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Finite and Infinite-Dimensional Models for Oversampled Filter Banks

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ABSTRACT A filter bank is a collection of functions $\{g_{m,n}\}$ of the form $g_{m,n}(k) = g_m(k - an)$. Such systems play an important role in digital signal processing and communication. For given analysis functions $g_{m,n}$ we want to find synthesis functions $\gamma_{m,n}(k) = \gamma_m(k - an)$ such that each signal f can be written as $f = \sum \langle f, \gamma_{m,n}\rangle g_{m,n}$. The mathematical theory addresses the questions when such $\gamma_{m,n}$ exist and how to construct them usually in infinite dimensions, whereas numerical methods have to operate with finite-dimensional models. In this chapter we discuss the relation between certain finite-dimensional models used for numerical procedures and infinite-dimensional filter bank theory. We show that the "synthesis filter bank" obtained by solving a finite-dimensional problem converges to the synthesis filter bank of the original infinite-dimensional problem in $\ell^2(\mathbb{Z})$. We give rates of approximations if the g_m satisfy certain decay properties (such as finite impulse response or polynomial decay) and also address the construction of paraunitary filter banks. Furthermore, we investigate the use of the periodic finite model and validate existing methods that exploit the polyphase representation. Finally for oversampled DFT filter banks a factorization of the frame operator is presented that improves upon the known factorization by means of the Zak transform.

1.1 Introduction

Filter banks play an important role in digital signal processing and communication [45, 47, 40]. A uniform filter bank – also known as time-discrete shift-invariant system in the mathematical literature [32, 28] – is a system of functions $\{g_{m,n}\}$ of the form

$$g_{m,n}(k) = g_m(k - na), \qquad k, n \in \mathbb{Z}, m = 0, 1, \dots, M - 1, \tag{1.1}$$

where $g_m \in \ell^2(\mathbb{Z})$ and $0 < a \in \mathbb{N}$.

For a given *analysis filter bank* $\{g_{m,n}\}$ we want to find a *synthesis filter bank* $\{\gamma_{m,n}\}$ with

$$\gamma_{m,n}(k) = \gamma_m(k - na), \qquad k, n \in \mathbb{Z}, m = 0, 1, \ldots, M-1, \tag{1.2}$$

such that any $f \in \boldsymbol{\ell}^2(\mathbb{Z})$ has an $\boldsymbol{\ell}^2(\mathbb{Z})$-convergent representation

$$f = \sum_{m=0}^{M-1} \sum_{n=-\infty}^{\infty} \langle f, \gamma_{m,n} \rangle g_{m,n} \,. \tag{1.3}$$

Of course, if $\{g_{m,n}\}$ is an orthogonal basis for $\boldsymbol{\ell}^2(\mathbb{Z})$ we have $\gamma_{m,n} = g_{m,n}$. In the case of a nonorthogonal filter bank and in particular for an oversampled filter bank (i.e. $a < M$) the determination of $\gamma_{m,n}$ is more involved. It is well-known that the theoretical answer when and how we can find such synthesis functions $\gamma_{m,n}$ can be given using the concept of frames in $\boldsymbol{\ell}^2(\mathbb{Z})$. When implementing filter banks on a computer however, we cannot directly make use of the infinite-dimensional theory, but have to set up a finite-dimensional model.

There is a variety of research papers dealing with oversampled filter banks in $\boldsymbol{L}^2(\mathbb{R})$ or $\boldsymbol{\ell}^2(\mathbb{Z})$, see [47, 32, 4, 5, 8, 9], there is also no lack of methods claiming to solve for the synthesis filter banks, e.g. see [40, 49, 30, 52, 41], those methods naturally operate in finite dimensions. But only little is known about the relation of the finite-dimensional solutions and the original solutions of the infinite dimensional model.

The purpose of this chapter is to clarify the relation between finite-dimensional models as they are used for actual implementation of filter bank algorithms and the infinte-dimensional filter bank theory. Whatever finite-dimensional model we choose for numerical purposes, such as for the computation of the synthesis functions $\gamma_{m,n}$, we have to ask if these approximations really converge to the original solution with increasing dimension. Moreover, whenever possible, we have to give a priori bounds on the rate of approximation. We will also investigate how to set up a valid finite-dimensional model that allows preserving important structural properties of filter banks such as polyphase representation.

The chapter is organized as follows. In the remaining part of the introduction we briefly review some basic facts on the connection between filter banks and frames. In Section 1.2 we describe two finite-dimensional models for filter banks and outline the finite section method. In Section 1.3 we prove the convergence of finite-dimensional approximations and give rates of approximation for filter banks whose analysis functions satisfy certain decay properties. We discuss the relation of the polyphase representation and finite-dimensional models in Section 1.4. Finally, in Section 1.6 we show how our results can be used to derive an efficient factorization of the frame operator for oversampled DFT filter banks, that improves upon the well-known Zak transform factorization.

Before we proceed, we introduce some notation. Let $x \in \ell^2(\mathbb{Z})$, then $\|x\|$ is the standard ℓ^2-norm of x. $\ell^2_a(\mathbb{Z})$ is the space of square summable sequences $\{x_k\}_{k\in\mathbb{Z}}$ where x_k is an $a \times a$ matrix. We denote the spectrum of an operator T by $\sigma(T)$. If T is positive definite and invertible then $\|T\| = \max\{\mu : \mu \in \sigma(T)\}$ and in this case $\mathrm{cond}(T) := \|T\|\|T^{-1}\|$ is the condition number of T.

1.1.1 Oversampled filter banks and frames

The concept of frames [14, 51] provides a natural environment to study oversampled filter banks [32, 8, 40, 4]. Since the connection between frame theory and filter banks is well understood, we only give a brief review from the filter bank view point, for more details see [47, 8, 5].

We say that the set $\{g_{m,n}\}$ is a frame for $\ell^2(\mathbb{Z})$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{m=0}^{M-1} \sum_{n=-\infty}^{\infty} |\langle f, g_{m,n}\rangle|^2 \le B\|f\|^2 , \qquad \forall f \in \ell^2(\mathbb{Z}) . \tag{1.4}$$

If $\{g_{m,n}\}$ is a frame for $\ell^2(\mathbb{Z})$, then there exists a so-called *dual frame*, defined by

$$\gamma_{m,n}(k) = (S^{-1} g_m)(k - na), \tag{1.5}$$

where S is the frame operator given by

$$Sf = \sum_{m=0}^{M-1} \sum_{n=-\infty}^{\infty} \langle f, g_{m,n}\rangle g_{m,n} . \tag{1.6}$$

If a filter bank satisfies (1.4) we also call it a *perfect reconstruction* filter bank [47]. If not mentioned otherwise, we will always assume in this chapter that the filter bank $\{g_{m,n}\}$ is a frame for $\ell^2(\mathbb{Z})$, in which case equation (1.3) always holds. In general there are many synthesis functions of the form (1.2) that satisfy condition (1.3). In this chapter we are restrict our attention to the ones defined by (1.5) (which are distinguished by the fact that they have minimal ℓ^2-norm [11]). Therefore we will be somewhat sloppy in our terminology and refer to the $\gamma_{m,n}$ defined in (1.5) as *the* synthesis filter bank or *the* dual frame, respectively.

