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Rates of convergence for the approximation of dual shift-invariant systems in $\boldsymbol{\ell}^2(\mathbb{Z})$

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Abstract

A shift-invariant system is a collection of functions $\{g_{m,n}\}$ of the form $g_{m,n}(k) = g_m(k - an)$. Such systems play an important role in time-frequency analysis and digital signal processing. A principal problem is to find a dual system $\gamma_{m,n}(k) = \gamma_m(k - an)$ such that each function f can be written as $f = \sum \langle f, \gamma_{m,n} \rangle g_{m,n}$. The mathematical theory usually addresses this problem in infinite dimensions (typically in $\boldsymbol{L}^2(\mathbb{R})$ or $\boldsymbol{\ell}^2(\mathbb{Z})$), whereas numerical methods have to operate with a finite-dimensional model. Exploiting the link between the frame operator and Laurent operators with matrix-valued symbol, we apply the finite section method to show that the dual functions obtained by solving a finite-dimensional problem converge to the dual functions of the original infinite-dimensional problem in $\boldsymbol{\ell}^2(\mathbb{Z})$. For compactly supported $g_{m,n}$ (FIR filter banks) we prove an exponential rate of convergence and derive explicit expressions for the involved constants. Further we investigate under which conditions one can replace the discrete model of the finite section method by the periodic discrete model, which is used in many numerical procedures. Again we provide explicit estimates for the speed of convergence. Some remarks on tight frames complete the paper.

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1 Introduction

Shift-invariant systems play an important role in time-frequency analysis and digital signal processing. In this paper we consider time-discrete shift-invariant systems, by which we mean functions $g_{m,n}$ of the form

$$g_{m,n}(k) = g_m(k - na), \qquad k, n \in \mathbb{Z}, m = 0, 1, \ldots, M-1\,, \tag{1}$$

where $g_m \in \boldsymbol{\ell}^2(\mathbb{Z})$ and $0 < a \in \mathbb{N}$. In the engineering literature such a system is known as *filter bank*.

Since the $g_{m,n}$ are in general not orthogonal, we are interested in finding dual systems

$$\gamma_{m,n}(k) = \gamma_m(k - na), \qquad k, n \in \mathbb{Z}, m = 0, 1, \ldots, M-1\,, \tag{2}$$

such that any $f \in \boldsymbol{\ell}^2(\mathbb{Z})$ has a $\boldsymbol{\ell}^2(\mathbb{Z})$-convergent representation

$$f = \sum_{m,n} \langle f, \gamma_{m,n}\rangle g_{m,n}\,. \tag{3}$$

The existence of such dual systems is guaranteed if $\{g_{m,n}\}$ is a frame [9]. If not otherwise stated we will assume in the sequel that $\{g_{m,n}\}$ is a frame for $\boldsymbol{\ell}^2(\mathbb{Z})$.

There is a variety of research papers dealing with shift-invariant systems in $\boldsymbol{L}^2(\mathbb{R})$ or $\boldsymbol{\ell}^2(\mathbb{Z})$, there is also no lack of methods claiming to efficiently solve for the duals, these methods naturally operate in finite dimensions. But only little is known about the relation of the finite-dimensional solutions and the original solutions of the infinite dimensional model. For practical purposes it is not enough to ensure that an approximation converges to the original solution, but one also should try to provide a priori bounds on the rate of convergence. The goal of this paper is to clarify the relation between the approximate dual functions computed by a finite-dimensional approach and the original dual functions in $\boldsymbol{\ell}^2(\mathbb{Z})$.

Throughout the paper we will use two key ideas to derive our results. We exploit the link between frame operators of shift-invariant systems and Laurent operators with matrix valued symbol to apply the finite section method for the proof of convergence. Furthermore we will discuss the rate of convergence making use of results of the off-diagonal entries of the inverse of certain types of matrices.

A detailed investigation of shift-invariant systems in $\boldsymbol{L}^2(\mathbb{R})$ and $\boldsymbol{L}^2(\mathbb{R}^d)$ can be found in the papers of Benedetto et al. [1, 3, 2] and in the work of Ron and Shen [30, 31]. In [28] Janssen analyses duality conditions for shift-invariant systems in $\boldsymbol{L}^2(\mathbb{R})$ and $\boldsymbol{\ell}^2(\mathbb{Z})$ with emphasis on Gabor systems. For an overview of the theory of filter banks the reader may consult [34].

The concept of frames [12, 36] provides a natural environment to study shift-invariant systems [1, 3, 30, 26]. We say that the set $\{g_{m,n}\}$ is a frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{m,n} |\langle f, g_{m,n}\rangle|^2 \le B\|f\|^2 , \qquad \forall f \in \boldsymbol{\ell}^2(\mathbb{Z}) . \tag{4}$$

If the shift-invariant system $\{g_{m,n}\}$ satisfies condition (4) we will also call it *shift-invariant frame.* The dual frame is given by

$$\gamma_{m,n}(k) = (S^{-1}g_m)(k - na) , \tag{5}$$

where S is the frame operator defined by

$$Sf = \sum_{m,n} \langle f, g_{m,n}\rangle g_{m,n} . \tag{6}$$

Among all dual systems satisfying (3) the duals defined by (5) have minimal $\boldsymbol{\ell}^2$-norm. In this paper we restrict our attention to duals with minimal $\boldsymbol{\ell}^2$-norm, therefore we will be somewhat sloppy in our terminology and refer to the $\gamma_{m,n}$ defined in (5) as *the* duals.

We define the analysis operator F by

$$F : f \in \boldsymbol{\ell}^2(\mathbb{Z}) \to Ff = \{\langle f, g_{m,n}\rangle\}_{m,n} \tag{7}$$

and the synthesis operator, which is just the adjoint operator of F, by

$$F^* : c \in \boldsymbol{\ell}^2(\mathbb{Z} \times \mathcal{I}) \to F^*c = \sum_{m,n} c_{m,n} g_{m,n} \tag{8}$$

where $\mathcal{I} := \{0, 1, \ldots, M-1\}$. The frame operator can be written as $S = F^*F$.

If the g_m are constructed from a single function g by modulations, i.e.,

$$g_m(k) = e^{2\pi imk/M} g(k) ,$$

we obtain the well-known *Gabor system* or *Weyl-Heisenberg system* [16, 14]. A nice property of Gabor frames is that the dual frame is also generated by a single function, we have [8]

$$\gamma_{m,n}(k) = e^{2\pi imk/M}\gamma(k-na) \qquad \text{with } \gamma = S^{-1}g \ .$$

In the filter bank community, systems of the form (1) are known as (oversampled) DFT filter banks [5].

