Computational frameworks for discrete Gabor analysis

Thomas Strohmer*

NUHAG – Numerical Harmonic Analysis Group
University of Vienna, Department of Mathematics, Austria

ABSTRACT

The Gabor transform yields a discrete representation of a signal in the phase space. Since the Gabor transform is non-orthogonal, efficient reconstruction of a signal from its phase space samples is not straightforward and involves the computation of the so-called *dual Gabor function*. We present a unifying approach to the derivation of numerical algorithms for discrete Gabor analysis, based on unitary matrix factorization. The factorization point of view is notably useful for the design of efficient numerical algorithms. This presentation is the first systematic account of its kind. In particular, it is shown that different algorithms for the computation of the dual window correspond to different factorizations of the frame operator. Simple number theoretic conditions on the time-frequency lattice parameters imply additional structural properties of the frame operator.

Keywords: Gabor transform, non-orthogonal series expansions, Kronecker product, conjugate gradients, circulant preconditioner

1 Introduction

A recurrent question in signal processing is how to best represent signals. When the signals are continuous it is usually desired to represent them by a discrete sequence. If the signal is band-limited the Shannon sampling theorem states that the signal can be uniquely represented by and stably reconstructed from its regular sampling values. A non-bandlimited signal still can be represented by a discrete sequence for instance by its samples in the phase space. In other words we compute the Gabor transform of a signal. For an introduction to Gabor analysis we refer to.[17] For appropriate analysis window and sampling density the signal can be reconstructed from its phase space samples using a so-called dual Gabor window.

In discrete Gabor expansions the main difficulty from a numerical point of view is an efficient computation of the so-called *dual window* γ. Recently a large number of algorithms has been developed for this purpose. However the poor mathematical and algorithmic notation has on the one hand concealed the simplicity and intrinsic beauty of Gabor analysis and on the other hand has led to a literature of duplicated results. We present a unifying framework for discrete Gabor analysis, our approach heavily relies on unitary matrix factorization, inspired by the book *Computational Frameworks for the Fast Fourier Transform* of van Loan.[36]

Finite and periodic discrete Gabor expansions have been extensively investigated, see e.g..[37,26,39,22,13,38,30,28]

Email: strohmer@@tyche.mat.univie.ac.at WWW:http://tyche.mat.univie.ac.at/ strohmer

However its rich structural properties have not yet been fully recognized. Hence a comprehensive elaboration of the structure of the Gabor transform and the inverse Gabor transform as well as the Gabor frame operator is a major aim of this contribution. More details can be found in.[35]

We are working with following finite-dimensional model. A discrete signal f of length L is a finite sequence $(f(0), f(1), \ldots, f(L-1)) \in \mathbb{C}^L$. The inner product of $f, g \in \mathbb{C}^L$ is $\langle f, g\rangle = \sum_{n=0}^{L-1} f(n)\overline{g(n)}$ and the ℓ^2-norm of f is given by $\|f\| = \sqrt{\langle f, f\rangle}$. Here and throughout this chapter we extend finite sequences in $\mathbb{C}^L$ to infinite sequences of period L by $f(k + lL) = f(k)$ for $k = 0, \ldots, L-1$ and $l \in \mathbb{Z}$.

We write $T_k f(t) := f(t + k)$ for the translation operator or time shift and $M_l f(t) := e^{2\pi i l t/L}, l \in \mathbb{Z}$ for the modulation operator or frequency shift. For a given function g the set of Gabor analysis functions $\{g_{mn}\}$ is given by

$$g_{mn} = M_{mb} T_{na} g \tag{1}$$

for $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$ where $Na = Mb = L$ with $a, b, N, M \in \mathbb{N}$. The parameters a and b represent time and frequency sampling intervals, and are also referred to as time and frequency shift parameters.

Discrete Gabor analysis can be understood in an algebraic setting. Let G be the $L \times MN$ matrix having g_{mn} as its $(m + nM)$-th column for $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$. We write $c_{m,n} = \langle f, g_{mn}\rangle$ and by slightly abusing our notation we treat the $N \times M$ array c also as column vector of length MN by stacking the columns of c. The *Gabor transform* of a function f is given by

$$G^* f = c\,.$$

The case $ab = L$ is usually referred to as *critically sampled* Gabor transform, whereas the choice $ab < L$ yields an *oversampled* Gabor transform, the ratio MN/L giving the oversampling rate. If we set $a = b = 1$ the Gabor transform of f coincides with the discrete short time Fourier transform of f.

We can reconstruct f from its *Gabor coefficients* $c_{m,n}$ by means of a *dual system* $\{\gamma_{m,n}\}$ via

$$f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{m,n} \gamma_{m,n}$$

where the set $\{\gamma_{m,n}\}$ is obtained from some *dual* window γ by equation (1). Defining Γ as the $L \times MN$ matrix having $\gamma_{m,n}$ as its $(m + nM)$-th column, the duality condition can be expressed as

$$G\Gamma^* = I_L \tag{2}$$

where I_L denotes the identity matrix on $\mathbb{C}^L$. The set $\{\gamma_{m,n}\}$ can be determined from a single dual function γ analogously to the set $\{g_{mn}\}$ in equation (1). If $ab < L$ there are in general many dual functions satisfying condition (2). Of particular importance is the dual with minimal norm, which we denote by ${}^\circ\gamma$.

