# Painless approximation of dual frames, with applications to shift-invariant systems

Thomas Strohmer
Department of Mathematics, University of California
Davis, CA 95616, USA

## ABSTRACT

We analyse the relation between infinite-dimensional frame theory and finite-dimensional models for frames as they are used for numerical algorithms. Special emphasis in this paper is on perfect reconstruction oversampled filter banks, also known as shift-invariant frames. For certain finite-dimensional models it is shown that the corresponding finite dual frame provides indeed an approximation of the dual frame of the original infinite-dimensional dual frame. For filter banks on $\boldsymbol{\ell}^2(\mathbb{Z})$ we derive error estimates for the approximation of the synthesis filter bank when the analysis filter bank satisfies certain decay conditions. We show how one has to design the finite-dimensional model to preserve important structural properties of filter banks, such as polyphase representation. Finally an efficient regularization method is presented to solve the ill-posed problem arising when approximating the dual frame on $\boldsymbol{L}^2(\mathbb{R})$ via truncated Gram matrix.

**Keywords:** frames, finite sections, filter banks, Toeplitz matrix, dual frame

## 1. INTRODUCTION

Frames have become an important and frequently used concept in digital signal processing and wavelet analysis.[1,2] Many deep mathematical results have been published in the last decade concerning the existence of frames (such as wavelet frames or Gabor frames) and the estimation of frame bounds. These results are almost exclusively derived in infinite-dimensional spaces of functions such as $\boldsymbol{\ell}^2(\mathbb{Z})$ or $\boldsymbol{L}^2(\mathbb{R})$. Any numerical implementation of frame methods however has to use a finite-dimensional model. Despite the increasing popularity of frames only little is known about the relation between infinite-dimensional frame theory and the finite-dimensional models used for numerical frame algorithms.

The purpose of this paper is to clarify this relation. The key problem thereby is the computation of the dual frame. How well do the finite-dimensional "dual frame functions" approximate the infinite-dimensional dual frame? Do they approximate it at all? Can we say anything about the approximation error? These are the questions we want to answer in this paper.

Filter banks play an important role in digital signal processing and communication .[3–5] Therefore special emphasis in this paper is given to the computation of dual filter bank frames. We will analyse which finite-dimensional model allows to preserve important structural properties of filter bank theory. Our results validate existing numerical filter bank algorithms based on the polyphase representation. Explicit convergence rates are derived for the approximation of synthesis filter banks when the analysis filter bank satisfies various types of decay.

Finally we discuss the computation of dual frames for $\boldsymbol{L}^2(\mathbb{R})$. Here a standard method to approximate the dual frame is based on the truncated frame Gram matrix. We propose an efficient regularization method to solve this ill-posed problem which in turn allows the approximation of the dual frame.

Before we proceed we briefly review some basic facts on frames and filter banks. Recall that a a set of functions $\{f_k\}_{k\in\mathbb{Z}}$ is said to be a frame for a separable Hilbert space $\boldsymbol{H}$ if there exist constants (the *frame bounds*) $A, B > 0$ such that

$$A\|f\|^2 \le \sum_k |\langle f, f_k\rangle|^2 \le B\|f\|^2 . \qquad \forall f \in \boldsymbol{H}. \tag{1}$$

Email: strohmer@math.ucdavis.edu

We define the *analysis operator* $T$ by

$$T : f \in \boldsymbol{H} \to Ff = \{\langle f, f_k \rangle\}_{k\in\mathbb{Z}} \,, \tag{2}$$

and the *synthesis operator*, which is just the adjoint operator of $T$, by

$$T^* : c \in \boldsymbol{\ell}^2(\mathbb{Z}) \to T^* c = \sum_k c_k f_k \,. \tag{3}$$

For a given frame $\{f_k\}_{k\in\mathbb{Z}}$ there exists a *dual frame* $\{h_k\}_{k\in\mathbb{Z}}$ such that any $f \in \boldsymbol{H}$ can be written as

$$f = \sum_{k\in\mathbb{Z}} \langle f, f_k \rangle h_k = \sum_{k\in\mathbb{Z}} \langle f, h_k \rangle f_k \,. \tag{4}$$

The "canonical" choice for the dual frame is

$$h_k = S^{-1} f_k \,, \tag{5}$$

where $S$ is the so-called frame operator

$$Sf = T^*Tf = \sum_{k\in\mathbb{Z}} \langle f, f_k \rangle f_k \,. \tag{6}$$

We will also make use of the operator $TT^*$ in form of its Gram matrix representation $R : \boldsymbol{\ell}^2(\mathbb{Z}) \to \boldsymbol{\ell}^2(\mathbb{Z})$ with entries $R_{k,l} = \langle f_k, f_l \rangle$. On $\mathcal{R}(T) = \mathcal{R}(R)$ the matrix $R$ is bounded by $AI \le R \le BI$ and invertible. On $\boldsymbol{\ell}^2(\mathbb{Z})$ this inverse extends to the *Moore-Penrose inverse* $R^+$.