For some special choices of $\{g_{m,n}\}$, closed-form solutions for the duals are known [8, 15, 27] (which does not necessarily mean that those closed-form solutions can be cheaply computed in practice). In general however the numerical computation of the dual frame elements γ_m involves the solution of the system of equations (5) of biinfinite order, independently of the choice of the basis for the matrix representation of S (be it the standard basis in $\boldsymbol{\ell}^2(\mathbb{Z})$ or the so-called polyphase representation used in the theory of filter banks). We certainly cannot solve a system of biinfinite order, but we can solve it approximately, based on a finite-dimensional model.

Whatever finite-dimensional model we choose to compute approximate duals, we have to ask if these approximations really converge to the original solution with increasing dimension and, whenever possible, we have to give a priori bounds on the rate of convergence. These are the problems we are going to attack in this paper.

Before we proceed we introduce some notation. Let $x \in \boldsymbol{\ell}^2(\mathbb{Z})$, then $\|x\|$ is the standard $\boldsymbol{\ell}^2$-norm of x. $\boldsymbol{\ell}^2_a(\mathbb{Z})$ is the space of square summable sequences $\{x_k\}_{k\in\mathbb{Z}}$ where x_k is an $a \times a$ matrix. We denote the spectrum of an operator T by $\sigma(T)$. If T is hermitian positive definite and invertible then $\|T\| = \max\{\mu : \mu \in \sigma(T)\}$ and in this case $\operatorname{cond}(T) := \|T\|\|T^{-1}\|$ is the condition number of T.

2 Dual frames and the finite section method

Due to the shift-invariance property (2) of the $\gamma_{m,n}$ we only have to compute approximations to the duals $\gamma_m = \gamma_{m,0}$ for $m = 0, \dots, M-1$, since all other $\gamma_{m,n}$ are just translations of γ_m. Equations (5) and (6) tell us that all shifted copies $g_{m,n}$ are required for the computation of the duals γ_m. But if the g_m have nice decay properties, such that most of their energy is contained in some interval around the origin, say, then one may hope that the γ_m will have the same properties. One could further argue that using only those functions

$g_{m,n}$ "living" in some interval around the origin, $[-N, N]$ say, should be sufficient to compute a good approximation to γ_m.

Let us put these vague ideas in a more precise mathematical setting. For $N \in \mathbb{N}$ and $x \in \boldsymbol{\ell}^2(\mathbb{Z})$ define the orthogonal projections P_N by

$$P_N x = (\dots, 0, 0, x_{-N}, x_{-N+1}, \dots, x_{N-1}, x_N, 0, 0, \dots) \tag{9}$$

and identify the image of P_N with the $2N+1$-dimensional space $\mathbb{C}^{2N+1}$. We construct a finite-dimensional approximation S_N of the frame operator S via the truncated frame elements $P_N g_{m,n}$ by

$$S_N = (P_N F^*)(F P_N)\,, \tag{10}$$

where F is defined in (7). The N-th approximation $\gamma_m^{(N)}$ to γ_m is then given by the solution of the finite-dimensional system of equations $S_N \gamma_m^{(N)} = P_N g_m$. In other words we truncate *all* frame elements $g_{m,n}$ to the interval $[-N, N]$. The questions that arise with such an approximation scheme are:

(i) Does $\gamma_m^{(N)}$ converge to γ_m for $N \to \infty$?

(ii) If the g_m satisfy certain decay conditions, can we give an estimate on the rate of convergence? In other words: given $\delta > 0$, can we provide an a priori estimate for the size of the interval $[-N, N]$, such that $\|\gamma_m - \gamma_m^{(N)}\| \le \delta$, where N depends on δ and the decay of g_m?

In numerical procedures people often use a periodic finite-dimensional model in order to approximate γ_m. Instead of the truncated functions $P_N g_m$ (which could be thought of as padded with zeros beyond the interval $[-N, N]$), the $P_N g_m$ are extended periodically across the interval $[-N, N]$. In other words all computations are done in the ring $\mathbb{Z}_{2N+1} = \mathbb{Z} \bmod (2N+1)$. In Section 5 we will investigate the behavior of the periodic approximate dual functions, when the length of the period approaches infinity.

2.1 The finite section method and Laurent operators

The *finite section method* or *projection method* is a classical technique to approximate the solution of infinite Toeplitz-type systems. For an introduction and many variations of the theme the reader is referred to the books of Gohberg and Fel'dman [19], Böttcher and Silbermann [6] and Hagen, Roch, and Silbermann [21]. Since we use the finite section method as a main tool throughout the paper, I briefly describe it in the specific context we will apply it.

Let $\mathbb{T}$ be the complex unit circle and let $\tau(z)$ be a continuous function on $\mathbb{T}$, $\tau \in \boldsymbol{C}(\mathbb{T})$. Let $\tau_k \in \mathbb{C}$ stand for the k-th Fourier coefficient of τ,

$$\tau_k = \int_0^1 \tau(e^{2\pi i\omega}) e^{-2\pi i k\omega}\, d\omega$$

then the operator $T(\tau)$ which acts on functions $x \in \boldsymbol{\ell}^2(\mathbb{Z})$ by

$$(Tx)_k = \sum_{j\in\mathbb{Z}} \tau_{j-k} x_j\,, \qquad k \in \mathbb{Z}\,, \tag{11}$$

is called *Laurent operator* or *multiplication operator* with *generating function* or *symbol* τ. A Laurent operator may also be characterized as a bounded linear operator acting on $\boldsymbol{\ell}^2(\mathbb{Z})$ which commutes with the forward shift operator on $\boldsymbol{\ell}^2(\mathbb{Z})$, cf. Chapter 13.2 in [17]. Since a Laurent operator can be represented by a biinfinite Toeplitz matrix, we will often identify $T(\tau)$ with its matrix representation.