G is a $L \times MN$ matrix and G^{-1} does not exist for $ab < L$. However we can always compute the pseudoinverse[34] G^+ of G. The rows of G are linear independent, provided that $\{g_{mn}\}$ constitutes a frame for $\mathbb{C}^L$. In this case G^+ becomes a left inverse which can be computed by $G^+ = G^*(GG^*)^{-1}$.

For perfect reconstruction of f the column rank of G must be L, which is equivalent to assuming that the set $\{g_{mn}\}$ constitutes a frame for $\mathbb{C}^L$. A necessary condition for this is $ab \leq L$. Let ${}^\circ\Gamma$ be the matrix containing the elements ${}^\circ\gamma_{m,n}$ of the dual frame associated with ${}^\circ\gamma$. Then it follows from the relation of frames and pseudo inverses[10] that ${}^\circ\Gamma$ coincides with the pseudo inverse of G, i.e. ${}^\circ\Gamma^* = G^+$.

Recall that the Gabor frame operator S is defined by

$$Sf = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle f, g_{mn}\rangle g_{mn}\,.$$

S is a positive definite Hermitian $L \times L$ matrix which can be expressed as $S = GG^*$.

It follows from standard linear algebra that the minimal norm dual ${}^\circ\gamma$ satisfies $S{}^\circ\gamma = g$. The computation of ${}^\circ\gamma$ involves the (pseudo-) inversion of G or, equivalently, the inversion of S. From a numerical analysis point of view we may either choose direct or iterative methods for this purpose. The central theme in the following sections is the idea that different algorithms correspond to different unitary matrix factorizations of the frame operator S and the matrix G, respectively. Efficient algorithms for the computation of dual windows are essentially based on these factorizations.

Wexler and Raz have obtained an elegant formulation of the duality condition of the systems $\{g_{mn}\}$ and $\{\gamma_{m,n}\}$, see.[37] We refer to Janssen's chapter in this book for a systematic discussion of the Wexler-Raz identity. Whereas the formulation of the duality condition in equation (2) holds for any frame and its dual, the Wexler-Raz duality condition is a specific characterization of duality for Gabor systems. In our algebraic setting we can describe this condition as follows.

Let H be the $L \times ab$ matrix with entries

$$H_{kl} = T_{mM} M_{nN} g(k)$$

where $l = m + na$ and $m = 0, \ldots, b-1$, $n = 0, \ldots, a-1$. Observe that H is of similar structure as G but with a and b replaced by M and N respectively.

Then the Wexler-Raz duality condition for g and γ can be expressed as

$$H^*\gamma = \frac{ab}{L}[1, 0, \ldots, 0]^T. \tag{3}$$

The minimal norm solution of (3) coincides with ${}^\circ\gamma$.

2 Factorization of the frame operator

One standard method to derive fast algorithms for the inversion of a matrix X is based on a unitary factorization $D = U^*XU$ where D is diagonal or at least close to diagonal. A few diagonal matrix notations are handy. If $d \in \mathbb{C}^N$, then $D = \mathrm{diag}(d) = \mathrm{diag}(d_0, \ldots, d_{N-1})$ is the $N \times N$ diagonal matrix with diagonal entries $d_0, \ldots, d_{N-1}$. If $D_0, \ldots, D_{N-1}$ are $p \times p$ matrices, then $D = \mathrm{diag}(D_0, \ldots, D_{N-1})$ is a $p \times p$ block diagonal matrix.

2.1 Permutations and Fourier transform

The simplest unitary matrices are permutation matrices. A *permutation matrix* is just the identity with its columns reordered. If P is a permutation and A is a matrix, then PA is a row permuted version of A and AP is a column permuted version of A. We will frequently make use of the mod p *perfect shuffle permutation*[36] whick plays also a key role in the FFT algorithm. Here is an example with $L = 15 = 3 \cdot 5 \equiv 3 \cdot 5$:

$$P_{3,15}^* x = [x_0 x_3 x_6 x_9 x_{12} | x_1 x_4 x_7 x_{10} x_{13} | x_2 x_5 x_8 x_{11} x_{14}]^* .$$

If $N = pq$ then $P_{p,N} = P_{q,M}^*$.

It is not surprising that the *Fourier matrix* is of fundamental importance in the factorization of the Gabor frame operator. By the Fourier matrix of order N we shall mean the (up to normalization) unitary $(N \times N)$ matrix F_N with entries $\omega_{mn} = e^{-2\pi imn/N}$.

Further we will frequently make use of circulant matrices. It is well known that a circulant matrix C_N is diagonalized by the Fourier matrix F_N. In particular let Π_N, defined by

$$\Pi_N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

be the $N \times N$ *basic circulant permutation matrix* then $\Pi_N = \frac{1}{N} F_N^* \Omega F_N$. where $\Omega = \mathrm{diag}(1, \omega, \omega^2, \dots, \omega^{N-1})$ contains the eigenvalues of Π_N, see.[11]

If $A \in C^{p \times q}$ and $B \in C^{m \times n}$ then the *Kronecker product* $A \otimes B$ is the p-by-q block matrix

$$A \otimes B = \begin{bmatrix} a_{00}B & \dots & a_{0q-1}B \\ \vdots & & \vdots \\ a_{p-10}B & \dots & a_{p-1q-1}B \end{bmatrix} \in C^{pm \times qn}.$$

If $N = mn$ and $A \in \mathbb{C}^{m \times m}$, then[36]

$$P_{m,N}^* (I_n \otimes A) = (A \otimes I_n) P_{m,N}^*. \tag{4}$$

If A and B are unitary matrices, then $A \otimes B$ is a unitary matrix, see.[36]

2.2 Matrix factorization and rational oversampling

We start by describing the factorization for general oversampling and show later how this factorization simplifies for special cases like integer oversampling.