In this paper we put special emphasis on so-called *shift-invariant frames* also known as *oversampled filter banks*.[4]

A uniform filter bank is a system of functions $\{g_{m,n}\}$ of the form

$$g_{m,n}(k) = g_m(k - na), \qquad k, n \in \mathbb{Z}, m = 0, 1, \dots, M-1, \tag{7}$$

where $g_m \in \boldsymbol{\ell}^2(\mathbb{Z})$ and $0 < a \in \mathbb{N}$.

For a given *analysis filter bank* $\{g_{m,n}\}$ we want to find a *synthesis filter bank* $\{\gamma_{m,n}\}$ with

$$\gamma_{m,n}(k) = \gamma_m(k - na), \qquad k, n \in \mathbb{Z}, m = 0, 1, \dots, M-1, \tag{8}$$

such that any $f \in \boldsymbol{\ell}^2(\mathbb{Z})$ has an $\boldsymbol{\ell}^2(\mathbb{Z})$-convergent representation

$$f = \sum_{m=0}^{M-1} \sum_{n=-\infty}^{\infty} \langle f, \gamma_{m,n} \rangle g_{m,n} \,. \tag{9}$$

If a filter bank satisfies (1) we also call it a *perfect reconstruction* filter bank.[4] If not mentioned otherwise, we will always assume in this chapter that the filter bank $\{g_{m,n}\}$ is a frame for $\boldsymbol{\ell}^2(\mathbb{Z})$, in which case equation (9) always holds.

Of course, if $\{g_{m,n}\}$ is an orthogonal basis for $\boldsymbol{\ell}^2(\mathbb{Z})$ we have $\gamma_{m,n} = g_{m,n}$. In the case of a nonorthogonal filter bank and in particular for an oversampled filter bank (i.e. $a < M$) the determination of $\gamma_{m,n}$ is more involved. It is well-known[4,6] that the theoretical answer when and how we can find such synthesis functions $\gamma_{m,n}$ can be given using the concept of frames in $\boldsymbol{\ell}^2(\mathbb{Z})$. When implementing filter banks on a computer however, we cannot directly make use of the infinite-dimensional theory, but have to set up a finite-dimensional model. How to choose a correct finite-dimensional model such that the corresponding finite synthesis functions actually converge to the synthesis functions of the original infinite-dimensional model is discussed in the following sections.

If the $g_m$ are constructed from a single function $g$ by complex modulations, i.e.,

$$g_m(k) = e^{2\pi i m k/M} g(k)$$

we obtain the *oversampled DFT filter bank*,[4,7,6] also known under the name *Gabor frame* or *Weyl-Heisenberg frame*.[8,5,4,9,6] A nice property of oversampled DFT filter banks is that the synthesis filter bank is also generated by a single function, we have[10]

$$\gamma_{m,n}(k) = e^{2\pi i mk/M}\gamma(k-na) \qquad \text{with } \gamma = S^{-1}g \; .$$

Let $G_m$ be the transfer function of the impulse response $g_m$. The *polyphase analysis matrix* $\mathbf{G}$ of the analysis filter bank is defined as

$$\mathbf{G}(e^{2\pi i\vartheta}) = \begin{bmatrix} \mathbf{G}_{00}(e^{2\pi i\vartheta}) & \dots & \mathbf{G}_{0(a-1)}(e^{2\pi i\vartheta}) \\ \mathbf{G}_{10}(e^{2\pi i\vartheta}) & \dots & \mathbf{G}_{1(a-1)}(e^{2\pi i\vartheta}) \\ \dots & \dots & \dots \\ \mathbf{G}_{(M-1)0}(e^{2\pi i\vartheta}) & \dots & \mathbf{G}_{(M-1)(a-1)}(e^{2\pi i\vartheta}) \end{bmatrix} , \tag{10}$$

where

$$\mathbf{G}_{mn}(e^{2\pi i\vartheta}) = \sum_{k=-\infty}^{\infty} g_m(n-ka)e^{2\pi i\vartheta k} , \qquad \vartheta \in [0,1) , \tag{11}$$

represents the $n$-th polyphase component of $\mathbf{G}(e^{2\pi i\vartheta})$, see [3,4]. The reader will recognize the *Zak transform*[11] in formula (11). A similar representation holds for the synthesis filter bank. Using equations (2),(10), and (11) we conclude that the matrix

$$\mathbf{S}(e^{2\pi i\vartheta}) = \mathbf{G}^H(e^{2\pi i\vartheta})\mathbf{G}(e^{2\pi i\vartheta}) \tag{12}$$

is a matrix representation of the frame operator $S$, cf..[12,6] In mathematical terminology this simply means that $S$ is a *Laurent operator* with matrix-valued symbol.[13] We will return to this interpretation in the following sections.

## 2. FINITE-DIMENSIONAL MODELS FOR FILTER BANKS

The numerical computation of the synthesis functions $\gamma_{m,n} = \gamma_m(\cdot - na)$ involves the solution of the biinfinite system of equations

$$Sg_m = \gamma_m \, . \tag{13}$$

We certainly cannot solve a system of biinfinite order, but we can solve it approximately, based on a finite-dimensional model.