If the frame bounds satisfy $A = B$, then the frame is called *tight* and the functions of the dual frame are given by $\gamma_{m,n} = \frac{1}{A} g_{m,n}$. In filter bank terminology we call $\{g_{m,n}\}$ in this case a *paraunitary* filter bank [45].

We define the analysis operator F by

$$F : f \in \ell^2(\mathbb{Z}) \to Ff = \{\langle f, g_{m,n}\rangle\}_{m,n} \tag{1.7}$$

and the synthesis operator, which is just the adjoint operator of F, by

$$F^* : c \in \ell^2(\mathbb{Z} \times \mathcal{I}) \to F^* c = \sum_{m,n} c_{m,n} g_{m,n} \tag{1.8}$$

where $\mathcal{I} := \{0, 1, \dots, M-1\}$. Clearly, the frame operator can be written as $S = F^* F$.

If the g_m are constructed from a single function g by complex modulations, i.e.,

$$g_m(k) = e^{2\pi i m k/M} g(k)$$

we obtain the *oversampled DFT filter bank* [47, 8, 4], also known under the name *Gabor frame* or *Weyl-Heisenberg frame* [19, 40, 47, 17, 4]. A nice property of oversampled DFT filter banks is that the synthesis filter bank is also generated by a single function, we have [10]

$$\gamma_{m,n}(k) = e^{2\pi i m k/M} \gamma(k - na) \qquad \text{with } \gamma = S^{-1} g \ .$$

Let G_m be the transfer function of the impulse response g_m. The *polyphase analysis matrix* $\mathbf{G}$ of the analysis filter bank is defined as

$$\mathbf{G}(e^{2\pi i\vartheta}) = \begin{bmatrix} \mathbf{G}_{00}(e^{2\pi i\vartheta}) & \dots & \mathbf{G}_{0(a-1)}(e^{2\pi i\vartheta}) \\ \mathbf{G}_{10}(e^{2\pi i\vartheta}) & \dots & \mathbf{G}_{1(a-1)}(e^{2\pi i\vartheta}) \\ \dots & \dots & \dots \\ \mathbf{G}_{(M-1)0}(e^{2\pi i\vartheta}) & \dots & \mathbf{G}_{(M-1)(a-1)}(e^{2\pi i\vartheta}) \end{bmatrix}, \tag{1.9}$$

where

$$\mathbf{G}_{mn}(e^{2\pi i\vartheta}) = \sum_{k=-\infty}^{\infty} g_m(n - ka) e^{2\pi i\vartheta k}, \qquad \vartheta \in [0,1), \tag{1.10}$$

represents the n-th polyphase component of $\mathbf{G}(e^{2\pi i\vartheta})$, see [45, 47]. The reader will recognize the *Zak transform* [29] in formula (1.10). A similar representation holds for the synthesis filter bank. Using equations (1.7),(1.9), and (1.10) we conclude that the matrix

$$\mathbf{S}(e^{2\pi i\vartheta}) = \mathbf{G}^H(e^{2\pi i\vartheta}) \mathbf{G}(e^{2\pi i\vartheta}) \tag{1.11}$$

is a matrix representation of the frame operator S, cf. [53, 4]. In mathematical terminology this simply means that S is a *Laurent operator* with matrix-valued symbol [43]. We will return to this interpretation in the following sections.

1.2 Finite-dimensional models for filter banks

The numerical computation of the synthesis functions $\gamma_{m,n} = \gamma_m(\cdot - na)$ involves the solution of the biinfinite system of equations

$$S g_m = \gamma_m \, . \tag{1.12}$$

We certainly cannot solve a system of biinfinite order, but we can solve it approximately, based on a finite-dimensional model.

Of course, the $\gamma_{m,n}$ can also be computed by means of the polyphase representation of the frame operator. But it is clear that a unitary transformation of S (or the analysis operator F respectively) does not avoid the difficulty of approximating an infinite-dimensional problem by a finite-dimensional model. We will return to the polyphase representation and its use in finite dimensional models in Section 1.4.

We will study three frequently used approaches for the implementation of oversampled filter banks. These approaches and the corresponding finite-dimensional models can be described as follows (we give a more precise mathematical formulation in the following sections):

(i) Truncate the functions $g_{m,n}$ to the interval $[-N, N]$ and pad them with zeros outside the interval $[-N, N]$. Then an approximation to γ_m is computed by applying formula (1.5) to the truncated system.

(ii) Establish the polyphase matrix $\mathbf{S}$ (or an approximation to it if the g_m have not finite impulse response) of the frame operator S, compute an approximation to its inverse by sampling and inverting $\mathbf{S}(e^{2\pi i\vartheta})$, and transform the result back in the time domain. As the reader may notice (or otherwise will see in the following sections) this approach actually makes implicitly use of following finite-dimensional model.

(iii) Truncate the functions $g_{m,n}$ to the interval $[-N, N]$ and extend the truncated functions periodically across the boundaries. An approximation to γ_m is computed based on formula (1.5) by inverting the corresponding truncated periodized frame operator.

We will see that each of these approaches has its advantages and disadvantages. On the one hand model (i) is somewhat easier to analyze and gives a better rate of convergence than those models used in (ii) and (iii). On the other hand model (iii) preserves perfect symbol calculus for Laurent operators which is important when we want to use the polyphase representation. Models (ii) and (iii) are also preferable from a numerical point of view, since they give rise to more efficient algorithms.

Note that for some special choices of $\{g_{m,n}\}$ closed-form solutions for $\gamma_{m,n}$ are known [10, 18, 31] (which does not necessarily mean that those closed-form solutions can be cheaply computed in practice).

We continue by first investigating approach (i) in more detail.

1.2.1 The finite section method for oversampled filter banks

Equations (1.5) and (1.6) tell us that all $g_{m,n}$ are required for the computation of the synthesis function γ_m. But if the g_m have nice decay properties, such that most of their energy is contained in some interval around the origin, say, then one may hope that the γ_m will have similar properties. One could further argue that using only those functions $g_{m,n}$ concentrated in some interval, $[-N, N]$ say, should be sufficient to compute a good approximation to γ_m. Let us put these vague ideas in a more precise mathematical setting.