As mentioned above in an algebraic setting the Gabor frame operator is the $N \times N$ matrix S given by $S = GG^*$.

The *Walnut representation*[21] of the Gabor frame operator tells us that S has a very specific structure. This structure was described in detail in[30] for the discrete setting. The entries of S are given by

$$S_{m,n} = \frac{N}{b} \sum_{k=0}^{\frac{N}{a}-1} T_{ka} g(m) \overline{T_{ka} g(n)}$$

if $|m - n| \bmod \frac{N}{b} = 0$ and $S_{m,n} = 0$ else.

THEOREM 2.1. *Let B_k denote the $b \times b$ matrix with entry*

$$(B_k)_{mn} = S_{k+m\frac{N}{b}, k+n\frac{N}{b}} \tag{5}$$

for $m, n = 0, \dots, b-1$ and $k = 0, \dots, \frac{N}{b} - 1$. Then S can be unitarily factorized into a block diagonal matrix with $P_{\frac{N}{b},N}$ as factorization matrix, thus we have

$$D_S = P_{M,L}^* S \; P_{M,L} = diag(B_0, \dots, B_{M-1}). \tag{6}$$

A block diagonal matrix can be inverted by inverting each block separately, see e.g. 19. Thus the computation of the $N \times N$ matrix S^{-1} can be reduced to the inversion of b blocks of size $\frac{N}{b} \times \frac{N}{b}$.

In Theorem 2.1 we have utilized the uniform sparsity of S. In the sequel we describe an alternative approach, focusing on the periodicity of the diagonal entries of S. This approach has been proposed by Qiu.[29]

LEMMA 2.2. *The Gabor frame operator S is a block circulant matrix of the form*

$$S = \begin{bmatrix} A_0 & A_1 & \dots & A_{\frac{N}{a}-1} \\ A_{\frac{N}{a}-1} & A_0 & \dots & A_{\frac{N}{a}-2} \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}$$

where the A_k are $a \times a$ matrices with $(A_k)_{m,n} = S_{ka+m,ka+n}$ for $k = 0, \dots, \frac{N}{a} - 1$ and $m = 0, \dots, a-1$, $n = 0, \dots, a-1$.

Observe that the blocks A_k are not circulant, thus S cannot be diagonalized by a straightforward Kronecker generalization of the diagonalization of circulant matrices. However we still can derive a diagonalization based on Fourier matrices.[29]

THEOREM 2.3. *S can be unitarily factorized into a block diagonal matrix via*

$$(F_N \otimes I_a)^* S (F_N \otimes I_a)$$

In other words, S is unitarily equivalent to $diag(\tilde{A}_0, \dots, \tilde{A}_{N-1})$ where

$$\tilde{A}_l = \sum_{k=0}^{N-1} N^{\frac{k}{2}} \omega^{lk} A_k \qquad \text{for } l = 0, \dots N-1 \,.$$

The block diagonalization described in Theorem 2.1 can be also derived from a factorization proposed by Prinz.[27] Extending the approach of Stewart, Potter and Ahalt[33] from critical sampling to rational oversampling, he factorizes the Gabor transform matrix G instead of S.

PROPOSITION 2.4. *The Gabor transform matrix G can be factorized into a block diagonal matrix D_G with $\frac{N}{b}$ rectangular blocks W_k of size $b \times \frac{N}{a}$ via*

$$D_G = P^*_{\frac{N}{b},N} G (I_{\frac{N}{a}} \otimes F_{\frac{N}{b}}) P_{a,M} \tag{7}$$

where $D_G = diag(W_0, \dots, W_{\frac{N}{b}-1})$ with $(W_k)_{mn} = \frac{N}{b} g(k - ma + n\frac{N}{b})$ for $m = 0, \dots, \frac{N}{a} - 1$, $n = 0, \dots, b-1$ and $k = 0, \dots, \frac{N}{b} - 1$. Obviously we have $D_S = D_G D_G^*$.

Applying the arguments used in the proof of Proposition 2.1 to H we obtain following result.

COROLLARY 2.5. *Let V_k denote the $N \times b$ matrix with entry*

$$(V_k)_{mn} = \sqrt{a} g(k - mM + na)$$

for $m = 0, \dots, N-1$, $n = 0, \dots, b-1$ and $k = 0, \dots, a-1$. Then H can be unitarily factorized into the block diagonal matrix

$$D_H = P^*_{a,L} H (I_b \otimes F_a) P^*_{b,ab} \tag{8}$$

where $D_H = diag(V_0, \dots, V_{a-1})$.

Since D_H is in general not invertible, we compute its left-inverse D_H^+ via $D_H^+ = (D_H^* D_H)^{-1} D_H^*$. $D_H^* D_H$ is a $ab \times ab$ block matrix with a blocks of size $b \times b$, whereas the $L \times L$ matrix D_S of equation (6) consists of M blocks of size b.