Of course, the $\gamma_{m,n}$ can also be computed by means of the polyphase representation of the frame operator. But it is clear that a unitary transformation of $S$ (or the analysis operator $F$ respectively) does not avoid the difficulty of approximating an infinite-dimensional problem by a finite-dimensional model. We will return to the polyphase representation and its use in finite dimensional models in Section 2.3. We will study three frequently used approaches for the implementation of oversampled filter banks. These approaches and the corresponding finite-dimensional models can be described as follows (we give a more precise mathematical formulation in the following sections):

(i) Truncate the functions $g_{m,n}$ to the interval $[-N,N]$ and pad them with zeros outside the interval $[-N,N]$. Then an approximation to $\gamma_m$ is computed by applying formula (5) to the truncated system.

(ii) Establish the polyphase matrix $\mathbf{S}$ (or an approximation to it if the $g_m$ have not finite impulse response) of the frame operator $S$, compute an approximation to its inverse by sampling and inverting $\mathbf{S}(e^{2\pi i\vartheta})$, and transform the result back in the time domain. As the reader may notice (or otherwise will see in the following sections) this approach actually makes implicitly use of following finite-dimensional model.

(iii) Truncate the functions $g_{m,n}$ to the interval $[-N,N]$ and extend the truncated functions periodically across the boundaries. An approximation to $\gamma_m$ is computed based on formula (5) by inverting the corresponding truncated periodized frame operator.

We will see that each of these approaches has its advantages and disadvantages. On the one hand model (i) is somewhat easier to analyze and gives a better rate of convergence than those models used in (ii) and (iii). On the other hand model (iii) preserves perfect symbol calculus for Laurent operators which is important when we want to use the polyphase representation. Models (ii) and (iii) are also preferable from a numerical point of view, since they give rise to more efficient algorithms.

Note that for some special choices of $\{g_{m,n}\}$ closed-form solutions for $\gamma_{m,n}$ are known[10,14,15] (which does not necessarily mean that those closed-form solutions can be cheaply computed in practice). We continue by first investigating approach (i) in more detail.

### 2.1. The finite section method for oversampled filter banks

Equations (5) and (6) tell us that all $g_{m,n}$ are required for the computation of the synthesis function $\gamma_m$. But if the $g_m$ have nice decay properties, such that most of their energy is contained in some interval around the origin, say, then one may hope that the $\gamma_m$ will have similar properties. One could further argue that using only those functions $g_{m,n}$ concentrated in some interval, $[-N, N]$ say, should be sufficient to compute a good approximation to $\gamma_m$.

Let us put these vague ideas in a more precise mathematical setting. For $N \in \mathbb{N}$ and $x \in \boldsymbol{\ell}^2(\mathbb{Z})$ define the orthogonal projections $P_N$ by

$$P_N x = (\dots, 0, 0, x_{-N}, x_{-N+1}, \dots, x_{N-1}, x_N, 0, 0, \dots) \tag{14}$$

and identify the image of $P_N$ with the $2N+1$-dimensional space $\mathbb{C}^{2N+1}$. We construct a finite-dimensional approximation $S_N$ of the frame operator $S$ via the truncated frame elements $P_N g_{m,n}$ by

$$S_N = (P_N F^*)(F P_N) \tag{15}$$

where $F$ is defined in (2). The $N$-th approximation $\gamma_m^{(N)}$ to $\gamma_m$ is then given by the solution of the finite-dimensional system of equations

$$S_N \gamma_m^{(N)} = P_N g_m \,. \tag{16}$$

Since $S_N = P_N F^* F P_N = P_N S P_N$, the corresponding system of equations $S_N \gamma_m^{(N)} = g_m^{(N)}$ is just a finite section of the biinfinite system $S\gamma = g$.

The questions that arise with such an approximation scheme are:
(i) Does $\gamma_m^{(N)}$ converge to $\gamma_m$ for $N \to \infty$? Or more general, is the finite section method is applicable to $S$?
(ii) If the $g_m$ satisfy certain decay conditions, can we give an estimate on the rate of convergence? In other words: given $\delta > 0$, can we provide an a priori estimate for the size of the interval $[-N, N]$, such that $\|\gamma_m - \gamma_m^{(N)}\| \le \delta$, where $N$ depends on $\delta$ and the decay of $g_m$?

Let $\tau$ be a continuous $a \times a$-matrix valued function on $\mathbb{T}$, $\tau \in \boldsymbol{C}_{a\times a}(\mathbb{T})$. Let $\tau_k \in \mathbb{C}^{a\times a}$ stand for the $k$-th Fourier coefficient of $\tau$, then the operator $T$ defined by the action

$$(Tx)_k = \sum_{j\in\mathbb{Z}} \tau_{j-k} x_j \,, \qquad x \in \boldsymbol{\ell}^2{}_a \,, \tag{17}$$

is called a *block Laurent operator* with generating function $\tau$. A block Laurent operator can be represented by a biinfinite block Toeplitz matrix where the entries $\tau_{k,l}$ are $a \times a$ matrices. For the approximate solution of biinfinite block Toeplitz systems we consider finite sections of the form