Consider the operator equation $Tx = y$ where $x, y \in \boldsymbol{\ell}^2(\mathbb{Z})$, or in matrix representation

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \tau_0 & \tau_1 & \tau_2 & \cdots \\ \cdots & \tau_{-1} & \tau_0 & \tau_1 & \cdots \\ \cdots & \tau_{-2} & \tau_{-1} & \tau_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix}. \tag{12}$$

For the approximate solution of the biinfinite Toeplitz system in (12) by the finite section method one considers the finite linear equations

$$T_N x^{(N)} = y^{(N)}\,, \tag{13}$$

where $y^{(N)} = (y_{-N}^{(N)}, \ldots, y_N^{(N)})$ and where T_N are the finite sections of the biinfinite Toeplitz matrix in (12):

$$T_0 = (\tau_0)\,,\; T_1 = \begin{bmatrix} \tau_0 & \tau_1 & \tau_2 \\ \tau_{-1} & \tau_0 & \tau_1 \\ \tau_{-2} & \tau_{-1} & \tau_0 \end{bmatrix},\; T_2 = \begin{bmatrix} \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ \tau_{-1} & \tau_0 & \tau_1 & \tau_2 & \tau_3 \\ \tau_{-2} & \tau_{-1} & \tau_0 & \tau_1 & \tau_2 \\ \tau_{-3} & \tau_{-2} & \tau_{-1} & \tau_0 & \tau_1 \\ \tau_{-4} & \tau_{-3} & \tau_{-2} & \tau_{-1} & \tau_0 \end{bmatrix}, \ldots .$$

Using the the orthogonal projections defined in (9) and identifying the image of P_N with the $2N+1$-dimensional space $\mathbb{C}^{2N+1}$ we can express the equations (13) in the form

$$P_N T P_N x^{(N)} = P_N y\,. \tag{14}$$

We say that the finite section method is *applicable* to T, if beginning with some $N \in \mathbb{N}$ for each $y \in \boldsymbol{\ell}^2(\mathbb{Z})$ the equation

$$P_N T P_N x^{(N)} = P_N y \tag{15}$$

has a unique solution $x^{(N)} \in \operatorname{Im} P_N$ and as $N \to \infty$ the vectors $x^{(N)}$ tend to the solution of $Tx = y$.

Now let τ be a continuous $a \times a$-matrix valued function on $\mathbb{T}$, $\tau \in \boldsymbol{C}_{a\times a}(\mathbb{T})$. Let $\tau_k \in \mathbb{C}^{a\times a}$ stand for the k-th Fourier coefficient of τ, then the operator T defined by the action

$$(Tx)_k = \sum_{j\in\mathbb{Z}} \tau_{j-k} x_j\,, \qquad x \in \boldsymbol{\ell}^2_a(\mathbb{Z}) \tag{16}$$

is called a *block Laurent operator* with generating function τ. A block Laurent operator can be represented by a biinfinite block Toeplitz matrix, where the τ_k in (12) are $a \times a$ matrices. Analogous to above, for the approximate solution of biinfinite block Toeplitz systems we consider finite sections of the form

$$P_N T P_N x^{(N)} = P_N y\,. \tag{17}$$

It is a well known fact and a direct consequence of the definition of shift-invariant systems that the associated frame operator S is a block Laurent operator, since S commutes with the shift operator $U_a(\varphi_k)_{k\in\mathbb{Z}} := (\varphi_{k-a})_{k\in\mathbb{Z}}$. Hence the operator equation $S\gamma_m = g_m$ can be expressed as a biinfinite block Toeplitz system of equations, where the blocks of the system matrix are of size $a \times a$. The blocks themselves are not Toeplitz. It is exactly this block-Laurent structure of S that is utilized in the polyphase representation proposed in the filter bank theory [34].

Let S_N be the frame operator associated with the truncated frame elements $P_N g_{m,n}$ as defined in (10). Since $S_N = P_N F^* F P_N = P_N S P_N$, the corresponding system of equations $S_N \gamma_m^{(N)} = g_m^{(N)}$ is just a finite section of the biinfinite system $S\gamma = g$. Hence it is natural to ask if we can approximate the duals γ_m by solving

$$P_N S P_N \gamma_m^{(N)} = P_N g \tag{18}$$

for increasing N, or in other words, we want to know if the finite section method is applicable to S. Moreover we are also interested in the rate of convergence, in case the finite section method applies to S.

3 Rate of convergence using the finite section method

If the functions g_m have a certain rate of decay, one would expect that the dual functions also share this behavior. We consider the following three types of decay for g_m and study first how a certain decay property translates to decay properties of the frame operator S.

a) *Exponential decay:* $|g_m(k)| \le c\lambda^{\alpha|k|}$ for $0 < \alpha < 1$, $\lambda \in (0,1)$ and some constant c.

b) *Polynomial decay:* $|g_m(k)| \le c(1+|k|)^{-\alpha}$ for $\alpha > 1$ and a constant c.

c) *Compact support:* $g_m(k) = 0$ for $|k| > s$.

In the sequel we will frequently make use of results by Jaffard on matrices with exponential and polynomial decay [23]. For the reader's convenience I state these results here, they can be found in somewhat more general form in [23].

Definition 3.1 *Let $T = (T_{k,l})_{k,l\in\mathcal{I}}$ be a matrix where the index set $\mathcal{I}$ is $\mathcal{I} = \mathbb{Z}, \mathbb{Z}^+$ or $\{0,\dots,N-1\}$.*
(i) T belongs to the space $\mathcal{E}_\alpha$ if the coefficients $T(k,l)$ satisfy

$$\forall \alpha_1 < \alpha, \quad |T(k,lo)| \le c\lambda^{\alpha|k-l|}\,, \tag{19}$$

where $\lambda \in (0,1)$ and c depends on α_1.
(ii) T belongs to the space $\mathcal{Q}_\alpha$ if the coefficients $T(k,l)$ satisfy

$$|T(k,l)| \le c|1+|k-l|)^{-\alpha}\,, \tag{20}$$

where $\alpha > 1$.

Lemma 3.2 [23, Proposition 1]. *The spaces $\mathcal{E}_\alpha$ and $\mathcal{Q}_\alpha$ are matrix algebras.*

Theorem 3.3 [23, Propositions 2 and 3]. *Let $T : \boldsymbol{\ell}^2(\mathcal{I}) \to \boldsymbol{\ell}^2(\mathcal{I})$ be an invertible operator, where $\mathcal{I}$ is $\mathcal{I} = \mathbb{Z}, \mathbb{Z}^+$ or $\{0, \ldots, N-1\}$:*
(i) If $T \in \mathcal{E}_\alpha$, then $T^{-1} \in \mathcal{E}_\alpha$.
(ii) If $T \in \mathcal{Q}_\alpha$, then $T^{-1} \in \mathcal{Q}_\alpha$.

The following proposition is a simple modification of well-known results used in connection with the construction of splines and wavelet bases.