It is an easy task to derive all factorizations above for the matrix H with the appropriate lattice parameters M, N. The advantage of introducing first the matrix G and S is that G^* represents the Gabor transform and therefore is the natural starting point for an algebraic and numerical treatment of discrete Gabor expansions.

2.3 Representation in the Fourier domain

In the equation $S\gamma = g$ the operator S acts on functions given in the time domain. Equivalently we could apply the frame operator in the frequency domain. Following proposition is an easy consequence of Parseval's formula and the Walnut representation.

PROPOSITION 2.6. *Denote $\hat{S} = F_L S F_L^*$, then the entries of $\hat{S}$*

are given by

$$\hat{S}_{mn} = \begin{cases} N \sum_{k=0}^{M-1} T_{kb} \hat{g}(m) \overline{T_{kb} \hat{g}(n)} & \text{if } |m-n| \mod N = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words if the set $\{g_{mb,na}\}$ constitutes a Gabor frame for $\mathbb{C}^L$, then $\{\hat{g}_{nb,ma}\}$ constitutes also a Gabor fram e for $\mathbb{C}^L$, see.[30]

Clearly all factorizations of S could equally well be applied to $\hat{S}$. We will make use of this fact in Section 5 in connection with the calculation of approximate inverses for S.

3 Finite Gabor Expansions and Number Theory

The unitary factorization of S into a block diagonal or block circulant matrix derived in Section 2 is based on the periodic and sparse structure of S which follows from the Walnut representation. These factorizations clearly hold in general and do not depend on number theoretic conditions on the lattice parameters a, b.

In this section we show that in a finite setting the Gabor frame operator exhibits an additional structure. Although finite and periodic discrete Gabor expansions have been extensively investigated, see e.g.,[37,1,22,28,39,38,26,13] its rich structural properties have not yet been fully recognized. A complete description of the structure of S is the purpose of this section. A first attempt to investigate Gabor expansions from a number theoretic point of view can be found in.[27]

3.1 Fundamental Factorization of the frame operator

In the sequel we will frequently use following notation. Let $q \in \mathbb{R}$, then $\lfloor q \rfloor$ denotes the largest integer $\leq q$ and $\lceil q \rceil$ is the smallest integer $\geq q$.

PROPOSITION 3.1. *Given a function g of length L and lattice parameters a, b. Denote the greatest common divisor (gcd) of a and M by c. Let W_k be the submatrices of the block diagonal factorization of G stated in Proposition 2.4, i.e. $(W_k)_{mn} = Mg(k + mM - na)$. Then the W_k satisfy following relation:*

$$\Pi_b^q W_k \Pi_N^l = W_{(k+la) \bmod M} \tag{9}$$

with $q = \lfloor \frac{lab}{L} \rfloor$. Thus there are (up to permutations) c different submatrices W_k for $k = 0, M-1$.

Next we show that a block circulant structure emerges in each submatrix W_k.

PROPOSITION 3.2. *Given a function g of length L and lattice parameters a, b. Denote $d = \gcd(b, N)$. Then W_k is a block circulant matrix with d generating blocks of size $\frac{b}{d} \times \frac{N}{d}$ for $k = 0, \dots, M-1$.*

Based on Proposition 3.1 and Proposition 3.2, we obtain a complete characterization of the structure of the frame operator and an estimate for the complexity for the computation of S^{-1}.

THEOREM 3.3 (FUNDAMENTAL FACTORIZATION OF $\boldsymbol{S}$). *The Gabor frame operator S can be unitarily factorized into the block diagonal matrix*

$$I_{\frac{M}{c}} \otimes diag(C_0, \dots, C_{cd-1}), \tag{10}$$

where the $\frac{b}{d} \times \frac{b}{d}$ submatrices C_j are defined by

$$diag(C_{dk}, \dots, C_{d(k+1)-1}) = (F_d \otimes I_{\frac{b}{d}})^* B_k (F_d \otimes I_{\frac{b}{d}})$$

for $k = 0, \dots, c-1$ and B_k is given by (5). Thus S can be inverted in $\mathcal{O}(cb(\frac{b^2}{d^2} + b \log d))$ operations.

Thus S is unitarily equivalent to a block-diagonal matrix with blocks of size $\frac{b}{\gcd(b,N)} \times \frac{b}{\gcd(b,N)}$ and the blocks have periodicity $\gcd(a, M)$. Using Theorem 3.3 we can easily compute the number of different eigenvalues of the frame operator.

COROLLARY 3.4. *S has at most $\gcd(ab, L)$ different eigenvalues. Furthermore if g is real-valued, the maximal number of eigenvalues reduces to $\frac{bc}{d}\lceil \frac{d+1}{2} \rceil$.*

There are two special cases of Theorem 3.3, which deserve to be mentioned separately. The first case is $\gcd(a, M) = 1$, the second is $\gcd(a, M) = a$. The second choice leads to the case of integer oversampling and will be treated in the next section. The first case provides a simple but nice factorization of S.

COROLLARY 3.5. *Assume that $\gcd(a, M) = 1$, then the Gabor frame operator S is unitarily equivalent to $I_M \otimes B_0$.*

Remark: Applying Proposition 3.1 to the matrix $D_H = \mathrm{diag}(V_0, \dots, V_{a-1})$ defined in (8) yields that the number of (up to permutations) different blocks V_k is given by $\gcd(a, M)$. Hence the costs to compute the dual ${}^\circ\gamma$ do not depend whether we choose the frame operator factorization of Theorem 2.1 or the factorization based on the Wexler-Raz condition. In Subsection 4 we will make use of the results presented here in connection with the conjugate gradient method.

