A Unified Approach to Numerical Algorithms for Discrete Gabor Expansions

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Abstract— The Gabor transform yields a discrete representation of a signal in the phase space [1, 2]. Since the Gabor transform is non-orthogonal, efficient reconstruction of a signal from its phase space samples is not straightforward and involves the computation of the so-called *dual Gabor function*. We present a unifying approach to the derivation of numerical algorithms for discrete Gabor analysis, based on unitary matrix factorization. The factorization point of view is notably useful for the design of efficient numerical algorithms. This presentation is the first systematic account of its kind. In particular, it is shown that different algorithms for the computation of the dual window correspond to different factorizations of the frame operator. Simple number theoretic conditions on the time-frequency lattice parameters imply additional structural properties of the frame operator.

I. Introduction

A recurrent question in signal processing is how to best represent signals. When the signals are continuous it is usually desired to represent them by a discrete sequence. If the signal is band-limited the Shannon sampling theorem states that the signal can be uniquely represented by and stably reconstructed from its regular sampling values. A signal $f \in L^2(\mathbb{R})$ can still be discretized by transforming f into a sequence of moments $\langle f, g_n \rangle$. A necessary requirement for unique representation and stable reconstruction of f is that the functions g_n constitute a frame for $L^2(\mathbb{R})$. In a landmarking paper Gabor [3] suggested to use modulated and shifted versions of a prototype function g (where g is usually a Gaussian-like function) as elementary representation function. Thus the (doubly-indexed) functions $g_{m,n}$ are given by

$$g_{m,n}(x) = g(x - an)e^{2\pi i bmx}$$

where a and b determine the time- and frequency sampling interval. In other words the *Gabor transform* of a signal f

$$f \mapsto \{\langle f, g_{m,n} \rangle\}_{m,n}$$

yields a time-frequency representation of f.

For an introduction to Gabor analysis we refer to [2]. For appropriate analysis window and sampling density the Gabor transform is invertible. However, since the Gabor transform is a *non-orthogonal transform*, the computation of the inverse Gabor transform is a non-trivial task, it involves the calculation of a so-called *dual Gabor window* γ. Thus in this paper we concentrate on efficient numerical methods for computing the dual window.

Recently a large number of algorithms has been developed for this purpose. However the poor mathematical and algorithmic notation has on the one hand concealed the simplicity and intrinsic beauty of Gabor analysis and on the other hand has led to a literature of duplicated results. We present a unifying framework for discrete Gabor analysis, our approach heavily relies on unitary matrix factorization, inspired by the book *Computational Frameworks for the Fast Fourier Transform* of van Loan [4].

Finite and periodic discrete Gabor expansions have been extensively investigated, see e.g. [5, 6, 7, 8, 9, 10, 11, 12]. However its rich structural properties have not yet been fully recognized. Hence a comprehensive elaboration of the structure of the Gabor transform and the inverse Gabor transform as well as the Gabor frame operator is a major aim of this contribution. More details can be found in [13].

We are working with following discrete model. A discrete signal f of length L is a finite sequence $(f(0), f(1), \dots, f(L-1)) \in \mathbb{C}^L$. The inner product of $f, g \in \mathbb{C}^L$ is $\langle f, g \rangle = \sum_{n=0}^{L-1} f(n)\overline{g(n)}$ and the ℓ^2-norm of f is given by $\|f\| = \sqrt{\langle f, f \rangle}$. Here and throughout this chapter we extend finite sequences in $\mathbb{C}^L$ to infinite sequences of period L by $f(k + lL) = f(k)$ for $k = 0, \dots, L-1$ and $l \in \mathbb{Z}$.

We write $T_k f(t) := f(t+k)$ for the translation operator or time shift and $M_l f(t) := e^{2\pi i l t/L}, l \in \mathbb{Z}$ for the modulation operator or frequency shift. For a given function g the set of Gabor analysis functions $\{g_{mn}\}$

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is given by

$$g_{mn} = M_{mb} T_{na} g \qquad (1)$$

for $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$ where $Na = Mb = L$ with $a, b, N, M \in \mathbb{N}$. The parameters a and b represent time and frequency sampling intervals, and are also referred to as time and frequency shift parameters.