For $N \in \mathbb{N}$ and $x \in \boldsymbol{\ell}^2(\mathbb{Z})$ define the orthogonal projections P_N by

$$P_N x = (\dots, 0, 0, x_{-N}, x_{-N+1}, \dots, x_{N-1}, x_N, 0, 0, \dots) \tag{1.13}$$

and identify the image of P_N with the $2N+1$-dimensional space $\mathbb{C}^{2N+1}$. We construct a finite-dimensional approximation S_N of the frame operator S via the truncated frame elements $P_N g_{m,n}$ by

$$S_N = (P_N F^*)(F P_N) \tag{1.14}$$

where F is defined in (1.7). The N-th approximation $\gamma_m^{(N)}$ to γ_m is then given by the solution of the finite-dimensional system of equations

$$S_N \gamma_m^{(N)} = P_N g_m \, . \tag{1.15}$$

Since $S_N = P_N F^* F P_N = P_N S P_N$, the corresponding system of equations $S_N \gamma_m^{(N)} = g_m^{(N)}$ is just a finite section of the biinfinite system $S\gamma = g$.

The questions that arise with such an approximation scheme are:
(i) Does $\gamma_m^{(N)}$ converge to γ_m for $N \to \infty$? Or more general, is the finite section method is applicable to S?
(ii) If the g_m satisfy certain decay conditions, can we give an estimate on the rate of convergence? In other words: given $\delta > 0$, can we provide an a priori estimate for the size of the interval $[-N, N]$, such that $\|\gamma_m - \gamma_m^{(N)}\| \le \delta$, where N depends on δ and the decay of g_m?

1.2.2 *Finite sections, Laurent operators, and polyphase matrices*

The *finite section method* or *projection method* is a classical technique to approximate the solution of infinite Toeplitz-type systems. For an introduction and many variations of the theme the reader is referred to the books of Gohberg and Fel'dman [21], Böttcher and Silbermann [6] and Hagen, Roch, and Silbermann [23]. Since we use the finite section method as a main tool throughout the paper, we briefly describe it in the specific context we will apply it.

Let $\mathbb{T}$ be the complex unit circle and let $\tau(z)$ be a continuous function on $\mathbb{T}$, $\tau \in \boldsymbol{C}(\mathbb{T})$. Let $\tau_k \in \mathbb{C}$ stand for the k-th Fourier coefficient of τ,

$$\tau_k = \int_0^1 \tau(e^{2\pi i\omega}) e^{-2\pi i k\omega} \, d\omega \, ,$$

then the operator $T(\tau)$ which acts on functions $x \in \boldsymbol{\ell}^2(\mathbb{Z})$ by

$$(Tx)_k = \sum_{j \in \mathbb{Z}} \tau_{j-k} x_j \,, \qquad k \in \mathbb{Z} \,, \tag{1.16}$$

is called *Laurent operator* or *multiplication operator* with *generating function* or *symbol* τ. A Laurent operator may also be characterized as a bounded linear operator acting on $\boldsymbol{\ell}^2(\mathbb{Z})$ which commutes with the forward shift operator U on $\boldsymbol{\ell}^2(\mathbb{Z})$, cf. Chapter 13.2 in [20]. Since a Laurent operator can be represented by a biinfinite Toeplitz matrix, we will often identify $T(\tau)$ with its matrix representation with respect to the standard basis in $\boldsymbol{\ell}^2(\mathbb{Z})$.

Consider the operator equation $Tx = y$, where $x, y \in \boldsymbol{\ell}^2(\mathbb{Z})$, or in matrix representation

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \tau_0 & \tau_1 & \tau_2 & \cdots \\ \cdots & \tau_{-1} & \tau_0 & \tau_1 & \cdots \\ \cdots & \tau_{-2} & \tau_{-1} & \tau_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix} . \tag{1.17}$$

For the approximate solution of the biinfinite Toeplitz system in (1.17) by the finite section method one considers the finite linear equations

$$T_N x^{(N)} = y^{(N)} \,, \tag{1.18}$$

where $y^{(N)} = (y_{-N}^{(N)}, \ldots, y_N^{(N)})$ and where T_N are the finite sections of the biinfinite Toeplitz matrix in (1.17):

$$T_0 = (\tau_0) \,, T_1 = \begin{bmatrix} \tau_0 & \tau_1 & \tau_2 \\ \tau_{-1} & \tau_0 & \tau_1 \\ \tau_{-2} & \tau_{-1} & \tau_0 \end{bmatrix} , T_2 = \begin{bmatrix} \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ \tau_{-1} & \tau_0 & \tau_1 & \tau_2 & \tau_3 \\ \tau_{-2} & \tau_{-1} & \tau_0 & \tau_1 & \tau_2 \\ \tau_{-3} & \tau_{-2} & \tau_{-1} & \tau_0 & \tau_1 \\ \tau_{-4} & \tau_{-3} & \tau_{-2} & \tau_{-1} & \tau_0 \end{bmatrix} , \ldots$$

Using the the orthogonal projections defined in (1.13) and identifying the image of P_N with the $(2N+1)$-dimensional space $\mathbb{C}^{2N+1}$ we can express the equations (1.18) in the form

$$P_N T P_N x^{(N)} = P_N y \,. \tag{1.19}$$

We say that the finite section method is *applicable* to T, if beginning with some $N \in \mathbb{N}$ for each $y \in \boldsymbol{\ell}^2(\mathbb{Z})$ the equation

$$P_N T P_N x^{(N)} = P_N y \tag{1.20}$$

has a unique solution $x^{(N)} \in \operatorname{Im} P_N$ and as $N \to \infty$ the vectors $x^{(N)}$ tend to the solution of $Tx = y$.

Now let τ be a continuous $a \times a$-matrix valued function on $\mathbb{T}$, $\tau \in \boldsymbol{C}_{a \times a}(\mathbb{T})$. Let $\tau_k \in \mathbb{C}^{a \times a}$ stand for the k-th Fourier coefficient of τ, then the operator T defined by the action

$$(Tx)_k = \sum_{j \in \mathbb{Z}} \tau_{j-k} x_j \,, \qquad x \in \boldsymbol{\ell}^2{}_a \,, \tag{1.21}$$

is called a *block Laurent operator* with generating function τ. A block Laurent operator can be represented by a biinfinite block Toeplitz matrix, where the τ_k in (1.17) are $a \times a$ matrices. Analogous to above, for the approximate solution of biinfinite block Toeplitz systems we consider finite sections of the form

$$P_N T P_N x^{(N)} = P_N y \,.$$

It follows directly from the definition of the functions $g_{m,n}$ in (1.1) that the filter bank analysis operator F defined in (1.7) satisfies $F = U_M F U_a^*$, where $U_a(\varphi_k)_{k \in \mathbb{Z}} := (\varphi_{k-a})_{k \in \mathbb{Z}}$ is the shift-operator of order a. Thus the associated frame operator S is a block Laurent operator and the operator equation $S\gamma_m = g_m$ can be expressed as a biinfinite block Toeplitz system of equations, where the blocks of the system matrix are of size $a \times a$. Note that the blocks themselves are not of Toeplitz type. It is exactly this block-Laurent property of S that is utilized in the polyphase representation, see also [43]. The symbol τ corresponds to the polyphase matrix $\mathbf{S}$ of equation (1.11).

