20.3dB

(d) adaptive weights:SNR=27.2dB

(e) SA-Voronoi: SNR = 29.4dB

(f) ABC: SNR = 29.9dB

Fig. 7. Nonuniformly sampled Lena and reconstructions