Discrete Gabor analysis can be understood in an algebraic setting. Let G be the $L \times MN$ matrix having g_{mn} as its $(m+nM)$-th column for $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$. We write $c_{m,n} = \langle f, g_{mn} \rangle$ and by slightly abusing our notation we treat the $N \times M$ array c also as column vector of length MN by stacking the columns of c. The *Gabor transform* of a function f is given by

$$G^* f = c\,.$$

The case $ab = L$ is usually referred to as *critically sampled* Gabor transform, whereas the choice $ab < L$ yields an *oversampled* Gabor transform, the ratio MN/L giving the oversampling rate. If we set $a = b = 1$ the Gabor transform of f coincides with the discrete short time Fourier transform of f.

We can reconstruct f from its *Gabor coefficients* $c_{m,n}$ by means of a *dual system* $\{\gamma_{m,n}\}$ via

$$f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{m,n} \gamma_{m,n}$$

where the set $\{\gamma_{m,n}\}$ is obtained from some *dual* window γ by equation (1). Defining Γ as the $L \times MN$ matrix having $\gamma_{m,n}$ as its $(m+nM)$-th column, the duality condition can be expressed as

$$G \Gamma^* = I_L \qquad (2)$$

where I_L denotes the identity matrix on $\mathbb{C}^L$. The set $\{\gamma_{m,n}\}$ can be determined from a single dual function γ analogously to the set $\{g_{mn}\}$ in equation (1). If $ab < L$ there are in general many dual functions satisfying condition (2). Of particular importance is the dual with minimal norm, which we denote by ${}^\circ\gamma$.

G is a $L \times MN$ matrix and G^{-1} does not exist for $ab < L$. However we can always compute the pseudoinverse [14] G^+ of G. The rows of G are linear independent, provided that $\{g_{mn}\}$ constitutes a frame for $\mathbb{C}^L$. In this case G^+ becomes a left inverse which can be computed by $G^+ = G^*(GG^*)^{-1}$.

For perfect reconstruction of f the column rank of G must be L, which is equivalent to assuming that the set $\{g_{mn}\}$ constitutes a frame for $\mathbb{C}^L$. A necessary condition for this is $ab \leq L$. Let ${}^\circ\Gamma$ be the matrix containing the elements ${}^\circ\gamma_{m,n}$ of the dual frame associated with ${}^\circ\gamma$. Then it follows from the relation of frames and pseudo inverses [15] that ${}^\circ\Gamma$ coincides with the pseudo inverse of G, i.e. ${}^\circ\Gamma^* = G^+$.

Recall that the Gabor frame operator S is defined by

$$Sf = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle f, g_{mn} \rangle g_{mn}\,.$$

S is a positive definite Hermitian $L \times L$ matrix which can be expressed as $S = GG^*$.

It follows from standard linear algebra that the minimal norm dual ${}^\circ\gamma$ satisfies $S\,{}^\circ\gamma = g$. The computation of ${}^\circ\gamma$ involves the (pseudo-) inversion of G or, equivalently, the inversion of S. From a numerical analysis point of view we may either choose direct or iterative methods for this purpose. The central theme in the following sections is the idea that different algorithms correspond to different unitary matrix factorizations of the frame operator S and the matrix G, respectively. Efficient algorithms for the computation of dual windows are essentially based on these factorizations.

Wexler and Raz have obtained an elegant formulation of the duality condition of the systems $\{g_{mn}\}$ and $\{\gamma_{m,n}\}$, see [5]. We refer to [16] for a systematic discussion of the Wexler-Raz identity. Whereas the formulation of the duality condition in equation (2) holds for any frame and its dual, the Wexler-Raz duality condition is a specific characterization of duality for Gabor systems. In our algebraic setting we can describe this condition as follows.

Let H be the $L \times ab$ matrix with entries

$$H_{kl} = T_{mM} M_{nN} g(k)$$

where $l = m + na$ and $m = 0, \ldots, b-1$, $n = 0, \ldots, a-1$. Observe that H is of similar structure as G but with a and b replaced by M and N respectively.

Then the Wexler-Raz duality condition for g and γ can be expressed as

$$H^* \gamma = \frac{ab}{L} [1, 0, \ldots, 0]^T. \qquad (3)$$

The minimal norm solution of (3) coincides with ${}^\circ\gamma$.

II. Factorization of the frame operator

One standard method to derive fast algorithms for the inversion of a matrix X is based on a unitary factorization $D = U^* X U$ where D is diagonal or at least close to diagonal. A few diagonal matrix notations are handy. If $d \in \mathbb{C}^N$, then $D = \mathrm{diag}(d) = \mathrm{diag}(d_0, \ldots, d_{N-1})$ is the $N \times N$ diagonal matrix with diagonal entries $d_0, \ldots, d_{N-1}$. If $D_0, \ldots, D_{N-1}$ are $p \times p$ matrices, then $D = \mathrm{diag}(D_0, \ldots, D_{N-1})$ is a $p \times p$ block diagonal matrix.