4 Conjugate Gradient Methods for Gabor Expansions

In this section we discuss an efficient iterative approach for computing ${}^\circ\gamma$, which heavily relies on unitary factorizations of S and H respectively. We have seen in Section 3 that S can be factorized into a block diagonal matrix where the block size depends on number theoretic relations of L, a, b. If the blocks are of large size, iterative methods for the computation of ${}^\circ\gamma$ may be preferable. Another reason why we consider iterative methods is that in applications it is often sufficient to compute only an approximate solution to the dual window. Iterative methods for Gabor expansions have also been discussed in.[18,5,6,12,13,30]

The conjugate gradient method is a particularly efficient method for solving Hermitian positive definite linear systems of equations. S is obviously Hermitian, and it is positive definite whenever the rank of the row space of G equals L. For a detailed introduction to the conjugate gradient method and its variants the reader is referred to.[19,4]

THEOREM 4.1 (THEOREM AND ALGORITHM). *Given a function g of length L, the time- and frequency shift parameters a, b such that $ab \leq L$ and the corresponding frame operator S Initialize $r_0 = g, c_0 = 0, n = 0$ and compute γ_n iteratively by the conjugate gradient method.*[19] *Then, assuming exact arithmetics, γ_n converges in at most $\gcd(ab, L)$ steps to the minimal norm dual ${}^\circ\gamma$. If in addition the window g is real and symmetric, CG terminates after at most $c\frac{b}{d}\lceil\frac{d+1}{2}\rceil$ iterations.*

Proof: The number of iterations needed to terminate for conjugate gradients is given by the number of different eigenvalues of S, see e.g..[4] Corollary 3.4 says that the number of different eigenvalues of S is at most $\gcd(ab, L)$ and for real-valued windows at most $c\frac{b}{d}\lceil\frac{d+1}{2}\rceil$. ❏

In each iteration, besides the inner products, a matrix vector multiplication Sq_n is required. It is this product, where the various unitary factorizations of S come into play. For clarity of presentation we restrict ourselves in the sequel to the factorization proposed in Theorem 2.1. Since $P_{M,L}$ is a unitary matrix, solving $S^\circ\gamma = g$ is equivalent to solving $D_S\tilde{\gamma} = \tilde{g}$ where D_S is the block diagonal matrix of (6), $\tilde{\gamma} = P^*_{M,L}{}^\circ\gamma$ and $\tilde{g} = P^*_{M,L}g$. Since the M blocks of D_S are of size $b \times b$, the product $D_S q_n$ can be computed in $\mathcal{O}(bL)$ operations. By Theorem 4.1 the maximal number of iterations is given by $\gcd(ab, L)$ the overall computational effort for CG is $\mathcal{O}(\gcd(ab, L)bL)$ operations. If the eigenvalues of D_S are additionally clustered (besides the clustering discussed in the proof above), CG will converge much faster.

Remarks:

- Note that we could also apply CG to $H\gamma = e_{ab}$ by switching to the system of normal equations.[4]
- For convergence properties of CG in presence of round off errors we refer to.[4]
- For $ab \ll L$ a good inital guess for γ_0 is $\gamma_0 = \frac{ab}{L}g$ where $\|g\| = 1$, since $\frac{L}{ab}{}^\circ\gamma$ converges to g for $ab \to 1$.
- CG has turned out to be an efficient method in conjunction with irregular Gabor expansions, cf..[16] For more details on irregular Gabor expansions we refer to.[14,15,20,23,32,2]

4.1 Block Conjugate gradient algorithm

Since $D_S = \mathrm{diag}(B_0, \ldots, B_{M-1})$, the solution of $D_S\tilde{\gamma} = \tilde{g}$ can be computed by solving each subsystem

$$B_k\tilde{\gamma}_k = \tilde{g}_k \tag{11}$$

separately. Here $\tilde{g}_k(n) = \tilde{g}(bk + n)$ for $n = 0, \ldots, b-1$, $k = 0, \ldots, M-1$. Note that the B_k are Hermitian, since S and hence D_S is Hermitian. From Theorem 3.3 we know that B_k coincides with B_{k+c} up to permutations, where $c\gcd(a, M)$. That means we can rearrange each subsystem (11) by permutations, such that the matrix of the k-th system is identical to that of the $k + c$-th system. Thus if $\gcd(a, M) = 1$, we have to invert only one matrix of size $b \times b$, followed by M matrix-vector multiplications to obtain ${}^\circ\gamma$. However if we solve the system $D_S\tilde{\gamma} = \tilde{g}$ iteratively by CG, we cannot restrict ourselves to the solution of a single system $B_k\tilde{\gamma}_k = \tilde{g}_k$, since $\tilde{g}_k \neq \tilde{g}_j$ for $k \neq j$ and therefore $\tilde{\gamma}_k \neq \tilde{\gamma}_j$. Thus we have to solve linear systems with multiple right-hand sides but with the same system matrix.