1.2.3 Polyphase matrices and "perfect symbol calculus" in finite dimensions

Let $T(a)$ and $T(b)$ be two (block) Laurent operators on $\boldsymbol{\ell}^2(\mathbb{Z})$, then

$$T(a)T(b) = T(ab) \,, \tag{1.22}$$

see e.g. [6]. This well-known relation is sometimes referred to as symbol calculus for Laurent operators [6]. Of course, also the inverse $[T(a)]^{-1}$ is a Laurent operator. When we consider the truncated operators $T_N(a)$ and $T_N(b)$ as described in (1.19), then $T_N(a)T_N(b) \neq T_N(ab)$. Furthermore $[T_N(a)]^{-1}$ is no longer a truncated Laurent operator. For oversampled filter banks this means that the truncated frame operator no longer possesses a polyphase representation.

Thus in order to preserve the polyphase property one often uses a periodic finite-dimensional for the approximate computation of $\gamma_{m,n}$. The functions $g_{m,n}$ are truncated to $[-N, N]$ and then extended periodically across the interval $[-N, N]$. In other words all computations are done in the ring $\mathbb{Z}_{2N+1} = \mathbb{Z} \bmod (2N+1)$.

Using this finite-dimensional model one can easily verify that the resulting "frame operator" can be represented by a block circulant matrix with non-circulant blocks, which in turn can be block-diagonalized by a finite Zak transform [12, 1, 41]. This is the finite analog of the polyphase representation (1.11). Increasing the length of the period $2N+1$ corresponds to increasing the density of sampling of the polyphase matrix $\mathbf{S}(e^{2\pi i\vartheta})$. We will discuss questions of convergence of this approach in detail in Section 1.4.

1.3 Convergence and rates of approximation for finite-dimensional approximations

We first show that the finite section method outlined in Section 1.2 applies to the frame operator of *any* frame in $\boldsymbol{\ell}^2(\mathbb{Z})$. Recall that a sequence of linear operators $T_n : \boldsymbol{B} \to \boldsymbol{B}$ converges strongly to T if for each $f \in \boldsymbol{B}$ the sequence $\boldsymbol{B}f$ converges to Tf. If the operators T_n are invertible it is naturally to ask under which conditions the operators T_n^{-1} converge strongly to T^{-1}. Following result is attributed to Kantorovich [39].

Proposition 1.3.1 *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of invertible operators on a Banach space $\boldsymbol{B}$, and assume that the sequence is uniformly bounded above and below, i.e, that there exist constants $0 < C_1, C_2 < \infty$ such that*

$$C_1 \|f\|_{\boldsymbol{B}} \le \|T_n f\|_{\boldsymbol{B}} \le C_2 \|f\|_{\boldsymbol{B}} \qquad \text{for all } n\in\mathbb{N} \text{ and } f \in \boldsymbol{B}. \tag{1.23}$$

Then $\{T_n\}$ converges in the strong operator topology to an invertible operator if and only if the same is true for $\{T_n^{-1}\}_n$, and then

$$\{\lim T_n\}^{-1} = \lim\{T_n^{-1}\}\,.$$

Now it is easy to prove following theorem.

Theorem 1.3.2 *Assume that $\{f_k\}_{k\in\mathbb{Z}}$ is a frame for $\boldsymbol{\ell}^2(\mathbb{Z})$ with frame bounds A, B and dual frame $\{h_k\}_{k\in\mathbb{Z}}$. Set $S_N = P_N S P_N$ and $h_k^{(N)} = S_N^{-1} P_N f_k$. Then the finite section method is applicable to S and in particular $\|h_k^{(N)} - h_k\| \to 0$ for $N \to \infty$.*

Proof: Clearly S and S_N are hermitian positive definite operators. Let $x \in \mathbb{C}^{2N+1}$ with $\|x\| = 1$. Then

$$\langle S_N x, x\rangle = \langle P_N S P_N x, x\rangle = \langle Sx, x\rangle \ge \frac{1}{\|S^{-1}\|} = A. \tag{1.24}$$

Along similar lines we get $\|S_N\| \le \|S\| \le B$. Hence $\sigma(S_N) \subseteq \sigma(S)$. In other words S_N is invertible and moreover the S_N are uniformly bounded by

$$A \le S_N \le B, \quad B^{-1} \le S_N^{-1} \le A^{-1}, \quad \text{for all } N \in \mathbb{N}. \tag{1.25}$$

Application of Proposition 1.3.1 completes the proof. ❑

1.3.1 Rates of approximation for oversampled filter banks

If the functions g_m of the analysis filter bank have a certain rate of decay, one would hope that the synthesis filter bank functions also share this behavior. We consider the following three types of decay for g_m and study first how a certain decay property translates to decay properties of the frame operator S.

a) *finite impulse response:* $g_m(k) = 0$ for $|k| > s$.
b) *Exponential decay:* $|g_m(k)| \leq c\lambda^{\alpha|k|}$ for $0 < \alpha < 1$, $\lambda \in (0,1)$ and some constant c.
c) *Polynomial decay:* $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ for $\alpha > 1$ and a constant c.

Note that if the $g_{m,n}$ have finite impulse response (FIR) one cannot that the $\gamma_{m,n}$ are also of FIR type. For DFT filter banks Bölcskei has shown [3] that the minimal norm duals are only of FIR type in very specific cases. But we can show that the synthesis functions are decaying exponentially, which is "almost as good as" FIR.

Proposition 1.3.3 *Let $\{g_{m,n}\}$ be a perfect reconstruction FIR filter bank for $\ell^2(\mathbb{Z})$ such that $g(k) = 0$ for $|k| > s$. Assume that $\|g_m\| = 1$ and let A, B be the frame bounds of the filter bank. Denote $\kappa = B/A$ and $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$. Then*

$$|S^{-1}_{k,l}| \leq D\lambda^{|k-l|} \tag{1.26}$$

and the decay of the $\gamma_{m,n}$ is bounded by

$$|\gamma_m(k)| \leq D\lambda^{|k|}\frac{\lambda^{-s}}{1-\lambda}, \tag{1.27}$$

where $\lambda = q^{\frac{1}{2s}}$ and $D = \frac{1}{A}\max\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\}$.

The proof of this result and of the result below can be found in [43] and makes use of the decay of the inverse of banded matrices, see [13].