A. Permutations and Fourier transform

The simplest unitary matrices are permutation matrices. A *permutation matrix* is just the identity with its columns reordered. If P is a permutation and A

is a matrix, then PA is a row permuted version of A and AP is a column permuted version of A. We will frequently make use of the mod p *perfect shuffle permutation* [4] which plays also a key role in the FFT algorithm. Observe that $P_{p,L} = P^*_{q,L}$. Here is an example with $L = 15 = 3 \cdot 5 \equiv 3 \cdot 5$:

$$P_{3,15}x =$$
$$[\, x_0\, x_5\, x_{10} \,|\, x_1\, x_6\, x_{11} \,|\, x_2\, x_7\, x_{12} \,|\, x_3\, x_8\, x_{13} \,|\, x_4\, x_9\, x_{14} \,]^*$$
$$P^*_{3,15}x =$$
$$[\, x_0\, x_3\, x_6\, x_9\, x_{12} \,|\, x_1\, x_4\, x_7\, x_{10}\, x_{13} \,|\, x_2\, x_5\, x_8\, x_{11}\, x_{14} \,]^* .$$

It is not surprising that the *Fourier matrix* is of fundamental importance in the factorization of the Gabor frame operator. By the Fourier matrix of order N we shall mean the (up to normalization) unitary $(N \times N)$ matrix F_N with entries $\omega_{mn} = e^{-2\pi i mn/N}$.

Further we will frequently make use of circulant matrices. It is well known that a circulant matrix C_N is diagonalized by the Fourier matrix F_N. In particular let Π_N, defined by

$$\Pi_N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

be the $N \times N$ *basic circulant permutation matrix* then $\Pi_N = \frac{1}{N} F_N^* \Omega F_N$. where $\Omega = \mathrm{diag}(1, \omega, \omega^2, \dots, \omega^{N-1})$ contains the eigenvalues of Π_N, see [17].

If $A \in C^{p \times q}$ and $B \in C^{m \times n}$ then the *Kronecker product* $A \otimes B$ is the p-by-q block matrix

$$A \otimes B = \begin{bmatrix} a_{00}B & \dots & a_{0q-1}B \\ \vdots & & \vdots \\ a_{p-10}B & \dots & a_{p-1q-1}B \end{bmatrix} \in C^{pm \times qn} .$$

If $N = mn$ and $A \in \mathbb{C}^{m \times m}$, then [4]

$$P^*_{m,N}(I_n \otimes A) = (A \otimes I_n) P^*_{m,N} . \qquad (4)$$

If A and B are unitary matrices, then $A \otimes B$ is a unitary matrix, see [4].

B. Matrix factorization and rational oversampling

We start by describing the factorization for general oversampling and show later how this factorization simplifies for special cases like integer oversampling.

As mentioned above in an algebraic setting the Gabor frame operator is the $N \times N$ matrix S given by $S = GG^*$.

The *Walnut representation* [18] of the Gabor frame operator tells us that S has a very specific structure. This structure was described in detail in [11] for the discrete setting. The entries of S are given by

$$S_{m,n} = \frac{N}{b} \sum_{k=0}^{\frac{N}{a}-1} T_{ka} g(m) \overline{T_{ka} g(n)}$$

if $|m - n| \bmod \frac{N}{b} = 0$ and $S_{m,n} = 0$ else.

Theorem 2.1 *Let B_k denote the $b \times b$ matrix with entry*

$$(B_k)_{mn} = S_{k+m\frac{N}{b},\,k+n\frac{N}{b}} \qquad (5)$$

for $m, n = 0, \dots, b-1$ and $k = 0, \dots, \frac{N}{b} - 1$. Then S can be unitarily factorized into a block diagonal matrix with $P_{\frac{N}{b},N}$ as factorization matrix, thus we have

$$D_S = P^*_{M,L} S \, P_{M,L} = diag(B_0, \dots, B_{M-1}). \qquad (6)$$

A block diagonal matrix can be inverted by inverting each block separately, see e.g. [19]. Thus the computation of the $N \times N$ matrix S^{-1} can be reduced to the inversion of b blocks of size $\frac{N}{b} \times \frac{N}{b}$.