$$P_N T P_N x^{(N)} = P_N y \,.$$

It follows directly from the definition of the functions $g_{m,n}$ in (7) that the filter bank analysis operator $F$ defined in (2) satisfies $F = U_M F U_a^*$, where $U_a(\varphi_k)_{k\in\mathbb{Z}} := (\varphi_{k-a})_{k\in\mathbb{Z}}$ is the shift-operator of order $a$. Thus the associated frame operator $S$ is a block Laurent operator and the operator equation $S\gamma_m = g_m$ can be expressed as a biinfinite block Toeplitz system of equations, where the blocks of the system matrix are of size $a \times a$. Note that the blocks themselves are not of Toeplitz type. It is exactly this block-Laurent property of $S$ that is utilized in the polyphase representation. The symbol $\tau$ corresponds to the polyphase matrix $\mathbf{S}$ of equation (12).

## 2.2. Rates of approximation for oversampled filter banks

We first show that the finite section method outlined in Section 2.1 applies to the frame operator of *any* frame in $\ell^2(\mathbb{Z})$. Recall that a sequence of linear operators $T_n : \boldsymbol{B} \to \boldsymbol{B}$ converges strongly to $T$ if for each $f \in \boldsymbol{B}$ the sequence $\boldsymbol{B}f$ converges to $Tf$. If the operators $T_n$ are invertible it is naturally to ask under which conditions the operators $T_n^{-1}$ converge strongly to $T^{-1}$. Following result is attributed to Kantorovich.[16]

PROPOSITION 2.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of invertible operators on a Banach space $\boldsymbol{B}$, and assume that the sequence is uniformly bounded above and below, i.e, that there exist constants $0 < C_1, C_2 < \infty$ such that*

$$C_1 \|f\|_{\boldsymbol{B}} \le \|T_n f\|_{\boldsymbol{B}} \le C_2 \|f\|_{\boldsymbol{B}} \qquad \text{for all } n\in\mathbb{N} \text{ and } f \in \boldsymbol{B}. \tag{18}$$

*Then $\{T_n\}$ converges in the strong operator topology to an invertible operator if and only if the same is true for $\{T_n^{-1}\}_n$, and then*

$$\{\lim T_n\}^{-1} = \lim\{T_n^{-1}\}\,.$$

Now it is easy to prove following theorem.[17]

THEOREM 2.2. *Assume that $\{f_k\}_{k\in\mathbb{Z}}$ is a frame for $\ell^2(\mathbb{Z})$ with frame bounds $A, B$ and dual frame $\{h_k\}_{k\in\mathbb{Z}}$. Set $S_N = P_N S P_N$ and $h_k^{(N)} = S_N^{-1} P_N f_k$. Then the finite section method is applicable to $S$ and in particular $\|h_k^{(N)} - h_k\| \to 0$ for $N \to \infty$.*

We consider the following three types of decay for $g_m$ and study first how a certain decay property translates to decay properties of the frame operator $S$.

a) *Finite impulse response:* $g_m(k) = 0$ for $|k| > s$.
b) *Exponential decay:* $|g_m(k)| \le c\lambda^{\alpha|k|}$ for $0 < \alpha < 1$, $\lambda \in (0, 1)$ and some constant $c$.
c) *Polynomial decay:* $|g_m(k)| \le c(1 + |k|)^{-\alpha}$ for $\alpha > 1$ and a constant $c$.

Note that if the $g_{m,n}$ have finite impulse response (FIR) one cannot that the $\gamma_{m,n}$ are also of FIR type. For DFT filter banks Bölcskei has shown[18] that the minimal norm duals are only of FIR type in very specific cases. But we can show that the synthesis functions are decaying exponentially, which is "almost as good as" FIR.

PROPOSITION 2.3. *Let $\{g_{m,n}\}$ be a perfect reconstruction FIR filter bank for $\ell^2(\mathbb{Z})$ such that $g(k) = 0$ for $|k| > s$. Assume that $\|g_m\| = 1$ and let $A, B$ be the frame bounds of the filter bank. Denote $\kappa = B/A$ and $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$. Then*

$$|S_{k,l}^{-1}| \le C\lambda^{|k-l|} \tag{19}$$

*and the decay of the $\gamma_{m,n}$ is bounded by*

$$|\gamma_m(k)| \le D\lambda^{|k|} \frac{\lambda^{-s}}{1-\lambda}\,, \tag{20}$$

*where $\lambda = q^{\frac{1}{2s}}$ and $D = \frac{1}{A}\max\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\}$.* The proof of this result and of the result below can be found in [13] and makes use of the decay of the inverse of banded matrices.[19]

For exponentially or polynomially decaying filter banks we can state similar results, however we could not derive explicit expressions of the involved constants.