For exponentially or polynomially decaying filter banks we can state similar results, however we could not derive explicit expressions of the involved constants.

Proposition 1.3.4 *Assume that the functions $g_{m,n}, m = 0,\dots,M-1$ constitute a perfect reconstruction filter bank for $\ell^2(\mathbb{Z})$ with frame operator S and synthesis filter bank $\{\gamma_m^{(N)}\}$.*
(a) Exponential decay: If $|g_m(k)| \leq c\lambda^{\alpha|k|}$ for $\alpha > 0$ and $\lambda \in (0,1)$. Then there exists an $\alpha_1 < \alpha$ and a constant c_1 depending on α_1 such that

$$|S_{k,l}| \leq c_1\lambda^{\alpha_1|k-l|}\,. \tag{1.28}$$

Furthermore the duals γ_m satisfy

$$|\gamma_m(k)| \leq c_2\lambda^{\alpha_2|k|}\,. \tag{1.29}$$

for some $\alpha_2 < \alpha$ *and a constant* c_2 *depending on* α_2.
(b) Polynomial decay: If $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ *for* $\alpha > 1$, *then there exists a constant* c_1 *such that*

$$|S_{k,l}| \leq c_1(1+|k-l|)^{-\alpha}\,. \tag{1.30}$$

Furthermore there is a constant c_2 *such that*

$$|\gamma_m(k)| \leq c_2(1+|k|)^{-\alpha}\,. \tag{1.31}$$

The proof, which relies on results of Jaffard concerning the decay of the inverse of certain matrices [26] can be found in [43].

Remark: Let $\{f_k\}_{k\in\mathbb{Z}}$ be a frame for $\boldsymbol{L}^2(\mathbb{R})$ with dual frame $\{\varphi_k\}_{k\in\mathbb{Z}}$. Then φ_k can be written as $\varphi_k = \sum_{k\in\mathbb{Z}} R^{-1}_{k,l} f_l$ where R is the Gram matrix with entries $R_{k,l} = \langle f_k, f_l\rangle$. R is invertible on the range of the frame analysis operator. Hence for filter banks on $\boldsymbol{L}^2(\mathbb{R})$ we can obtain decay results for the synthesis functions similar to the ones in Propositions 1.3.3 and 1.3.4 by considering the decay of the inverse of R. This will be discussed in more detail elsewhere.

We are now able to answer the question about the approximation of synthesis filter banks in $\boldsymbol{\ell}^2(\mathbb{Z})$ by the finite section method [43].

Theorem 1.3.5 *Assume that* $\{g_{m,n}\}$ *is a perfect reconstruction filter bank for* $\boldsymbol{\ell}^2(\mathbb{Z})$ *with* $\|g_m\| = 1$ *and frame bounds* A, B. *Set* $S_N = P_N S P_N$ *and* $\gamma_m^{(N)} = S_N^{-1} P_N g_m$ *for* $m = 0, 1, \ldots, M-1$ *and denote* $\kappa = B/A$. *Then following statements hold.*
(i) The finite section method is applicable to S *and* $\gamma_m^{(N)}$ *converges to* γ_m *for* $N \to \infty$.
(ii) If the g_m *are compactly supported with* $g_m(k) = 0$ *for* $|k| > s$ *and* $N > 2s$, *then the rate of convergence can be estimated by*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \sqrt{2} C \lambda^N (\lambda^s - \lambda^{s+1})^{-3} \tag{1.32}$$

where $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, $\lambda = q^{\frac{1}{2s}}$ *and*

$$C = \frac{1}{A}\max\{2\kappa, (1+\sqrt{\kappa})^2\}\,. \tag{1.33}$$

(iii) If $|g_m(k)| \leq c\lambda^{\alpha|k|}$ *for* $\alpha > 0$ *and* $\lambda \in (0,1)$, *then there exists an* $\alpha_1 < \alpha$ *and a constant* c_1 *depending on* α_1 *such that*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \kappa c_1 \lambda^{\alpha_1 N}\,. \tag{1.34}$$

(iv) If $|g_m(k)| \leq c(1+|k|)^{-\alpha}$ *for* $\alpha > 1$, *then there exists a constant* c_1 *such that*

$$\|\gamma_m - \gamma_m^{(N)}\| \leq \kappa c_1 (1+|N|)^{-\alpha}\,. \tag{1.35}$$

Proof: Statement (i) follows from Theorem 1.3.2.

In order to estimate the rate of convergence for (ii), (iii) and (iv) consider

$$\begin{aligned}\|\gamma_m - \gamma_m^{(N)}\| &= \|S^{-1}g_m - S_N^{-1}P_N g_m\| = \|S^{-1}g_m - S^{-1}SS_N^{-1}P_N g_m\| \\ &\le \|S^{-1}\| \left(\|g_m - S_N S_N^{-1}P_N g_m\| + \|S_N S_N^{-1}P_N g_m - SS_N^{-1}P_N g_m\|\right) \\ &\le A^{-1}\left(\|g_m - P_N g_m\| + \|(S_N - S)S_N^{-1}P_N g_m\|\right) . \end{aligned} \tag{1.36}$$

(ii): Recall that S is a $2s$-banded matrix, i.e., $S_{k,l} = 0$ for $|k-l| > 2s$. By Proposition 1.3.3 we can estimate the decay of the entries of S^{-1} by

$$|S^{-1}_{k,l}| \le D\lambda^{|k-l|} \tag{1.37}$$

where C and λ are is as in Proposition 1.3.3. Equation (1.25) implies

$$\operatorname{cond}(S_N) \le \operatorname{cond}(S) ,$$

hence we can use the same λ and the same constant D in order to bound the decay of the entries of S_N^{-1} independently of N and obtain

$$|(S_N^{-1})_{k,l}| \le D\lambda^{|k-l|} . \tag{1.38}$$

Since $\gamma_m^{(N)} = S_N^{-1}P_N g_m$ it follows from Proposition 1.3.3 that

$$|(\gamma_m^{(N)})_k| \le D\lambda^{|k|}\frac{\lambda^{-s}}{1-\lambda} , \qquad k \in \mathbb{Z} . \tag{1.39}$$

In the next step we estimate $\|(S_N - S)S_N^{-1}P_N g_m\|$. Write $e_m^{(N)} = (S_N - S)\gamma_m^{(N)}$ and observe that $\|e_m^{(N)}\|^2 = \sum_{k=-\infty}^{\infty}|(e_m^{(N)})_k|^2 = 2\sum_{k=0}^{\infty}|(e_m^{(N)})_k|^2$ (note that $(e_m^{(N)})_0 = 0$). For $k = 0, 1., \dots$ consider

$$(e_m^{(N)})_k = \sum_{l=-\infty}^{\infty}(S_N - S)_{k,l}(\gamma_m^{(N)})_l = \sum_{l=k-2s}^{k+2s}(S_N - S)_{k,l}(\gamma_m^{(N)})_l$$