In Theorem 2.1 we have utilized the uniform sparsity of S. In the sequel we describe an alternative approach, focusing on the periodicity of the diagonal entries of S. This approach has been proposed by Qiu [20].

Lemma 2.2 *The Gabor frame operator S is a block circulant matrix of the form*

$$S = \begin{bmatrix} A_0 & A_1 & \dots & A_{\frac{N}{a}-1} \\ A_{\frac{N}{a}-1} & A_0 & \dots & A_{\frac{N}{a}-2} \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}$$

where the A_k are $a \times a$ matrices with $(A_k)_{m,n} = S_{ka+m,ka+n}$ for $k = 0, \dots, \frac{N}{a} - 1$ and $m = 0, \dots, a-1$, $n = 0, \dots, a-1$.

Observe that the blocks A_k are not circulant, thus S cannot be diagonalized by a straightforward Kronecker generalization of the diagonalization of circulant matrices. However we still can derive a diagonalization based on Fourier matrices [20].

Theorem 2.3 *S can be unitarily factorized into a block diagonal matrix via*

$$(F_N \otimes I_a)^* S (F_N \otimes I_a)$$

In other words, S is unitarily equivalent to $diag(\tilde{A}_0, \dots, \tilde{A}_{N-1})$ where

$$\tilde{A}_l = \sum_{k=0}^{N-1} N^{\frac{k}{2}} \omega^{lk} A_k \qquad \text{for } l = 0, \dots N-1 .$$

The block diagonalization described in Theorem 2.1 can be also derived from a factorization proposed by Prinz [21]. Extending the approach of Stewart, Potter and Ahalt [22] from critical sampling to rational oversampling, he factorizes the Gabor transform matrix G instead of S.

Proposition 2.4 *The Gabor transform matrix G can be factorized into a block diagonal matrix D_G with $\frac{N}{b}$ rectangular blocks W_k of size $b \times \frac{N}{a}$ via*

$$D_G = P^*_{\frac{N}{b},N} G (I_{\frac{N}{a}} \otimes F_{\frac{N}{b}}) P_{a,M} \qquad (7)$$

where $D_G = diag(W_0, \dots, W_{\frac{N}{b}-1})$ with $(W_k)_{mn} = \frac{N}{b} g(k - ma + n\frac{N}{b})$ for $m = 0, \dots, \frac{N}{a}-1$, $n = 0, \dots, b-1$ and $k = 0, \dots, \frac{N}{b} - 1$.
Obviously we have $D_S = D_G D_G^*$.

Applying the arguments used in the proof of Proposition 2.1 to H we obtain following result.

Corollary 2.5 *Let V_k denote the $N \times b$ matrix with entry*

$$(V_k)_{mn} = \sqrt{a} g(k - mM + na)$$

for $m = 0, \dots, N-1, n = 0, \dots, b-1$ and $k = 0, \dots, a-1$. Then H can be unitarily factorized into the block diagonal matrix

$$D_H = P^*_{a,L} H (I_b \otimes F_a) P^*_{b,ab} \qquad (8)$$

where $D_H = diag(V_0, \dots, V_{a-1})$.

Since D_H is in general not invertible, we compute its left-inverse D_H^+ via $D_H^+ = (D_H^* D_H)^{-1} D_H^*$. $D_H^* D_H$ is a $ab \times ab$ block matrix with a blocks of size $b \times b$, whereas the $L \times L$ matrix D_S of equation (6) consists of M blocks of size b.

It is an easy task to derive all factorizations above for the matrix H with the appropriate lattice parameters M, N. The advantage of introducing first the matrix G and S is that G^* represents the Gabor transform and therefore is the natural starting point for an algebraic and numerical treatment of discrete Gabor expansions.

III. Finite Gabor Expansions and Number Theory

The unitary factorization of S into a block diagonal or block circulant matrix derived in Section 2 is based on the periodic and sparse structure of S which follows from the Walnut representation. These factorizations clearly hold in general and do not depend on number theoretic conditions on the lattice parameters a, b.

In this section we show that in a finite setting the Gabor frame operator exhibits an additional structure. Although finite and periodic discrete Gabor expansions have been extensively investigated, see e.g. [5, 8, 12, 7, 10, 6, 9], its rich structural properties have not yet been fully recognized. A complete description of the structure of S is the purpose of this section. We will show in Subsection 3.2 that the well known diagonalization of the frame operator in the cases of critical and integer oversampling (cf. e.g. [23, 24, 25]) are simple consequences of Theorem 3.3. A first attempt to investigate Gabor expansions from a number theoretic point of view can be found in [21].