PROPOSITION 2.4. *Assume that the functions $g_{m,n}, m = 0, \dots, M-1$ constitute a perfect reconstruction filter bank for $\ell^2(\mathbb{Z})$ with frame operator $S$ and synthesis filter bank $\{\gamma_m^{(N)}\}$.*
*(a) Exponential decay: If $|g_m(k)| \le c\lambda^{\alpha|k|}$ for $\alpha > 0$ and $\lambda \in (0, 1)$. Then there exists an $\alpha_1 < \alpha$ and a constant $c_1$ depending on $\alpha_1$ such that*

$$|S_{k,l}| \le c_1 \lambda^{\alpha_1|k-l|}\,. \tag{21}$$

*Furthermore the duals $\gamma_m$ satisfy*

$$|\gamma_m(k)| \leq c_2 \lambda^{\alpha_2 |k|} . \tag{22}$$

*for some $\alpha_2 < \alpha$ and a constant $c_2$ depending on $\alpha_2$.*
*(b) Polynomial decay: If $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ for $\alpha > 1$, then there exists a constant $c_1$ such that*

$$|S_{k,l}| \leq c_1 (1 + |k-l|)^{-\alpha} . \tag{23}$$

*Furthermore there is a constant $c_2$ such that*

$$|\gamma_m(k)| \leq c_2 (1+|k|)^{-\alpha} . \tag{24}$$

The proof, which relies on results of Jaffard concerning the decay of the inverse of certain matrices[20] can be found in [13].

**Remark:** Let $\{f_k\}_{k\in\mathbb{Z}}$ be a frame for $\boldsymbol{L}^2(\mathbb{R})$ with dual frame $\{\varphi_k\}_{k\in\mathbb{Z}}$. Then $\varphi_k$ can be written as $\varphi_k = \sum_{k\in\mathbb{Z}} R^{-1}_{k,l} f_l$ where $R$ is the Gram matrix with entries $R_{k,l} = \langle f_k, f_l \rangle$. $R$ is invertible on the range of the frame analysis operator. Hence for filter banks on $\boldsymbol{L}^2(\mathbb{R})$ we can obtain decay results for the synthesis functions similar to the ones in Propositions 2.3 and 2.4 by considering the decay of the inverse of $R$. This will be discussed in more detail elsewhere.

We are now able to answer the question about the approximation of synthesis filter banks in $\boldsymbol{\ell}^2(\mathbb{Z})$ by the finite section method.[13]

Theorem 2.5. *Assume that $\{g_{m,n}\}$ is a perfect reconstruction filter bank for $\boldsymbol{\ell}^2(\mathbb{Z})$ with $\|g_m\| = 1$ and frame bounds $A, B$. Set $S_N = P_N S P_N$ and $\gamma_m^{(N)} = S_N^{-1} P_N g_m$ for $m = 0, 1, \dots, M-1$ and denote $\kappa = B/A$. Then following statements hold.*
*(i) The finite section method is applicable to $S$ and $\gamma_m^{(N)}$ converges to $\gamma_m$ for $N \to \infty$.*
*(ii) If the $g_m$ are compactly supported with $g_m(k) = 0$ for $|k| > s$ and $N > 2s$, then the rate of convergence can be estimated by*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \sqrt{2} C \lambda^N (\lambda^s - \lambda^{s+1})^{-3} \tag{25}$$

*where $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, $\lambda = q^{\frac{1}{2s}}$ and*

$$C = \frac{1}{A} \max\{2\kappa, (1+\sqrt{\kappa})^2\} . \tag{26}$$

*(iii) If $|g_m(k)| \leq c\lambda^{\alpha|k|}$ for $\alpha > 0$ and $\lambda \in (0,1)$, then there exists an $\alpha_1 < \alpha$ and a constant $c_1$ depending on $\alpha_1$ such that*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \kappa c_1 \lambda^{\alpha_1 N} . \tag{27}$$

*(iv) If $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ for $\alpha > 1$, then there exists a constant $c_1$ such that*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \kappa c_1 (1+|N|)^{-\alpha} . \tag{28}$$

### 2.3. Convergence using polyphase representation

Let $T(a)$ and $T(b)$ be two (block) Laurent operators on $\boldsymbol{\ell}^2(\mathbb{Z})$, then[21]

$$T(a)T(b) = T(ab) . \tag{29}$$

This well-known relation is sometimes referred to as symbol calculus for Laurent operators.[21] Of course, also the inverse $[T(a)]^{-1}$ is a Laurent operator. When we consider the truncated operators $T_N(a)$ and $T_N(b)$, then

$T_N(a)T_N(b) \neq T_N(ab)$. Furthermore $[T_N(a)]^{-1}$ is no longer a truncated Laurent operator. For oversampled filter banks this means that the truncated frame operator no longer possesses a polyphase representation.

We can overcome this drawback by a periodical extension of the truncated functions $P_N g_{m,n}$. In fact, the periodic extension of finite signals is more than just a simple and convenient way to handle boundary problems. It is in some sense the "canonical" way to set up a discrete model, when translations come into play, since it allows to preserve important mathematical properties of the infinite dimensional problem.

In this section we will clarify the question under which conditions the approximate synthesis functions computed within $\mathbb{Z}_{2N+1}$ converge to the synthesis functions $\gamma_{m,n}$ of $\ell^2(\mathbb{Z})$ when the length of the period tends to infinity.

A positive answer to this question also validates the approach of approximating the synthesis functions by means of sampling and inverting the polyphase matrix, as described in Section 2.1.