Unfortunately CG does not return the inverse B_k^{-1}, thus we would have to apply CG to each subsystem (11) separately. Hence the question arises if the CG method can be modified such that we can take profit out of the fact that system matrix does not change in (11). This leads immediately to the so-called *block conjugate gradient method* (BCG).[25]

The block CG method solves the matrix equation

$$AX = Y \tag{12}$$

in at most $\lceil n/p \rceil$ iterations and may involve less work than applying the conjugate gradient method p times.[25] For our problem, the matrix A in (12) is given by B_k, X and Y contain permuted versions of the vectors $\tilde{\gamma}_k$ and $\tilde{g}_k$, respectively. More details on the block conjugate gradient method (e.g. efficient parallel implementations) can be found in.[25,9,24]

5 Preconditioners and Approximate Inverses

The convergence rate of the conjugate gradient method depends on the condition number of the matrix and how clustered the spectrum of the matrix is. If the spectrum of S is not clustered or if the condition number $\kappa(S)$ is large the convergence will be slow. One way to speed up the convergence rate of the method is to precondition the system. Thus instead of solving $S\gamma = g$ we solve the preconditioned system $C^{-1}S\gamma = C^{-1}g$. The preconditioning matrix C should satisfy following criteria

- $\kappa(C^{-1}S) \ll \kappa(S)$.
- The spectrum of $C^{-1}S$ should be clustered.
- C should be constructed within $\mathcal{O}(L \log L)$ operations.
- $Cx = y$ should be solved in $\mathcal{O}(L \log L)$ operations.

Circulant matrices play an important role as preconditioners, see[7] and the references cited there for a detailed discussion of preconditioners. Among the circulant preconditioners, T. Chan's optimal Frobenius norm preconditioner, originally proposed for Toeplitz systems,[8] is particularly useful. For a Toeplitz matrix S it is defined to be the minimizer of

$$\|C - S\|_F$$

over all $L \times L$ circulant matrices C. Here $\|\,.\,\|_F$ denotes the Frobenius norm of a matrix.

T. Chan's preconditioner is still useful, if S is not a Toeplitz matrix. The k-th diagonal of $C(S)$ are easily obtained by the arithmetic average of the k-th diagonal of S, where the diagonals of S are wrapped around. More precisely the entries of the $_k$-th diagonal of $C(S)$ is given by

$$c_k = \frac{1}{L} \sum_{m-n=k \bmod L} S_{m,n} \qquad \text{for } k = 0, \ldots, L-1.$$

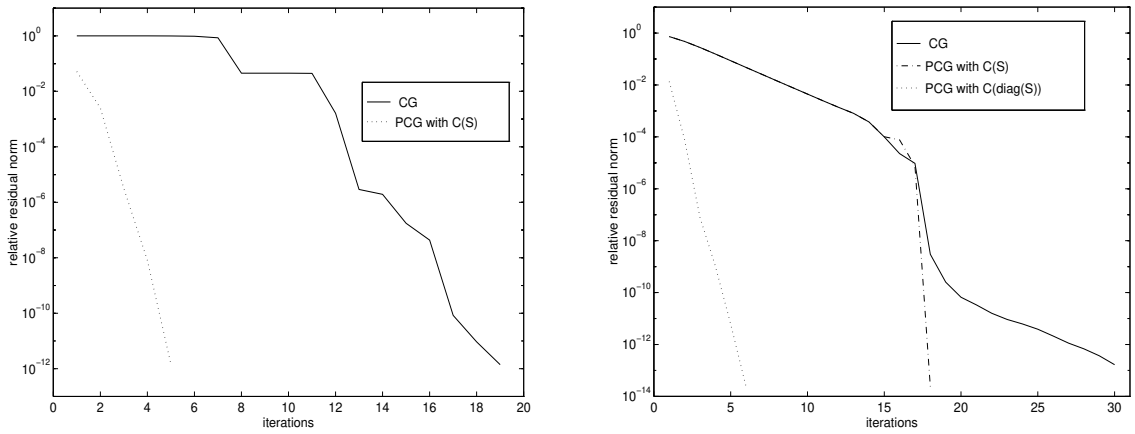
T. Chan's preconditioner is particularly useful in the context of Gabor analysis, due to following observations.

If the window g has small support compared to the size of the time-frequency lattice parameters, then the frame operator S is just a multiplication operator, which is to say a diagonal matrix. This fact was first observed by Walnut, see e.g.,[3] its discrete analogue can be found in.[31]

PROPOSITION 5.1. *Given a window g, and lattice parameters a and b. If the length s_g of the support of g satisfies $s_g \leq M$, then the frame operator S is a diagonal matrix. If the length $s_{\hat{g}}$ of the support of $\hat{g}$ satisfies $s_{\hat{g}} \leq N$ then S is a circulant matrix.*

Proof: We assume without loss of generality that the support of g is located at the first consecutive integers $\{0, 1, \ldots, s_g - 1\}$, i.e. $g(k) = 0$ for $s_g \leq k \leq L-1$. The support condition $s \leq M$ implies

$$\sum_{k=0}^{N-1} T_{ka}\, g(m) \overline{T_{ka}\, g(n)} = 0, \qquad \text{for } |m-n| \geq M \geq s_g - 1\,,$$



(a) (b)

Figure 1: Convergence rate of PCG for two different Gabor setups. In the first experiment we have used $\alpha = 2, a = 15, b = 15, L = 240$, in the second experiment we have used $\alpha = 8, a = 30, b = 6, L = 240$.