A. Fundamental Factorization of the frame operator

In the sequel we will frequently use following notation. Let $q \in \mathbb{R}$, then $\lfloor q \rfloor$ denotes the largest integer $\leq q$ and $\lceil q \rceil$ is the smallest integer $\geq q$.

Proposition 3.1 *Given a function g of length L and lattice parameters a, b. Denote the greatest common divisor (gcd) of a and M by c. Let W_k be the submatrices of the block diagonal factorization of G stated in Proposition 2.4, i.e. $(W_k)_{mn} = Mg(k + mM - na)$. Then the W_k satisfy following relation:*

$$\Pi_b^q W_k \Pi_N^l = W_{(k+la) \bmod M} \qquad (9)$$

with $q = \lfloor \frac{lab}{L} \rfloor$. Thus there are (up to permutations) c different submatrices W_k for $k = 0, M-1$.

Next we show that a block circulant structure emerges in each submatrix W_k.

Proposition 3.2 *Given a function g of length L and lattice parameters a, b. Denote $d = \gcd(b, N)$. Then W_k is a block circulant matrix with d generating blocks of size $\frac{b}{d} \times \frac{N}{d}$ for $k = 0, \dots, M-1$.*

Based on Proposition 3.1 and Proposition 3.2, we obtain a complete characterization of the structure of the frame operator and an estimate for the complexity for the computation of S^{-1}.

Theorem 3.3 (Fundamental Factorization of S) *The Gabor frame operator S can be unitarily factorized into the block diagonal matrix*

$$I_{\frac{M}{c}} \otimes diag(C_0, \dots, C_{cd-1}), \qquad (10)$$

where the $\frac{b}{d} \times \frac{b}{d}$ submatrices C_j are defined by

$$diag(C_{dk}, \dots, C_{d(k+1)-1}) = (F_d \otimes I_{\frac{b}{d}})^* B_k (F_d \otimes I_{\frac{b}{d}})$$

for $k = 0, \dots, c-1$ and B_k is given by (5). Thus S can be inverted in $\mathcal{O}(cb(\frac{b^2}{d^2} + b \log d))$ operations.

Thus S is unitarily equivalent to a block-diagonal matrix with blocks of size $\frac{b}{\gcd(b,N)} \times \frac{b}{\gcd(b,N)}$ and the blocks have periodicity $\gcd(a, M)$. Using Theorem 3.3 we can easily compute the number of different eigenvalues of the frame operator.

Corollary 3.4 *S has at most $\gcd(ab, L)$ different eigenvalues. Furthermore if g is real-valued, the maximal number of eigenvalues reduces to $\frac{bc}{d} \lceil \frac{d+1}{2} \rceil$.*

There are two special cases of Theorem 3.3, which deserve to be mentioned separately. The first case is $\gcd(a, M) = 1$, the second is $\gcd(a, M) = a$. The second choice leads to the case of integer oversampling and will be treated in the next section. The first case provides a simple but nice factorization of S.

Corollary 3.5 *Assume that $\gcd(a, M) = 1$, then the Gabor frame operator S is unitarily equivalent to $I_M \otimes B_0$.*

Remark: Applying Proposition 3.1 to the matrix $D_H = \mathrm{diag}(V_0, \dots, V_{a-1})$ defined in (8) yields that the number of (up to permutations) different blocks V_k is given by $\gcd(a, M)$. Hence the costs to compute the dual $^\circ\gamma$ do not depend whether we choose the frame operator factorization of Theorem 2.1 or the factorization based on the Wexler-Raz condition.

B. Critical Sampling and Integer Oversampling

In this section we show that in the case of critical sampling and integer oversampling G and S respectively can be unitarily diagonalized via properly chosen block Fourier matrices. These diagonalizations of S and G have been discovered by different authors, mostly via the Zak transform. Thus we will also show the analogy of our approach with Zak transform based methods.

The choice $ab = L$ is called *critical sampling*, whereas choosing $abq = L$ for $q \in \mathbb{N}$ is referred to as *integer oversampling.*

As an immediate consequence of Theorem 3.1 we obtain following result for critical sampling.

Corollary 3.6 *Assume that $ab = L$, then G can be unitarily factorized into a diagonal matrix with $(I_a \otimes F_b)$ as factorization matrix.*

This result has been derived by different people, mostly via the Zak transform, cf. [25, 26]. We show the analogy of the Zak transform approach with our point of view.