Let the functions $g_m$ be compactly supported, such that $g_m(k) = 0$ for $|k| > s$. We construct our periodic model, as usual, by extending the truncated functions $P_N g_m$ periodically beyond the interval $[-N, N]$ for $N > s$ by setting

$$ {}^P g_m(k + lN) := g_m(k) \qquad \text{for } k = -N, \dots, N, \ , l \in \mathbb{Z}\,. \tag{30} $$

The other elements of our periodic finite filter bank are now given by

$$ {}^P g_{m,n}(k) = {}^P g_m(k - na) \tag{31} $$

where the shift, according to our model, is understood as circulant shift. The periodicity of the ${}^P g_{m,n}$ implies that the corresponding frame operator is a block circulant matrix, rather than merely a block Toeplitz matrix. Observe that the entries of ${}^P S_N$ deviate from the entries of $S_N$ only in the lower right and upper left corner of the matrix.

In other words we have replaced the block Toeplitz matrix $S_N$ in the system $S_N \gamma_m^{(N)} = g_m^{(N)}$ by a block circulant matrix. However replacing a block Toeplitz system by a block circulant matrix in a linear system of equations will in general significantly affect the solution.

Moreover it is a priori not clear under which conditions invertibility of $S_N$ implies invertibility of ${}^P S_N$. For instance take following Hermitian positive definite Toeplitz matrix $T$ and its "circulant completion" ${}^P T$:

$$ T = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \qquad {}^P T = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} $$

then $T$ is invertible, whereas ${}^P T$ is obviously not invertible.

The following theorem justifies the usage of the periodic model for the approximation of synthesis filter banks.[13] The theorem also provides a way to estimate the frame bounds of a filter bank by computing the eigenvalues of the sampled polyphase matrix, see [17,22] for more details.

THEOREM 2.6. *Let $\{g_{m,n}\}$ be a perfect reconstruction oversampled FIR filter bank for $\ell^2(\mathbb{Z})$ with frame bounds $A, B$. Assume that $\|g_m\| = 1$ and that $g_m(k) = 0$ for $|k| > s$. Let $N > 2s$ and let ${}^P g_{m,n}$ be the periodized functions as defined in* (30) *and* (31) *with frame operator ${}^P S_N$. Then ${}^P S_N$ is invertible and ${}^P \gamma_m^{(N)} = {}^P S_N^{-1}\, {}^P g_m$ converges to $\gamma_m$ for $N \to \infty$. If $N > 3s$ then the rate of convergence can be estimated by*

$$ \|P_N \gamma_m - {}^P \gamma_m^{(N)}\| \leq 3\sqrt{2} C \lambda^N (\lambda^s - \lambda^{s+1})^{-3} \tag{32} $$

*where* $\kappa = \frac{B}{A}, q = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}$, $\lambda = q^{\frac{1}{2s}}$ *and*

$$ C = \frac{1}{A} \max\{2\kappa, (1 + \sqrt{\kappa})^2\} \tag{33} $$

Similar results hold if the analysis functions $g_{m,n}$ have exponential or polynomial decay, since in this case we can approximate $g_{m,n}$ arbitrarily well by compactly supported functions.

## 2.4. Finite-dimensional approximation of paraunitary filter banks via $S^{-\frac{1}{2}}$

Since paraunitary filter banks are tight frames (i.e. the frame bounds $A, B$ coincide) the frame operator is given by $S = AI$ where $I$ is the identity operator on $\ell^2(\mathbb{Z})$. In this case the frame bound $A$ can also be interpreted as oversampling factor if $\|g_m\| = 1$, see.[2] Paraunitary filter banks are very attractive due to their noise reduction property. The reconstruction error variance is inversely proportional to the oversampling factor.[23,2]

Hence the construction of tight frames is of great interest in theory and in applications. Daubechies[10] has shown that for a given frame, say $\{f_k\}$ the set of functions $\psi_k$ constructed by

$$\psi_k := S^{-\frac{1}{2}} f_k \tag{34}$$

constitutes a tight frame. Recall that $S$ is positive definite, hence $S^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are uniquely determined. Moreover $S^{\frac{1}{2}}$ commutes with the same operators that commute with $S$. For filter banks this implies that the tight frame generated by $\{S^{-\frac{1}{2}} g_{m,n}\}$ also constitutes a system of the form (7).

The tight frame constructed via formula (34) has a natural interpretation in linear algebra. Let $\{f_k\}_{k=1}^m$ be a given frame in $\mathbb{C}^n$ with frame operator $S$. The tight frame $\{\hat{\varphi}_k\}_{k=1}^m$ that minimizes

$$\sum_{k=1}^{m} \|f_k - \varphi_k\|^2 \tag{35}$$

among all tight frames $\{\varphi_k\}_{k=1}^m$ is given by $\hat{\varphi}_k := (\mathrm{trace}(S^{\frac{1}{2}})/n) S^{-\frac{1}{2}} f_k$. In the linear algebra community this fact is known in the following context. Let $T$ be the $n \times m$ matrix having the $f_k$ as its $k$-th column. Let $\mathcal{U}_{n,m}$ be the set of all matrices $U$ such that $UU^H = cI$, where $I$ is the $n \times n$ identity matrix and $c \in \mathbb{R}$ (in other words $U$ is a multiple of a matrix with unitary rows). Denote the Frobenius or Hilbert-Schmidt norm of a matrix by $\|.\|_F$. Then the matrix that solves

$$\min_{U \in \mathcal{U}_{n,m}} \|T - U\|_F\,,$$

is given by

$$\hat{U} = \frac{\mathrm{trace}(P)}{n} V\,,$$

where $T = PV$ is the polar decomposition of $T$ with $P = (TT^H)^{\frac{1}{2}}$, see e.g. Chapter 7.4 in [24]. Although this is a classical result in linear algebra it seems to be not very well-known in the frame- and signal processing community.