whence

$$S_{m,n} = 0 \quad \text{for } |m-n| \geq M$$

thus S is a diagonal matrix. The second assertion follows from Lemma 2.6. ∎

Evidently, if $\hat{g}$ satisfies the "small support" condition, $C(S) = S$. An immediate consequence of Lemma 2.6 is

$$C(\hat{S}) = F \operatorname{diag}(S) F^* .$$

In general g will not satisfy one of the conditions of Proposition 5.1 in a strict sense. If we assume that the Fourier transform of g is a rapidly decaying function, such that $\hat{g}(k)$ is small outside the interval $[-\frac{N}{2}, \frac{N}{2}]$, then as a consequence of Lemma 2.6, the off-diagonal entries of $S_{\hat{g}}$ will also be small and $S_{\hat{g}}$ will be strictly diagonal dominant. Since $S = F_L \hat{S} F_L^*$, it follows that the eigenvalues of $C^{-1}(S)S$ will cluster around 1, if the length of the support of $\hat{g}$ approaches N. Thus for rapidly decaying window having most of the energy of its Fourier transform concentrated within the interval $[-\frac{N}{2}, \frac{N}{2}]$, $C^{-1}S$ will serve as an approximate inverse of S. By duality the same arguments hold for a narrow window g and in this case the eigenvalues of $C^{-1}(\operatorname{diag}(S))$ will cluster around 1, if the length of the support of g approaches M.

Note that $C(S)$ can be constructed and inverted in $\mathcal{O}(b \log b)$ operations, thus using the preconditioned conjugate gradients method with $C(S)$ as preconditioner only slightly increases the computational effort in one iteration step.

Figure 3 illustrates the efficiency of applying the preconditioners $C(S)$ and $C(\operatorname{diag}(S))$. We compute the dual ${}^\circ\gamma$ by the preconditioned conjugate gradient method (PCG) for two different situations. In the first experiment we have used a Gaussian window which is well localized in the frequency domain ($L = 240, \alpha = 2$ lattice parameters are $a = b = 15$). Having in mind the observations discussed above, we can expect that applying the preconditioner $C(S)$ will considerably improve the clustering of the eigenvalues of S. The convergence rates of CG and PCG are displayed in Figure 3(a).

In the second experiment we have used a narrow Gaussian window ($L = 240, \alpha = 8$, lattice parameters are $a = 30, b = 6$). $C(S)$ still improves the clustering of the singular values of S, but evidently $C(\mathrm{diag}(S))$ has the edge in this situation. The corresponding convergence rates are shown in Figure 3(b).

One might argue that for the case where g and $\hat{g}$ are approximately equally well concentrated, a "double preconditioning" strategy might useful. That means we first compute $A = C^{-1}(S)S$ followed by applying $C(\mathrm{diag}(A))$. However since $C^{-1}(S)S$ is no longer Hermitian and therefore does not induce an inner product and CG cannot be applied directly. If we still want to use CG, we can switch to the system of normal equations.[19] Since this approach is handicapped by squaring of the condition number, it will considerably reduce the positive effect of the additional preconditioner. However double preconditioning may still be useful in some cases to obtain an approximate solution to ${}^{\circ}\gamma$ or a good initial guess γ_0 for the CG method.

Acknowledgement

The author wants to thank Hans G. Feichtinger for fruitful discussion on this topic. The work presented in this paper has been partially supported by the Hochschuljubiläumsstiftung der Stadt Wien H-00199/95 and project S7001-MAT of the Austrian Science foundation FWF.

6 REFERENCES

[1] L. Auslander, I.C. Gertner, and R. Tolimieri. The Discrete Zak Transform Application to Time-Frequency Analysis and Synthesis of Nonstationary Signals. *IEEE Trans. Signal Proc.*, 39:825–835, 1991.

[2] J.J. Benedetto, C. Heil, and D.F. Walnut. Differentiation and the Balian–Low Theorem. *J. Four. Anal. Appl.*, 1(4):355–403, 1995.

[3] J.J. Benedetto and D.F. Walnut. Gabor frames for L^2 and related spaces. In J.J. Benedetto and M.W. Frazier, editors, *Wavelets: Mathematics and Applications*, pages 97–162. CRC Press, Boca Raton, 1994.

[4] A. Björck. *Numerical Methods for Least Squares Problems.* SIAM, 1996.

[5] R.N. Braithwaite and M.P. Beddoes. Iterative Methods for Solving the Gabor expansion: Considerations of Convergence. *IEEE Trans. Image Proc.*, 1/2:243–244, 1992.

[6] R.N. Braithwaite and M.P. Beddoes. Selecting the projection function for an iterative Gabor expansion. In *Proc. Conf. SPIE 2034, Mathematical Imaging: Wavelet Applications and Image Processing, San Diego, 1993*, SPIE, 1993.

[7] R. Chan and M. Ng. Conjgate gradient methods for Toeplitz systems. *SIAM Review*, 38(3):427–482, 1996.

[8] T. Chan. An optimal circulant preconditioner for Toeplitz systems. *SIAM J. Sci. Stat. Comput.*, 9:766–771, 1989.

[9] T. Chan and W. Wan. Analysis of Projection Methods for Solving Linear Systems with Multiple Right-hand Sides. *SIAM J. Sci. Comp.*, to appear.

[10] O. Christensen. Frames and pseudo-inverses. *J. Math. Anal. Appl.*, 195:401–414, 1995.