Proposition 3.4 *Assume that the functions $g_{m,n}, m = 0, \ldots, M-1$ constitute a shift-invariant frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with frame operator S and duals $\gamma_m^{(N)}$.*
(a) Exponential decay: If $|g_m(k)| \leq c\lambda^{\alpha|k|}$ for $\alpha > 0$ and $\lambda \in (0,1)$, then there exists an α_1 with $0 < \alpha_1 < \alpha$ and a constant c_1 depending on α_1 such that

$$|S_{k,l}| \leq c_1 \lambda^{\alpha_1|k-l|} . \tag{21}$$

Furthermore, the duals γ_m satisfy

$$|\gamma_m(k)| \leq c_2 \lambda^{\alpha_2|k|} \tag{22}$$

for some $0 < \alpha_2 < \alpha$ and a constant c_2 depending on α_2.
(b) Polynomial decay: If $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ for $\alpha > 1$, then there exists a constant c_1 such that

$$|S_{k,l}| \leq c_1(1+|k-l|)^{-\alpha} . \tag{23}$$

Furthermore, there is a constant c_2 such that

$$|\gamma_m(k)| \leq c_2(1+|k|)^{-\alpha} . \tag{24}$$

Proof: (a) Consider

$$S_{k,l} = \sum_{m=0}^{M-1} \sum_{n=\infty}^{\infty} g_{m,n}(k) g_{m,n}(l) . \tag{25}$$

Due to the exponential decay of the g_m we have

$$|S_{k,l}| \leq Mc^2 \sum_{n=-\infty}^{\infty} \lambda^{\alpha|k-na|} \lambda^{\alpha|l-na|} . \tag{26}$$

Statement (21) follows from (26) exactly as in Jaffard's proof of Proposition 1 in [23]. Now we can apply Theorem 3.3 to conclude that $S^{-1} \in \mathcal{E}_\alpha$. Since $\gamma_m = S^{-1} g_m$, relation (22) follows in the same way as (21).

Case (b) can be proved similar to case (a). ❑

Remark: We can obtain similar results for the decay of the duals for shift-invariant systems on $\boldsymbol{L}^2(\mathbb{R})$ by considering the decay of the inverse of the Gram matrix $R = (\langle g_{m,n}, g_{m',n'} \rangle)_{m,n,m',n'}$. This will be discussed in more detail elsewhere.

I admit that (22) and (24) are not very satisfactory from a practical point of view, since both estimates do not provide explicit expressions for the involved constants. I hope to address this problem in my future work.

Fortunately, for compactly supported g_m we can do better. Compact support of the g_m allows us to compute the transform coefficients $\langle f, g_{m,n} \rangle$ exactly and also the matrix entries of S can be calculated by finite sums. In filter bank design the analysis filters are often designed to be compactly supported (FIR filter banks). Note however that compact support of the g_m does not imply compact support of the duals. For Gabor frames Bölcskei has shown in [4] that the minimal norm duals are only compactly supported in very specific cases. It is well known that the redundancy of frames provides some design freedom for the dual systems, which for instance can be used to construct dual functions with compact support [35, 10, 4]. The following results allow us to obtain a good estimate for the decay of the duals $\gamma_{m,n}$ for compactly supported $g_{m,n}$. But before we can state the results we need some preparation.

Recall that a matrix T is called s-banded, if $T_{k,l} = 0$ for $|k-l| > s, k, l \in \mathbb{Z}$. The following theorem about the decay of the inverse of a band matrix is due to Demko, Moss, and Smith [11].

Theorem 3.5 *Let T be the matrix of a bounded and invertible operator on $\boldsymbol{\ell}^2(\mathcal{I}$, where $\mathcal{I} = \mathbb{Z}, \mathbb{Z}^+$ or $\{0, 1, \dots, N-1\}$, and assume T to be hermitian positive definite and s-banded. Let $B = \|T\|$ and $A = \|T^{-1}\|$ and set $\kappa = B/A, q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ and $\lambda = q^{\frac{1}{s}}$. Then we have*

$$|T^{-1}_{k,l}| \leq D \lambda^{|k-l|}, \tag{27}$$

where

$$D = \frac{1}{A} \max\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\}. \tag{28}$$

Furthermore, we shall need following corollary.

Corollary 3.6 *Let T be a matrix whose entries decay exponentially off the diagonal, i.e.,*

$$|T_{k,l}| \le c\lambda^{|k-l|}, \qquad k,l \in \mathbb{Z}, \tag{29}$$

and let y be a sequence in $\ell^2(\mathbb{Z})$ with $\|y\| = 1$ and with compact support such that $y_k = 0$ for $|k| > s$. Denote $\varphi = Ty$, then

$$|\varphi_k| \le \frac{c\lambda^{-s}}{1-\lambda}\lambda^{|k|}. \tag{30}$$

Proof: We have

$$\varphi_k = \sum_{l=-\infty}^{\infty} T_{k,l} y_l = \sum_{l=-s}^{s} T_{k,l} y_l, \qquad k \in \mathbb{Z}.$$

For reasons of symmetry we can restrict ourselves to the case φ_k with $k \ge 0$. First we consider the case $k \ge s$:

$$|\varphi_k| \le \sum_{l=-s}^{s} c\lambda^{|k-l|} = c\lambda^k \sum_{j=0}^{2s} \lambda^{s-j} = c\lambda^k \frac{\lambda^{-s} - \lambda^{s+1}}{1-\lambda}.$$

Now we consider the case $0 \le k < s$:

$$\begin{aligned} |\varphi_k| &\le \sum_{l=-s}^{s} c\lambda^{|k-l|} = \sum_{l=-s}^{k-1} c\lambda^{k-l} + \sum_{l=k}^{s} c\lambda^{l-k} \\ &= c\lambda^k \frac{\lambda^{-k+1} - \lambda^{s+1} + \lambda^k - \lambda^{s-2k+1}}{1-\lambda}. \end{aligned}$$

Now we show that for $0 \le k \le s$ the following relation holds:

$$c\lambda^k \frac{\lambda^{-k+1} - \lambda^{s+1} + \lambda^k - \lambda^{s-2k+1}}{1-\lambda} \le c\lambda^k \frac{\lambda^{-s} - \lambda^{s+1}}{1-\lambda}.$$

Set $h(x) = \lambda^{-x} + \lambda^{x+1} - \lambda - 1$ for $x = 0, 1, \dots$ and note that

$$\begin{aligned} &\lambda^{-(x+1)} > \lambda^{x+1} \\ \Leftrightarrow\ &\lambda^{-(x+1)} - \lambda^{-x} > \lambda^{x+1} - \lambda^{x+2} \\ \Leftrightarrow\ &\lambda^{x+2} + \lambda^{-(x+1)} - \lambda - 1 > \lambda^{x+1} + \lambda^{-x} - \lambda - 1, \end{aligned}$$

whence $h(x)$ is strictly monotonically increasing. Since $h(0) = 0$ and by setting $x = s - k$ it readily follows that

$$\lambda^{s+1} - \lambda^{-k+1} + \lambda^{s-2k+1} - \lambda^{-k} \le \lambda^{s+1} - \lambda^{-s}$$

which implies (3) and hence

$$|\varphi_k| \le c\lambda^{|k|}\frac{\lambda^{-s}}{1-\lambda}, \qquad k \in \mathbb{Z}. \tag{31}$$

❏

Now it is easy to estimate the decay of the dual functions $\gamma_{m,n}$ for compactly supported g_m.