Clearly $(e_m^{(N)})_k = 0$ for $k \le N$, hence we only have to consider the case $k > N$:

$$\begin{aligned}|(e_m^{(N)})_k| &= |\sum_{l=k-2s}^{k+2s}(S_N - S)_{k,l}(\gamma_m^{(N)})_l| \le B\frac{D\lambda^{-s}}{1-\lambda}\sum_{l=k-2s}^{k+2s}\lambda^{|l|} \\ &\le \frac{BD\lambda^{-s}}{1-\lambda}\frac{\lambda^{k-2s} - \lambda^{k+2s}}{1-\lambda} \le \frac{BD}{(1-\lambda)^2}\lambda^{k-3s} ,\end{aligned}$$

whence

$$|(e_m^{(N)})_k|^2 \le \left(\frac{BD}{(1-\lambda)^2}\right)^2 \lambda^{2k-6s} .$$

Due to $(S_N - S)_{k,l} = 0$ for $|k-l| > 2s$ and since $(\varphi_m^{(N)})_k = 0$ for $|k| > N$ we obtain $\sum_{k=0}^{\infty} |(e_m^{(N)})_k|^2 = \sum_{k=N+1}^{N+2s} |(e_m^{(N)})_k|^2$ and therefore

$$\begin{aligned}
\|e_m^{(N)}\|^2 &= 2 \sum_{k=N+1}^{N+2s} |(e_m^{(N)})_k|^2 \leq 2 \frac{B^2 D^2}{(1-\lambda)^4} \lambda^{-6s} \sum_{k=N+1}^{N+2s} \lambda^{2k} \\
&\leq \frac{2B^2D^2}{(1-\lambda)^4} \lambda^{2N} \frac{\lambda^{2-6s} - \lambda^{2-2s}}{1-\lambda^2} \leq \left(\sqrt{2} BD\lambda^N \frac{\lambda^{-3s}}{(1-\lambda)^3} \right)^2 ,
\end{aligned}$$

which implies

$$\|e_m^{(N)}\| \leq \sqrt{2} BD\lambda^N (\lambda^s - \lambda^{s+1})^{-3} . \tag{1.40}$$

Combining (1.40) with (1.36) yields the bound (1.32).
(iii): In this case

$$\|g_m - P_N g_m\|^2 \leq 2c \sum_{k=N+1}^{\infty} \lambda^{\alpha k} . \tag{1.41}$$

The result follows now along similar lines as in the proof of statement (ii) by using (1.36) and applying Proposition 1.3.4.
(iv): We have

$$\|g_m - P_N g_m\|^2 \leq 2c \sum_{k=N+1}^{\infty} (1+k)^{-\alpha} \leq 2c \int_N^{+\infty} (1+x)^{-\alpha}\, dx \leq \frac{2c(1+N)^{-\alpha}}{\alpha - 1} .$$

The rest is as in (iii). ❑

Remarks:

(i) In spite of these results, good a priori estimates of the frame bounds as given e.g. in [11, 31] play an important role.

(ii) A different approach to approximate the inverse of the frame operator by finite dimensional methods is due to Casazza and Christensen. In [7] they consider a more general setting, however at the cost that their estimates are less explicit and therefore less useful in applications.

1.4 Convergence using polyphase representation

As already explained in Section 1.2.3 the finite-dimensional model analyzed in Section 1.2 does not enable us to exploit the polyphase approach for the representation and inversion of the truncated frame operator S_N. As mentioned in Section 1.2.3 we can overcome this drawback by a periodical extension of the truncated functions $P_N g_{m,n}$. In fact, the periodic extension of finite signals is more than just a simple and convenient way to handle boundary problems. It is in some sense the "canonical" way to set up a discrete model, when translations come into play, since it allows to preserve important mathematical properties of the infinite dimensional problem.

More generally speaking one preserves the underlying group structure of the original problem. By doing so, we can immediately apply the abstract results derived for shift-invariant systems [16, 22, 44]. In fact, sometimes – e.g. in the case of oversampled DFT filter banks, see Section 1.6 – we gain even more structural properties compared to the infinite dimensional problem, due to the finiteness of the underlying abelian group.

Despite the fact that the periodic model is widely used in connection with the implementation of oversampled filter banks (sometimes implicitly by using the polyphase approach), almost no attempt has been made to investigate the relation between the synthesis functions of the periodic finite model and the synthesis functions of the infinite dimensional problem.

In this section we will clarify the question under which conditions the approximate synthesis functions computed within $\mathbb{Z}_{2N+1}$ converge to the synthesis functions $\gamma_{m,n}$ of $\ell^2(\mathbb{Z})$ when the length of the period tends to infinity.

A positive answer to this question also validates the approach of approximating the synthesis functions by means of sampling and inverting the polyphase matrix, as described in Section 1.2.

Let the functions g_m be compactly supported, such that $g_m(k) = 0$ for $|k| > s$. We construct our periodic model, as usual, by extending the truncated functions $P_N g_m$ periodically beyond the interval $[-N, N]$ for $N > s$ by setting

$$ {}^{P}g_m(k + lN) := g_m(k) \qquad \text{for } k = -N, \dots, N, \ , l \in \mathbb{Z} \, . \tag{1.42} $$

The other elements of our periodic finite filter bank are now given by

$$ {}^{P}g_{m,n}(k) = {}^{P}g_m(k - na) \tag{1.43} $$

where the shift, according to our model, is understood as circulant shift. The periodicity of the ${}^{P}g_{m,n}$ implies that the corresponding frame operator is a block circulant matrix, rather than merely a block Toeplitz matrix. Observe that the entries of ${}^{P}S_N$ deviate from the entries of S_N only in the lower right and upper left corner of the matrix.

In other words we have replaced the block Toeplitz matrix S_N in the system $S_N \gamma_m^{(N)} = g_m^{(N)}$ by a block circulant matrix. However replacing a block Toeplitz system by a block circulant matrix in a linear system of equations will in general significantly affect the solution.

Moreover it is a priori not clear under which conditions invertibility of S_N implies invertibility of ${}^{P}S_N$. For instance take following Hermitian positive definite Toeplitz matrix T and its "circulant completion" ${}^PT$:

$$T = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \qquad {}^PT = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

then T is invertible, whereas ${}^PT$ is obviously not invertible.

The following theorem justifies the usage of the periodic model for the approximation of synthesis filter banks, cf. [43]. The theorem also provides a way to estimate the frame bounds of a filter bank by computing the eigenvalues of the sampled polyphase matrix, see [42] for more details.