Now we return to frames $\{f_k\}_{k \in \mathbb{Z}}$ for $\ell^2(\mathbb{Z})$ and want to investigate if the tight frames constructed via equation (34) can be approximated by the finite section method. We want to approximate the elements of the tight frame $\varphi_k := S^{-\frac{1}{2}} f_k$ by the solution of the finite sections

$$(P_N S P_N)^{\frac{1}{2}} \varphi_k^{(N)} = P_N f_k\,.$$

The applicability of the finite section method follows directly from Proposition 2.1. Considering the rate of approximation one can easily derive estimates similar to the ones given in Propositions 2.3 and 2.4 to bound the decay of $\varphi_k$ by using Lemma 2 in [25].

We state the result for FIR filter banks, the versions for other types of decay and for the periodic model are left to the reader.

Corollary 2.7. *Let $\{g_{m,n}\}$ be a perfect reconstruction filter bank for $\ell^2(\mathbb{Z})$ with frame operator $S$ and frame bounds $A, B$. Assume that $\|g_m\| = 1$ and set $\varphi_m = S^{-\frac{1}{2}} g_m$ and $\varphi_m^{(N)} = (P_N S P_N)^{-\frac{1}{2}} P_N g_m$. Then the finite section method is applicable to $S^{\frac{1}{2}}$. If moreover $g$ is compactly supported with $g(k) = 0$ for $|k| > s$ then*

$$\|\varphi_m - \varphi_m^{(N)}\| \le K \beta^N \qquad N = 0, 1, \ldots$$

*for $\beta \subset (0,1)$ and some constant $K > 0$, where $\beta$ and $K$ depend on the frame bounds $A, B$ and the support length $s$, but are independent of $N$.*

## 3. APPROXIMATION OF DUAL FILTER BANKS IN $L^2(\mathbb{R})$

If the frame elements $f_k$ are defined on $\mathbb{Z}$ we can resort to the finite section method, based on Theorem 2.2. However if $f_k$ is defined on $\mathbb{R}$ different methods have to be used, since $S$ can no longer be represented by a infinite-dimensional matrix. Several finite-dimensional approaches have been proposed for this purpose.[26,27] The numerical efficiency and practicability (for instance how to determine the required "truncation" parameters) of these methods is not clear yet.

A standard method for a theoretical determination of the dual frame is based on the frame Gram matrix. Recall that any $f$ can be written as

$$f(x) = \sum_{k\in\mathbb{Z}} c_k f_k \qquad \text{where } \; Rc = b\,, \tag{36}$$

with $b = \{b_k\} = \{\langle f, f_k\rangle\}$ and $R$ is the bi-infinite Gram matrix with entries $R_{k,l} = \langle f_k, f_l\rangle$.

In order to to compute a finite-dimensional approximation to $c = \{c_k\}_{k\in\mathbb{Z}}$ we use the finite section method as in Section 2.1. Setting $R_N = P_N R P_N$ and $b^{(N)} = P_N b$, we obtain the $n$-th approximation $c^{(N)}$ to $c$ by solving

$$R_N c^{(N)} = b^{(N)}\,. \tag{37}$$

It is clear that using the truncated frame $\{f_k\}_{k=-N}^{N}$ in (36) for an approximate reconstruction of $f$ leads to the same system of equations.

If the set $\{f_k\}_{k\in\mathbb{Z}}$ is an exact frame then we have following well-known result.

Lemma 3.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be an exact frame for $\boldsymbol{H}$ with frame bounds $A, B$ and $Rc = b$ and $R_N c^{(N)} = b^{(N)}$ as defined above. Then $R_N^{-1}$ converges strongly to $R^{-1}$ and hence $c^{(N)} \to c$ for $N \to \infty$.*

A short proof of this result is obtained analogous to the proof of Theorem 2.2.

If $\{f_k\}_{k\in\mathbb{Z}}$ is a non-exact frame the situation is more delicate. In this case it follows from standard linear algebra that $\sigma(R) \subseteq \{0 \cup [A, B]\}$, or expressed in words, the singular values associated with $\mathcal{N}(R)^\perp$ are bounded away from zero. Here $\sigma(R)$ denotes the spectrum of $R$. For the truncated matrices $R_N$ we have

$$\sigma(R_N) \subseteq \{(0, B]\}$$

and the smallest of the singular values will go to zero for $N \to \infty$, see [27]. This behavior of the spectrum of the truncated matrices $R_N$ causes serious stability problems.