Proposition 3.7 *Let $\{g_{m,n}\}$ be a shift-invariant frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with $\|g_m\| = 1$ and assume that $g_m(k) = 0$ for $|k| > s$. Let A, B be the frame bounds and set $\kappa = B/A$ and $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$. Then*

$$|\gamma_m(k)| \le D\lambda^{|k|}\frac{\lambda^{-s}}{1-\lambda}, \tag{32}$$

where $\lambda = q^{\frac{1}{2s}}$ and $D = \frac{1}{A}\max\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\}$.

Proof. Observe that S is a $2s$-banded matrix. Now the result is an immediate consequence of Theorem 3.5 and Corollary 3.6. ❏

Laurent operators with matrix valued symbol are a special case of so-called "paired" block Toeplitz operators studied by Gohberg and Fel'dman in [19] and Gohberg and Kaashoek in [18]. Following result is a special case of Theorem 4.1 in [18] (see also the comments following Theorem 4.1 in [18]) or of Theorem 9.3 in [19], respectively.

Proposition 3.8 *Let $\boldsymbol{H}$ be a separable Hilbert space, and let A be an invertible block Laurent operator on $\boldsymbol{\ell}^2(\boldsymbol{H})$ defined by $A = [\tau_{i-j}]_{i,j=-\infty}^{\infty}$ where τ is a continuous matrix-valued function on the unit circle. Let P_N be as in (9). Then the finite section method is applicable to A if and only if the block Toeplitz operators $T = [\tau_{i-j}]_{i,j=0}^{\infty}$ and its associate $\tilde{T} = [\tau_{j-i}]_{i,j=0}^{\infty}$ are invertible.*

We are now able to answer the question about the approximation of dual shift-invariant frames by duals computed via the finite section method.

Theorem 3.9 *Assume that* $\{g_{m,n}\}$ *is a shift-invariant frame for* $\ell^2(\mathbb{Z})$ *with* $\|g_m\| = 1$ *and frame bounds* A, B. *Set* $S_N = P_N S P_N$ *and* $\gamma_m^{(N)} = S_N^{-1} P_N g_m$ *for* $m = 0, 1, \ldots, M-1$ *and denote* $\kappa = B/A$. *Then the following statements hold:*
(i) The finite section method is applicable to S *and* $\gamma_m^{(N)}$ *converges to* γ_m *for* $N \to \infty$.
(ii) If the g_m *are compactly supported with* $g_m(k) = 0$ *for* $|k| > s$ *and* $N > 2s$, *then the rate of convergence can be estimated by*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \sqrt{2} C \lambda^N (\lambda^s - \lambda^{s+1})^{-3}, \tag{33}$$

where $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, $\lambda = q^{\frac{1}{2s}}$ *and*

$$C = \frac{1}{A} \max\{2\kappa, (1+\sqrt{\kappa})^2\}. \tag{34}$$

(iii) If $|g_m(k)| \leq c\lambda^{\alpha|k|}$ *for* $\alpha > 0$ *and* $\lambda \in (0,1)$, *then there exists an* $\alpha_1 < \alpha$ *and a constant* c_1 *depending on* α_1 *such that*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \kappa c_1 \lambda^{\alpha_1 N}. \tag{35}$$

(iv) If $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ *for* $\alpha > 1$, *then there exists a constant* c_1 *such that*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \kappa c_1 (1+|N|)^{-\alpha}. \tag{36}$$

Proof: (i): Recall that S is a block Laurent operator. It follows from Proposition 3.8 that S_N^{-1} converges to S^{-1} strongly, i.e.,

$$\|S_N^{-1}x - SIx\| \to 0 \quad \forall x \in \ell^2(\mathbb{Z}), \text{ for } N \to \infty,$$

if the generating function of S is continuous on the unit circle $\mathbb{T}$ and if the operators QSQ and QS^*Q are invertible on $\ell^2(\mathbb{Z}_+)$, where the projection Q is defined by

$$Qx = (\ldots, 0, x_0, x_1, \ldots).$$

The invertibility of QSQ is a direct consequence of the positive-definiteness of S, since for $x \in \operatorname{Im} Q$, $\|x\| = 1$, we have $\langle QSQx, x\rangle = \langle Sx, x\rangle > 0$, hence QSQ is invertible. In fact using the same argument we conclude that S_N is invertible for all $N = 0, 1, 2, \ldots$.

The continuity of the generating function of S follows from the fact that S is banded, thus the generating function is a trigonometric matrix polynomial and therefore continuous on the unit circle $\mathbb{T}$. Hence the projection method is applicable to S for any right hand side $\in \boldsymbol{\ell}^2(\mathbb{Z})$ and the first part of the theorem is proved.

In order to estimate the rate of convergence for (ii), (iii) and (iv) consider

$$\begin{aligned}\|\gamma_m - \gamma_m^{(N)}\| &= \|S^{-1}g_m - S_N^{-1}P_N g_m\| = \|S^{-1}g_m - S^{-1}SS_N^{-1}P_N g_m\| \\ &\le \|S^{-1}\| \left(\|g_m - S_N S_N^{-1}P_N g_m\| + \|S_N S_N^{-1}P_N g_m - SS_N^{-1}P_N g_m\|\right) \\ &\le A^{-1}\left(\|g_m - P_N g_m\| + \|(S_N - S)S_N^{-1}P_N g_m\|\right) . \end{aligned} \tag{37}$$

(ii): Recall that S is a $2s$-banded matrix, i.e., $S_{k,l} = 0$ for $|k-l| > 2s$. Applying Theorem 3.5 we can estimate the decay of the entries of S^{-1} by

$$|S^{-1}_{k,l}| \le D\lambda^{|k-l|} , \tag{38}$$

where D is as in Theorem 3.5 with $\lambda = q^{\frac{1}{2s}}$.