Theorem 1.4.1 *Let $\{g_{m,n}\}$ be a perfect reconstruction oversampled FIR filter bank for $\ell^2(\mathbb{Z})$ with frame bounds A, B. Assume that $\|g_m\| = 1$ and that $g_m(k) = 0$ for $|k| > s$. Let $N > 2s$ and let ${}^Pg_{m,n}$ be the periodized functions as defined in (1.42) and (1.43) with frame operator ${}^PS_N$. Then ${}^PS_N$ is invertible and ${}^P\gamma_m^{(N)} = {}^PS_N^{-1}\,{}^Pg_m$ converges to γ_m for $N \to \infty$. If $N > 3s$ then the rate of convergence can be estimated by*

$$\|P_N\gamma_m - {}^P\gamma_m^{(N)}\| \le 3\sqrt{2}C\lambda^N(\lambda^s - \lambda^{s+1})^{-3} \tag{1.44}$$

where $\kappa = \frac{B}{A}, q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, $\lambda = q^{\frac{1}{2s}}$ and

$$C = \frac{1}{A}\max\{2\kappa, (1+\sqrt{\kappa})^2\} \tag{1.45}$$

Proof: First we show that invertibility of S implies invertibility of ${}^PS_N$. The results about the spectrum of block circulant matrices in the proof of Theorem 3 in [54] in combination with Corollary 1 in [54] imply that

$$\sigma({}^PS_N) \subseteq \sigma(S)\,.$$

$\sigma(S)$ is bounded away from zero since $A > 0$, hence it follows that ${}^PS_N$ is invertible.

To prove (1.44) we write

$$\|P_N\gamma_m - {}^P\gamma_m^{(N)}\| \le \|\gamma_m - \gamma_m^{(N)}\| + \|P_N\gamma_m^{(N)} - {}^P\gamma_m^{(N)}\|\,. \tag{1.46}$$

We already have estimated $\|\gamma_m - \gamma_m^{(N)}\|$ in (1.32) of Theorem 1.3.5, so it remains to estimate $\|P_N\gamma_m^{(N)} - {}^P\gamma_m^{(N)}\|$. Proceeding analogously to the calculation preceding equation (1.36) we can write

$$\|\gamma_m^{(N)} - {}^P\gamma_m^{(N)}\| \le A^{-1}\|({}^PS_N - S_N)\,{}^PS_N^{-1}P_Ng_m\|\,. \tag{1.47}$$

Note that ${}^PS_N$ is three-band matrix, with one band centered at the main diagonal, and two other bands of width $2s$ located at the lower left and upper right corner of the matrix. It follows from Proposition 5.1 in [13] that the entries of ${}^P S_N^{-1}$ decay exponentially off the diagonal and off the lower right and upper left corner. More precisely

$$|({}^P S_N^{-1})_{k,l}| \le \begin{cases} C\lambda^{|k-l|} & \text{if } 0 \le |k-l| \le N \\ C\lambda^{2N+1-|k-l|} & \text{if } N+1 \le |k-l| \le 2N \end{cases} \tag{1.48}$$

where C and λ are as in Proposition 1.3.3.

Before we show that $\varphi_m^{(N)} := {}^P S_N^{-1}\, {}^P g_m$ also decays exponentially, observe that due to the construction of ${}^P S_N$ the matrix $E_N := {}^P S_N - S_N$ is a sparse matrix having non-zero entries only in the upper right and the lower left corner, as illustrated in (1.49).

$$E_N = \begin{array}{c} \begin{array}{cccccc} & & & {\scriptstyle N-2s+1} & & {\scriptstyle N} \end{array} \\ \begin{bmatrix} & & & \times & \cdots & \times \\ & & & & \ddots & \vdots \\ & & & & & \times \\ & & 0 & & & \\ \times & & & & & \\ \vdots & \ddots & & & & \\ \times & \cdots & \times & & & \end{bmatrix} \end{array} . \tag{1.49}$$

Denoting $e_m^{(N)} = E_N {}^P S_N^{-1}\, {}^P g_m$, it readily follows from the sparsity of E_N that only the first $2s$ and the last $2s$ components of e_N are non-zero, hence only the decay of the first $2s$ and last $2s$ components of $\varphi_m^{(N)} = {}^P S_N^{-1}\, {}^P g_m$ is of interest.

Since ${}^P S_N$ is a three-band matrix, the decay behavior of the entries of ${}^P S_N{}^{-1}$ is a little bit more complicated than for ordinary band matrices. We split the analysis of the decay of the entries of $\varphi_m^{(N)}$ into two steps by considering first the entries $(\varphi_m^{(N)})_k$ for $k = N-2s, \dots, N-m-1$ and then those for $k = N-m, \dots, N-1$.

Case $k = N-2s+1, \dots, N-s$:

$$(\varphi_m^{(N)})_k = \sum_{l=-N}^{N} ({}^P S_N^{-1})_{k,l} ({}^P g_m)_l = \sum_{l=-s}^{s} ({}^P S_N^{-1})_{k,l} ({}^P g_m)_l$$

due to the compact support of g_m. Hence

$$|(\varphi_m^{(N)})_k| \le |\sum_{l=-s}^{s} ({}^P S_N^{-1})_{k,l}| \le D \sum_{l=-s}^{s} \lambda^{|k-l|} .$$

The assumption $N > 3s$ implies that $k \ge l$ hence

$$D \sum_{l=-s}^{s} \lambda^{|k-l|} = D \sum_{l=-s}^{s} \lambda^{k-l} \le \frac{D}{1-\lambda} \lambda^{k-s} \tag{1.50}$$

Case $k = N-m, \ldots, N-1$:

$$|(\varphi_m^{(N)})_k| \le |\sum_{l=-s}^{s} ({}^P S_N^{-1})_{k,l}| \le \sum_{l=-s}^{k-N-1} |({}^P S_N^{-1})_{k,l}| + \sum_{l=k-N}^{s} |({}^P S_N^{-1})_{k,l}|$$
$$\le D \sum_{l=-s}^{k-N-1} \lambda^{2N+1-k+l} + D \sum_{l=k-N}^{s} \lambda^{k-l} \le \frac{2D}{1-\lambda}\lambda^{k-s}.$$

Note that $\|e_m^{(N)}\|^2 = \sum_{k=-N}^{N} |(e_m^{(N)})_k|^2 = 2\sum_{k=N-2s+1}^{N} |(e_m^{(N)})_k|^2$. Due to the special sparsity structure of E_N as illustrated in (1.49) we can write

$$|(e_m^{(N)})_k| \le \sum_{l=2N-2s+1+k}^{N} |(E_N)_{k,l}||\varphi_l| \le \frac{2BD}{1-\lambda}\lambda^{-s} \sum_{l=2N-2s+1+k}^{N} \lambda^l$$
$$\le \frac{2BD}{(1-\lambda)^2}\lambda^{-s}(\lambda^{2N-2s+1+k} - \lambda N + 1 \le \frac{2BD}{(1-\lambda)^2}\lambda^{k-3s+2N}.$$