[11] P.J. Davis. *Circulant Matrices.* John Wiley, 1979.

[12] H.G. Feichtinger. Optimal iterative algorithms in Gabor analysis. In *Proc. IEEE-SP Int. Symp. on Time-Frequency and Time-Scale Analysis*, pages 44–47, Philadelphia/PA, Oct. 1994.

[13] H.G. Feichtinger, O. Christensen, and T. Strohmer. A group-theoretical approach to Gabor analysis. *Opt. Engrg.*, 34:1697–1704, 1995.

[14] H.G. Feichtinger and K. Gröchenig. Gabor wavelets and the Heisenberg group: Gabor Expansions and Short Time Fourier transform from the group theoretical point of view. In C.K. Chui, editor, *Wavelets – A Tutorial in Theory and Applications*, pages 359–397. Academic Press, Boston, 1992.

[15] H.G. Feichtinger and K. Gröchenig. Non-Orthogonal wavelet and Gabor expansions, and group representations. In G. Beylkin, R. Coifman, I. Daubechies, et al., editors, *Wavelets and their applications*, pages 353–376. Jones and Bartlett, 20 Park Plaza, Boston, MA 02116, USA, 1992.

[16] H.G. Feichtinger, W. Kozek, and T. Strohmer. Reconstruction of signals from irregular samples of its short time Fourier transform. In *Proc. SPIE'95: Wavelets Applications in Signal and Image Processing*, San Diego, 1995. to appear.

[17] H.G. Feichtinger and T. Strohmer, editors. *Gabor Analysis and Algorithms: Theory and Applications*. Birkhäuser, Boston, 1997. to appear.

[18] T. Genossar and M. Porat. Can one evaluate the Gabor expansion using Gabor's iterative algorithm? *IEEE Trans. Sign. Proc.*, 40(8):1852–1861, 1992.

[19] G.H. Golub and C.F. van Loan. *Matrix Computations, third ed.* Johns Hopkins, London, Baltimore, 1996.

[20] K. Gröchenig. Irregular sampling of wavelet and short time Fourier transforms. *Constructive Approx. Theory.*, 9:283–297, 1993.

[21] C. Heil and D. Walnut. Continuous and discrete wavelet transforms. *SIAM Review*, 31(4):628–666, 1989.

[22] A.J.E.M. Janssen. Duality and biorthogonality for discrete-time Weyl-Heisenberg systems. Technical Report Nat.Lab. UR 0002/94, PRL Eindhoven, Eindhoven, February 1994.

[23] H.J. Landau. On the density of phase-space expansions. *IEEE Trans. Info. Theory*, 39:1152–1156, 1993.

[24] A.A. Nikishin and A. Y. Yeremin. Variable block CG algorithms for solving large sparse symmetric positive definite linear systems on parallel computers, I: General iterative scheme. *SIAM J. Matrix Anal. Appl.*, 16(4):1135–1153, 1995.

[25] D.P. O'Leary. The Block Conjugate Gradient Algorithm and Related Methods. *Linear Algebra and its Applications*, 29:293–322, 1980.

[26] R.S. Orr. Derivation of the Finite Discrete Gabor Transform by periodization and sampling. *Signal Proc.*, 34(1):85–97, 1993.

[27] P. Prinz. Calculating the dual Gabor window for general sampling sets. *IEEE Trans. Signal Proc.*, 44(8):2078–2082, 1996.

[28] S. Qian and D. Chen. *Joint Time-Frequency Analysis: Method and Application*. Prentice Hall, Englewood Cliffs, NJ, 1996.

[29] S. Qiu. Block-Circulant Gabor-matrix structure and discrete Gabor transforms. *Opt. Engrg.*, pages 2872–2878, 1995.

[30] S. Qiu and H.G. Feichtinger. Discrete Gabor structure and optimal representation. *IEEE Trans. Signal Proc.*, 43(10):2258–2268, October 1995.

[31] S. Qiu, H.G. Feichtinger, and T. Strohmer. Inexpensive Gabor Decompositions. In A.F. Laine and M. Unser, editors, *Proc. SPIE'94: Wavelets Applications in Signal and Image Processing*, pages 286–295, San Diego, 1994.

[32] J. Ramanathan and T. Steger. Incompleteness of sparse coherent states. *Appl. Comp. Harm. Anal.*, 2:148–153, 1995.

[33] D.F. Stewart, L.C. Potter, and S.C. Ahalt. Computationally attractive real Gabor transforms. *IEEE Trans. SSP*, pages 77–84, January 1995.

[34] G. Strang. *Linear Algebra and Its Applications*. Academic Press, New York, second edition, 1980.

[35] T. Strohmer. Numerical algorithms for discrete Gabor expansions. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*. Birkhäuser, Boston, 1997. to appear.

[36] C.F. van Loan. *Computational Frameworks for the Fast Fourier Transform*. SIAM, 1992.

[37] J. Wexler and S. Raz. Discrete Gabor Expansions. *Signal Processing*, 21(3):207–221, November 1990.

[38] J. Yao, P. Krolak, and C. Steele. The Generalized Gabor Transform. *IEEE Trans. Image Proc.*, 4(7):978–988, July 1995.

[39] M. Zibulski and Y.Y. Zeevi. Frame analysis of the discrete Gabor scheme. *IEEE Trans. Signal Proc.*, 42(4):942–945, 1994.