Cauchy's Interlace Theorem [22] implies that $\mathrm{cond}(S_N) \le \mathrm{cond}(S_M)$ for $N \le M$. A similar result holds with respect to the infinite-dimensional matrix S. Let $x \in \mathbb{C}^{2N+1}$ with $\|x\| = 1$. Then

$$\langle S_N x, x\rangle = \langle P_N S P_N x, x\rangle = \langle Sx, x\rangle \ge \frac{1}{\|S^{-1}\|} = A. \tag{39}$$

Along similar lines we get $\|S_N\| \le \|S\| \le B$. Hence $\sigma(S_N) \subseteq \sigma(S)$ and consequently $\mathrm{cond}(S_N) \le \mathrm{cond}(S)$. Therefore we can use the same λ and the same constant D in order to bound the decay of the entries of S_N^{-1} independently of N and obtain

$$|(S_N^{-1})_{k,l}| \le D\lambda^{|k-l|} . \tag{40}$$

Since $\gamma_m^{(N)} = S_N^{-1}P_N g_m$ it follows from Corollary 3.6 that

$$|(\gamma_m^{(N)})_k| \le D\lambda^{|k|}\frac{\lambda^{-s}}{1-\lambda} , \qquad k \in \mathbb{Z} . \tag{41}$$

In the next step we estimate $\|(S_N - S)S_N^{-1}P_N g_m\|$. Write $e_m^{(N)} = (S_N - S)\gamma_m^{(N)}$ and observe that $\|e_m^{(N)}\|^2 = \sum_{k=-\infty}^{\infty} |(e_m^{(N)})_k|^2 = 2\sum_{k=0}^{\infty} |(e_m^{(N)})_k|^2$ (note that $(e_m^{(N)})_0 = 0$). For $k = 0, 1., \dots$ consider

$$(e_m^{(N)})_k = \sum_{l=-\infty}^{\infty} (S_N - S)_{k,l}(\gamma_m^{(N)})_l = \sum_{l=k-2s}^{k+2s} (S_N - S)_{k,l}(\gamma_m^{(N)})_l .$$

Clearly $(e_m^{(N)})_k = 0$ for $k \le N$, hence we only have to consider the case $k > N$:

$$|(e_m^{(N)})_k| = |\sum_{l=k-2s}^{k+2s} (S_N - S)_{k,l}(\gamma_m^{(N)})_l| \le B\frac{D\lambda^{-s}}{1-\lambda}\sum_{l=k-2s}^{k+2s} \lambda^{|l|}$$
$$\le \frac{BD\lambda^{-s}}{1-\lambda}\frac{\lambda^{k-2s}-\lambda^{k+2s}}{1-\lambda} \le \frac{BD}{(1-\lambda)^2}\lambda^{k-3s},$$

whence

$$|(e_m^{(N)})_k|^2 \le \left(\frac{BD}{(1-\lambda)^2}\right)^2 \lambda^{2k-6s}.$$

Due to $(S_N - S)_{k,l} = 0$ for $|k-l| > 2s$ and since $(\gamma_m^{(N)})_k = 0$ for $|k| > N$ we obtain $\sum_{k=0}^{\infty} |(e_m^{(N)})_k|^2 = \sum_{k=N+1}^{N+2s} |(e_m^{(N)})_k|^2$ and therefore

$$\|e_m^{(N)}\|^2 = 2\sum_{k=N+1}^{N+2s} |(e_m^{(N)})_k|^2 \le 2\frac{B^2D^2}{(1-\lambda)^4}\lambda^{-6s}\sum_{k=N+1}^{N+2s}\lambda^{2k}$$
$$\le \frac{2B^2D^2}{(1-\lambda)^4}\lambda^{2N}\frac{\lambda^{2-6s}-\lambda^{2-2s}}{1-\lambda^2} \le \left(\sqrt{2}BD\lambda^N\frac{\lambda^{-3s}}{(1-\lambda)^3}\right)^2$$

which implies

$$\|e_m^{(N)}\| \le \sqrt{2}BD\lambda^N(\lambda^s - \lambda^{s+1})^{-3}. \tag{42}$$

Combining (42) with (37) yields the bound (33).
(iii): In this case

$$\|g_m - P_N g_m\|^2 \le 2c\sum_{k=N+1}^{\infty}\lambda^{\alpha k}. \tag{43}$$

The result follows now along similar lines as in the proof of statement (ii) by using (37) and applying Proposition 3.4.
(iv): We have

$$\|g_m - P_N g_m\|^2 \le 2c\sum_{k=N+1}^{\infty}(1+k)^{-\alpha} \le 2c\int_N^{+\infty}(1+x)^{-\alpha}\,dx \le \frac{2c}{\alpha-1}(1+N)^{-\alpha}.$$

The rest is as in (iii). ❑

Remarks:

(i) In spite of these results, good a priori estimates of the frame bounds as given e.g. in [9, 25] play an important role.

(ii) A different approach to approximate the inverse of the frame operator by finite dimensional methods is due to Casazza and Christensen. In [7] they consider a more general setting, however at the cost that their estimates are less explicit and therefore less useful in applications.

4 Approximation of $S^{-\frac{1}{2}}$ and tight frames

An important special case of frames are so-called *tight frames*, which are defined in terms of equality of the upper and lower frame bound, i.e., $A = B$. Besides the nice stability property $cond(S) = 1$, tight frames are distinguished by a simple expression for the duals, which are just scaled versions of the original frame elements, more precisely one has $\gamma_{m,n} = \frac{1}{A} g_{m,n}$. Due to this close relationship to orthogonal bases the construction of tight frames is of great interest in theory and in applications. Daubechies [8] has shown that for a given frame, say $\{\varphi_k\}$, the set of functions $\{S^{-\frac{1}{2}}\varphi_k\}$ constitutes a tight frame. Recall that S is positive definite, hence $S^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are uniquely determined.

Recall that $S^{\frac{1}{2}}$ commutes with the same operators that commute with S. For shift-invariant systems this implies that the tight frame generated by $\varphi_{m,n} = S^{-\frac{1}{2}} g_{m,n}$ also constitutes a shift-invariant system. We want to approximate the elements of the tight frame $\{\varphi_{m,n}\}$ by the solutions of the finite sections

$$(P_N S P_N)^{\frac{1}{2}} \varphi_m^{(N)} = P_N g_m \, .$$

One can easily check that the entries of $S^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ decay exponentially (polynomially) off the diagonal if the g_m have compact support or exponential (polynomial) decay, see e.g. [24, 29]. The applicability of the finite section method follows now from Theorem 5.1 in [18]. Along the same lines of the proof of Theorem 3.9 one can show that $\varphi_m^{(N)} := (P_N S P_N)^{-\frac{1}{2}} P_N g_m$ converges exponentially fast to φ_m. Since we do not have explicit expressions for the constants in the estimate of the decay of the entries of $S^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ we only can give following qualitative result for the approximation error.

Theorem 4.1 *Let $\{g_{m,n}\}$ be a shift-invariant frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with frame operator S and frame bounds A, B. Assume that $\|g\| = 1$ and set $\varphi_m =$*

$S^{-\frac{1}{2}}g_m$ and $\varphi_m^{(N)} = (P_N S P_N)^{-\frac{1}{2}} P_N g_m$. Then the finite section method is applicable to $S^{\frac{1}{2}}$. If moreover g is compactly supported with $g(k) = 0$ for $|k| > s$ then

$$\|\varphi_m - \varphi_m^{(N)}\| \le K\beta^N \qquad N = 0, 1, \dots$$

for $\beta \in (0,1)$ and some constant $K > 0$, where β and K depend on the frame bounds A, B and the support length s, but are independent of N.