Recall that for the parameter a the discrete Zak transform of g is a $(a \times N)$ matrix given by

$$(\mathcal{Z}_a g)(k, n) = \frac{1}{\sqrt{L}} \sum_{l=0}^{b-1} g(k - la) e^{2\pi i la n/L}$$

for $k = 0, \dots, a-1$, $n = 0, \dots, N-1$. Observe that the n-th element of the first row of $W_k F_b$ is given by $\sum_{l=0}^{b-1} g(k - la) e^{-2\pi i n l/b}$ which coincides up to conjugation with $(\mathcal{Z}_a g)(k, n)$.

For $abq = L$ it is known that the dual window $^\circ\gamma$ satisfies the relation

$$(\mathcal{Z}_a^\circ \gamma)(k, n) = \frac{(\mathcal{Z}_a g)(k, m)}{a \sum_{l=0}^{q-1} |(\mathcal{Z}_a g)(k - la, n)|^2} \qquad (11)$$

see [25].

For critical sampling equation (11) reduces to

$$(\mathcal{Z}_a^\circ \gamma)(k, n) = \frac{1}{a (\mathcal{Z}_a g)^*(k, n)} . \qquad (12)$$

Equation (12) just describes simultaneous inversion of the circulant matrices W_k by the Fourier matrix F_b. Thus $(\mathcal{Z}_a^\circ \gamma)(k, n)$ is equal to the n-th entry of the first row of $W_k^{-1} F_b$.

Similar results hold for integer oversampling. Note that the choice $abq = L$ implies $\gcd(a, M) = a$ and $\gcd(b, N) = b$. For $q > 1$ the matrix G and hence the blocks W_k are not invertible. We compute the left inverses of the blocks W_k by $W_k^+ = W_k^* (W_k W_k^*)^{-1}$. According to Theorem 2.1 S can be block-diagonalized with blocks $B_k = W_k W_k^*$. It follows from Theorem 3.3 that the $b \times b$ blocks B_k are circulant and therefore can be diagonalized by the Fourier matrix F_b. Summing up we have following corollary.

Corollary 3.7 *Assume that $abq = L$ with $q \in \mathbb{N}$. Then S is unitarily diagonalized by $(I_{aq} \otimes F_b)$.*

The analogy to the Zak transform based approach for computing $^\circ\gamma$ becomes obvious, when we compute $W_k^* (W_k W_k^*)^{-1}$. Due to the block circulant structure of W_k and the circulant structure of $W_k W_k^*$ the matrix W_k^+ can be calculated efficiently by Fourier transforms.

Following lemma follows immediately from Proposition 3.2 and the properties of circulant matrices [17].

Lemma 3.8 *In case of integer oversampling (with oversampling rate q) the columns of a block W_k can always be reordered such that*

$$W_k P_{q,bq} = \begin{bmatrix} C_0 & C_1 & \dots & C_{q-1} \end{bmatrix} \qquad (13)$$

where the C_k are circulant $(b \times b)$ matrices.

Hence the entries of the circulant matrix $W_k W_k^*$ can be computed by Fourier transforms of length b. In particular for fixed k let c_j denote the first column of the circulant block C_j for $j = 0, \dots, q-1$ according to (13) and $\tilde{c_j}(m) = |(F_b c_j)(m)|^2$, then the first column of $W_k W_k^*$ is given by $\sum_{j=0}^{q-1} (F_q^* \tilde{c_j})$. Since $P_{q,bq}$ is unitary, $W_k P_{q,bq} P_{q,bq}^* W_k = W_k W_k^*$. The product $W^* (W W^*)^{-1}$ can be computed in a similar manner. These calculations are exactly the same arising in equation (11).

Undisputed, the Zak transform has shed light on several theoretical aspects of Gabor analysis [23, 24]. However from a numerical point of view algorithms involving the Zak transform are a roundabout way. FFT and circulant matrices – both well established tools in the mathematical and engineering community – are sufficient for the derivation of fast algorithms to compute the dual window γ, especially since efficient numerical implementations of the Zak transform are essentially based on FFT. Moreover the Zak transform is not an appropriate tool to exploit the full structural properties of S for general rational oversampling.

Remark: A detailed discussion of iterative methods for computing the dual window can be found in [13]. The presented factorizations of the Gabor frame operator are especially useful in conjunction with the conjugate gradient method and for the derivation of efficient preconditioners.

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