To see this let $K = U\Sigma V^*$ be the singular value decomposition of a matrix $K$ with $\Sigma = \operatorname{diag}(\lambda_{-N}, \ldots, \lambda_N)$. Then the Moore-Penrose inverse of $K$ is $K^+ = V\Sigma^+ U^*$, where

$$\Sigma^+ = \operatorname{diag}(\lambda_k^+)\,, \quad \lambda_k^+ = \begin{cases} 1/\lambda_k & \text{if } \lambda_k \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

For $R_N$ this means that those singular values close to zero will give rise to extremely large coefficients in $R_N^+$. In fact $\|R_N^+\| \to \infty$ for $N \to \infty$. Consequently $c^{(N)}$ does not converge to $c$ and a stable approximation of the dual frame is not possible.

Practically $\|R_N^+\|$ is always bounded due to finite precision arithmetics, but it is clear that it will lead to meaningless results for large $n$. Moreover, if the sampling values are perturbed by noise, then those error components which correspond to small singular values $\lambda_k$ are amplified by the (then large) factors $1/\lambda_k$. Although for a given $R_N$ these amplifications are theoretically bounded, they may be practically unacceptable large.

Such phenomena are well-known in regularization theory.[28] A standard technique to compute a stable solution for an ill-conditioned system is to use a truncated singular value decomposition (TSVD).[28] This means in our case we compute a regularized pseudo-inverse $R_N^{+,\tau} = V\Sigma^{+,\tau}U^*$ where

$$\Sigma^{+,\tau} = \operatorname{diag}(d_k^+)\,, \quad d_k^+ = \begin{cases} 1/\lambda_k & \text{if } \lambda_k \geq \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

In[27] it is shown that if we allow the truncation level $\tau$ to vary point-wise for each $n$ then the regularized inverses $R_N^{+,\tau}$ converge strongly to $R^+$ for $N \to \infty$ and consequently $\lim_{N\to\infty} \|f - f^{(N)}\| = 0$, where

$$f^{(N)}(t) = \sum_{k=-N}^{N} c_k^{(N,\tau)} f_k$$

with

$$c^{(N,\tau)} = R_N^{+,\tau} b^{(N)} .$$

The disadvantages of this approach are obvious: In general the matrix $R_N$ does not have any particular structure, thus the computational costs for the singular value decomposition are $\mathcal{O}(N^3)$ which is prohibitively large in many realistic problems. Furthermore the optimal truncation level $\tau$, depending on the dimension $N$, the machine precision $\varepsilon$ of the computer, and the sampling set, is not known a priori and has to be determined for each $N$ independently.

Since $\tau$ is of vital importance for the quality of the reconstruction, but no theoretical explanations how to choose $\tau$ are given in the sampling literature, we briefly discuss this issue.

Based on the results in Sections 4 and 6 in [29] (in particular the remarks following Theorem 4.1 therein) a reasonable choice for $\tau$ is

$$\tau \subseteq [(\delta/2)^{1/3}, (\delta)^{1/2}] , \tag{40}$$

where $\delta = \varepsilon$ is the machine precision. Extensive numerical simulations confirm this choice. In case of noisy data the noise level is usually much higher than the round-off level $\varepsilon$. In this case $\delta$ should be chosen such that $\|y - \tilde{y}\| \le \delta$ where $\tilde{y}$ represents perturbed data. If the noise level $\delta$ is not known it has to be estimated, which however is a difficult problem and will not be discussed here. The reader is referred to[29] for more details.

### 3.1. CG and regularization of the truncated frame method

As mentioned above one way to stabilize the solution of $R_N c^{(N)} = b^{(N)}$ is a truncated singular value decomposition, where the truncation level serves as regularization parameter. For large $N$ the costs of the singular value decomposition become prohibitive for practical purposes.

We propose the conjugate gradient method[30] to solve $R_N c^{(N)} = b^{(N)}$. It is in general much more efficient than a TSVD or Tikhonov regularization, and at the same time it can also be used as regularization method.

The standard error analysis for CG cannot be used in our case, since the matrix is ill-conditioned. Rather we have to resort to the error analysis developed in.[31,32]

When solving a linear system $Ax = y$ by CG for noisy data $y^\delta$ following happens. The iterates $x_k$ of CG may diverge for $k \to \infty$, however the error propagation remains limited in the beginning of the iteration. The quality of the approximation therefore depends on how many iterative steps can be performed until the iterates turn to diverge. The idea is now to stop the iteration at about the point where divergence sets in. In other words the iterations count is the regularization parameter which remains to be controlled by an appropriate stopping rule.[33,32] A detailed discussion of stopping rules can be found in [[32]].

For filter banks on $\boldsymbol{L}^2(\mathbb{R})$ the Gram matrix $R$ becomes a block Toeplitz matrix, the blocks themselves however are in general not of Toeplitz type. But the matrix-vector multiplication can still be carried out efficiently by block-FFT techniques. Thus the coefficients $c^{(N,\delta)}$ can be efficiently computed in $\mathcal{O}(kN \log N)$ operations, where $k$ is the number of iterations.

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