5 Rate of convergence by "canonical" approximation using a periodic model

We have seen in Section 3 that the functions $\gamma_{m,n}^{(N)}$ which are the duals to the truncated functions $P_N g_{m,n}$ converge exponentially fast to the duals $\gamma_{m,n}$ of the infinite-dimensional problem. So from a theoretical point of view we know that we can compute a good approximation to γ_m by solving the finite system $S_N \gamma_m^{(N)} = g_m^{(N)}$. From a numerical point of view, however, we are interested in solving $S_N \gamma_m^{(N)} = g_m^{(N)}$ as efficient as possible.

Unfortunately due to the truncation of S we loose some structural properties of the problem. For instance the inverse of S or the product S^2 still have block Toeplitz structure, whereas S_N^{-1} and S_N^2 are no longer of Toeplitz type. Certainly one could apply one of the fast solvers for Toeplitz systems, but we can also try to find a more canonical way to design a finite dimensional model, which allows preserving the perfect symbol calculus of Laurent operators. This is one motivation to use a periodic model for the design of numerical algorithms. The periodic model is implicitly used in algorithms that compute the duals g_m by sampling and inverting the polyphase matrix in filter bank theory [34, 5].

Despite the fact that the periodic model is widely used in connection with shift-invariant systems, almost no attempt has been made to investigate the relation between the duals of the periodic finite model and the duals of the infinite dimensional problem.

One exception is [25] where Janssen shows that if the $g_{m,n}$ constitute a Gabor frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with frame bounds A, B and if $g_{m,n} \in \boldsymbol{\ell}^1(\mathbb{Z})$, then the periodized functions $g_{m,n}^{per}(k) = \sum_{j=-\infty}^{\infty} g_{m,n}(k - jL)$ constitute a frame for $\mathbb{C}^L$ with frame bounds A, B. Furthermore the duals for the periodized frame can be obtained by periodizing the duals of the original frame elements. The drawback of this otherwise very appealing result is that the length of the period L has to be the least common multiple of the time shift parameter a and the number of frequency channels M. This condition can be a serious restriction in practice. For instance take $a = 2, M = 3$, then the periodized functions are of length 6, which may be much too short for many applications.

In this section we will clarify the question under which conditions the duals computed within the periodic model converge to the duals $\gamma_{m,n}$ when the length of the period tends to infinity.

Let the functions g_m be compactly supported, such that $g_m(k) = 0$ for $|k| > s$. We construct our periodic model, as usual, by extending the truncated functions $P_N g_m$ periodically beyond the interval $[-N, N]$ for $N > s$ by setting

$$ {}^{P}g_m(k + lN) := g_m(k) \qquad \text{for } k = -N, \dots, N \ , l \in \mathbb{Z} \, . \tag{44} $$

The other elements of our periodic finite shift-invariant system are now given by

$$ {}^{P}g_{m,n}(k) = {}^{P}g_m(k - na) \, , \tag{45} $$

where the shift, according to our model, is understood as circulant shift. The periodicity of the ${}^{P}g_{m,n}$ implies that the corresponding frame operator is a block circulant matrix, rather than merely a block Toeplitz matrix. Observe that the entries of ${}^{P}S_N$ deviate from the entries of S_N only in the lower right and upper left corner of the matrix.

In other words we have replaced the block Toeplitz matrix S_N in the system $S_N \gamma_m^{(N)} = g_m^{(N)}$ by a block circulant matrix. However replacing a block Toeplitz system by a block circulant matrix in a linear system of equations will in general significantly affect the solution.

Moreover it is a priori not clear under which conditions invertibility of S_N implies invertibility of ${}^{P}S_N$. For instance take following hermitian positive definite Toeplitz matrix T and its "circulant completion" ${}^{P}T$:

$$ T = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} , \qquad {}^{P}T = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} . $$

Here T is invertible, whereas ${}^{P}T$ is obviously not invertible.

The following theorem justifies the usage of the circulant model for the approximation of dual shift-invariant systems. Implicitly the theorem also provides an estimate of the error we make when we approximate the solution of a block Toeplitz system by solving the related block circulant system.

Theorem 5.1 *Let $\{g_{m,n}\}$ be a shift-invariant frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with frame bounds A, B. Assume that $\|g_m\| = 1$ and that the g_m are compactly supported with $g_m(k) = 0$ for $|k| > s$. Let $N > 2s$ and let ${}^P g_{m,n}$ be the periodized functions as defined in* (44) *and* (45) *with frame operator ${}^P S_N$. Then ${}^P S_N$ is invertible and ${}^P\gamma_m^{(N)} = {}^P S_N^{-1} {}^P g_m$ converges to γ_m for $N \to \infty$. If $N > 3s$ then the rate of convergence can be estimated by*

$$\|P_N \gamma_m - {}^P\gamma_m^{(N)}\| \le 3\sqrt{2} C \lambda^N (\lambda^s - \lambda^{s+1})^{-3}, \tag{46}$$

where $\kappa = \frac{B}{A}, q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, $\lambda = q^{\frac{1}{2s}}$ and

$$C = \frac{1}{A} \max\{2\kappa, (1+\sqrt{\kappa})^2\}. \tag{47}$$

Proof: First we show that invertibility of S implies invertibility of ${}^P S_N$. The results about the spectrum of block circulant matrices in the proof of Theorem 3 in [37] in combination with Corollary 1 in [37] imply that

$$\sigma({}^P S_N) \subseteq \sigma(S).$$

$\sigma(S)$ is bounded away from zero since $A > 0$, hence it follows that ${}^P S_N$ is invertible.

To prove (46) we write

$$\|P_N \gamma_m - {}^P\gamma_m^{(N)}\| \le \|\gamma_m - \gamma_m^{(N)}\| + \|P_N \gamma_m^{(N)} - {}^P\gamma_m^{(N)}\|. \tag{48}$$

We already have estimated $\|\gamma_m - \gamma_m^{(N)}\|$ in (33) of Theorem 3.9, so it remains to estimate $\|P_N \gamma_m^{(N)} - {}^P\gamma_m^{(N)}\|$. Proceeding analogously to the calculation preceding equation (37) we can write

$$\|\gamma_m^{(N)} - {}^P\gamma_m^{(N)}\| \le A^{-1} \|({}^P S_N - S_N) {}^P S_N^{-1} P_N g_m\|. \tag{49}$$

Note that ${}^P S_N$ is three-band matrix, with one band centered at the main diagonal, and two other bands of width $2s$ located at the lower left and upper right corner of the matrix. It follows from Proposition 5.1 in [11] that the entries of ${}^P S_N^{-1}$ decay exponentially off the diagonal and off the lower right and upper left corner. More precisely,

$$|({}^P S_N^{-1})_{k,l}| \le \begin{cases} D\lambda^{|k-l|} & \text{if } 0 \le |k-l| \le N \\ D\lambda^{2N+1-|k-l|} & \text{if } N+1 \le |k-l| \le 2N\,, \end{cases} \tag{50}$$

where D is as in Theorem 3.5 with $\lambda = q^{\frac{1}{2s}}$.

Before we show that $\varphi_m^{(N)} := {}^P S_N^{-1}\,{}^P g_m$ also decays exponentially, observe that due to the construction of ${}^P S_N$ the matrix $E_N := {}^P S_N - S_N$ is a sparse matrix having non-zero entries only in the upper right and the lower left corner, as illustrated in (51).

$$E_N = \begin{array}{c} \begin{array}{cccccc} & & & {\scriptstyle N-2s+1} & & {\scriptstyle N} \end{array} \\ \begin{bmatrix} & & & \times & \cdots & \times \\ & & & & \ddots & \vdots \\ & & & & & \times \\ & & 0 & & & \\ \times & & & & & \\ \vdots & \ddots & & & & \\ \times & \cdots & \times & & & \end{bmatrix} \end{array} . \tag{51}$$

Denoting $e_m^{(N)} = E_N\,{}^P S_N^{-1}\,{}^P g_m$, it readily follows from the sparsity of E_N that only the first $2s$ and the last $2s$ components of e_N are non-zero, hence only the decay of the first $2s$ and last $2s$ components of $\varphi_m^{(N)} = {}^P S_N^{-1}\,{}^P g_m$ is of interest.

Since ${}^P S_N$ is a three-band matrix, the decay behavior of the entries of ${}^P S_N{}^{-1}$ is a little bit more complicated than for ordinary band matrices. We split the analysis of the decay of the entries of $\varphi_m^{(N)}$ into two steps by considering first the entries $(\varphi_m^{(N)})_k$ for $k = N-2s, \dots, N-m-1$ and then those for $k = N-m, \dots, N-1$.

Case $k = N-2s+1, \dots, N-s$:

$$(\varphi_m^{(N)})_k = \sum_{l=-N}^{N} ({}^P S_N^{-1})_{k,l} ({}^P g_m)_l = \sum_{l=-s}^{s} ({}^P S_N^{-1})_{k,l} ({}^P g_m)_l\,,$$

due to the compact support of g_m. Hence

$$|(\varphi_m^{(N)})_k| \leq |\sum_{l=-s}^{s} ({}^P S_N^{-1})_{k,l}| \leq D \sum_{l=-s}^{s} \lambda^{|k-l|} . \tag{52}$$

The assumption $N > 3s$ implies that $k \geq l$, hence

$$D \sum_{l=-s}^{s} \lambda^{|k-l|} = D \sum_{l=-s}^{s} \lambda^{k-l} = D\lambda^k \frac{\lambda^{-s} - \lambda^{s+1}}{1-\lambda} \leq \frac{D}{1-\lambda} \lambda^{k-s} . \tag{53}$$

Case $k = N - m, \dots, N - 1$:

$$\begin{aligned} |(\varphi_m^{(N)})_k| &\leq |\sum_{l=-s}^{s} ({}^P S_N^{-1})_{k,l}| \leq \sum_{l=-s}^{k-N-1} |({}^P S_N^{-1})_{k,l}| + \sum_{l=k-N}^{s} |({}^P S_N^{-1})_{k,l}| \\ &\leq D \sum_{l=-s}^{k-N-1} \lambda^{2N+1-k+l} + D \sum_{l=k-N}^{s} \lambda^{k-l} \leq \frac{2D}{1-\lambda} \lambda^{k-s} . \end{aligned} \tag{54}$$

Note that $\|e_m^{(N)}\|^2 = \sum_{k=-N}^{N} |(e_m^{(N)})_k|^2 = 2 \sum_{k=N-2s+1}^{N} |(e_m^{(N)})_k|^2$. Due to the special sparsity structure of E_N as illustrated in (51) we can write

$$\begin{aligned} |(e_m^{(N)})_k| &\leq \sum_{l=2N-2s+1+k}^{N} |(E_N)_{k,l}||\varphi_l| \leq \frac{2BD}{1-\lambda} \lambda^{-s} \sum_{l=2N-2s+1+k}^{N} \lambda^l \\ &\leq \frac{2BD}{(1-\lambda)^2} \lambda^{-s} (\lambda^{2N-2s+1+k} - \lambda N + 1 \leq \frac{2BD}{(1-\lambda)^2} \lambda^{k-3s+2N} . \end{aligned}$$

It follows that

$$|(e_m^{(N)})_k|^2 \leq \left(\frac{2BD}{(1-\lambda)^2} \right)^2 \lambda^{2k+4N-6s}$$

and

$$\begin{aligned} \|e_m^{(N)}\|^2 = 2 \sum_{k=-N}^{-N+2s-1} |(e_m^{(N)})_k|^2 \leq 2 \frac{(2BD)^2}{(1-\lambda)^4} \sum_{k=-N}^{-N+2s-1} \lambda^{2k-6s+4N} \\ \leq 2 \frac{(2BD)^2}{(1-\lambda)^4} \frac{\lambda^{2N-6s} - \lambda^{2N-2s}}{1-\lambda} \leq \left(2\sqrt{2} \frac{BD}{(1-\lambda)^3} \lambda^{N-3s} \right)^2 \end{aligned}$$

and therefore

$$\|e_m^{(N)}\| \le 2\sqrt{2}\frac{BD\lambda^{-3s}}{(1-\lambda)^3}\lambda^N = 2\sqrt{2}BD(\lambda^s - \lambda^{s+1})^{-3}\lambda^N ,$$

which together with (49) yields the desired estimate (46). ❑

The corresponding variant of Theorem 4.1 for the approximation of a tight frame via the periodic model is left to the reader.

Remark: Theorem 5.1 validates the approach proposed in filter bank theory to compute the duals γ_m by sampling and inverting the polyphase matrix.

In fact, the periodic extension of finite signals is more than just a simple and convenient way to handle boundary problems. It is in some sense the "canonical" way to set up a discrete model, when translations come into play, since it allows to preserve important mathematical properties of the infinite dimensional problem.

More generally speaking one preserves the underlying group structure of the original problem. By doing so, we can immediately apply the abstract results derived for shift-invariant systems [13, 20, 33]. In fact, sometimes, (e.g. in the case of Gabor analysis [32]) we gain even more structural properties compared to the infinite dimensional problem, due to the finiteness of the underlying abelian group.

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