It follows

$$|(e_m^{(N)})_k|^2 \le \left(\frac{2BD}{(1-\lambda)^2}\right)^2 \lambda^{2k+4N-6s}$$

and

$$\|e_m^{(N)}\|^2 = 2\sum_{k=-N}^{-N+2s-1} |(e_m^{(N)})_k|^2 \le 2\frac{(2BD)^2}{(1-\lambda)^4} \sum_{k=-N}^{-N+2s-1} \lambda^{2k-6s+4N}$$
$$\le 2\frac{(2BD)^2}{(1-\lambda)^4}\frac{\lambda^{2N-6s} - \lambda^{2N-2s}}{1-\lambda} \le \left(2\sqrt{2}\frac{BD}{(1-\lambda)^3}\lambda^{N-3s}\right)^2$$

and therefore

$$\|e_m^{(N)}\| \le 2\sqrt{2}\frac{BD\lambda^{-3s}}{(1-\lambda)^3}\lambda^N = 2\sqrt{2}BD(\lambda^s - \lambda^{s+1})^{-3}\lambda^N$$

which together with (1.47) yields the desired estimate (1.44). ❑

Similar results hold if the analysis functions $g_{m,n}$ have exponential or polynomial decay, since in this case we can approximate $g_{m,n}$ arbitrarily well by compactly supported functions.

1.5 Finite-dimensional approximation of paraunitary filter banks via $S^{-\frac{1}{2}}$

Since paraunitary filter banks are tight frames (i.e. the frame bounds A, B coincide) the frame operator is given by $S = AI$ where I is the identity operator on $\ell^2(\mathbb{Z})$. In this case the frame bound A can also be interpreted as oversampling factor if $\|g_m\| = 1$, see [11]. Paraunitary filter banks are very attractive due to their noise reduction property. The reconstruction error variance is inversely proportional to the oversampling factor [33, 11].

Hence the construction of tight frames is of great interest in theory and in applications. Daubechies [10] has shown that for a given frame, say $\{f_k\}$ the set of functions ψ_k constructed by

$$\psi_k := S^{-\frac{1}{2}} f_k \tag{1.51}$$

constitutes a tight frame. Recall that S is positive definite, hence $S^{\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are uniquely determined. Moreover $S^{\frac{1}{2}}$ commutes with the same operators that commute with S. For filter banks this implies that the tight frame generated by $\{S^{-\frac{1}{2}} g_{m,n}\}$ also constitutes a system of the form (1.1).

The tight frame constructed via formula (1.51) has a natural interpretation in linear algebra. Let $\{f_k\}_{k=1}^m$ be a given frame in $\mathbb{C}^n$ with frame operator S. The tight frame $\{\hat{\varphi}_k\}_{k=1}^m$ that minimizes

$$\sum_{k=1}^{m} \|f_k - \varphi_k\|^2 \tag{1.52}$$

among all tight frames $\{\varphi_k\}_{k=1}^m$ is given by $\hat{\varphi}_k := (\text{trace}(S^{\frac{1}{2}})/n) S^{-\frac{1}{2}} f_k$. In the linear algebra community this fact is known in the following context. Let T be the $n \times m$ matrix having the f_k as its k-th column. Let $\mathcal{U}_{n,m}$ be the set of all matrices U such that $UU^H = cI$, where I is the $n \times n$ identity matrix and $c \in \mathbb{R}$ (in other words U is a multiple of a matrix with unitary rows). Denote the Frobenius or Hilbert-Schmidt norm of a matrix by $\|.\|_F$. Then the matrix that solves

$$\min_{U \in \mathcal{U}_{n,m}} \|T - U\|_F \,,$$

is given by

$$\hat{U} = \frac{\text{trace}(P)}{n} V \,,$$

where $T = PV$ is the polar decomposition of T with $P = (TT^H)^{\frac{1}{2}}$, see e.g. Chapter 7.4 in [25]. Although this is a classical result in linear algebra it seems to be not very well-known in the frame- and signal processing community.

Now we return to frames $\{f_k\}_{k \in \mathbb{Z}}$ for $\ell^2(\mathbb{Z})$ and want to investigate if the tight frames constructed via equation (1.51) can be approximated by the finite section method. We want to approximate the elements of the tight frame $\varphi_k := S^{-\frac{1}{2}} f_k$ by the solution of the finite sections

$$(P_N S P_N)^{\frac{1}{2}} \varphi_k^{(N)} = P_N f_k \,.$$

The applicability of the finite section method follows directly from Proposition 1.3.1. Considering the rate of approximation one can easily derive estimates similar to the ones given in Propositions 1.3.3 and 1.3.4 to bound the decay of φ_k by using Lemma 2 in [27].

We state the result for FIR filter banks, the versions for other types of decay and for the periodic model are left to the reader.

Corollary 1.5.1 *Let $\{g_{m,n}\}$ be a perfect reconstruction filter bank for $\ell^2(\mathbb{Z})$ with frame operator S and frame bounds A, B. Assume that $\|g_m\| = 1$ and set $\varphi_m = S^{-\frac{1}{2}} g_m$ and $\varphi_m^{(N)} = (P_N S P_N)^{-\frac{1}{2}} P_N g_m$. Then the finite section method is applicable to $S^{\frac{1}{2}}$. If moreover g is compactly supported with $g(k) = 0$ for $|k| > s$ then*

$$\|\varphi_m - \varphi_m^{(N)}\| \leq K\beta^N \qquad N = 0, 1, \ldots$$

for $\beta \subset (0,1)$ and some constant $K > 0$, where β and K depend on the frame bounds A, B and the support length s, but are independent of N.

1.6 Oversampled DFT filter banks - beyond polyphase representation

If we construct the impulse responses g_m by complex modulation of a single function g via

$$g_m(k) = e^{2\pi i m k/M} g(k)$$

we obtain the *oversampled DFT filter bank* [47, 8, 4].

DFT filter banks and cosine modulated filter banks allow efficient FFT- or DCT-based implementations. For DFT filter banks it is clear that the subband signals $\langle f, g_{m,n}\rangle$ can be efficiently computed by FFT-based methods. Of course, the same is true for the reconstruction of f from its subband signals, once the synthesis filter bank $\{\gamma_{m,n}\}$ is known. Clearly the $\gamma_{m,n}$ can be computed a priori and stored for later usage. Nevertheless it is useful to have algorithms at hand that allow for a fast computation of the synthesis functions $\gamma_{m,n}$. Observe that we only have to calculate $\gamma := \gamma_{0,0}$, due to (1.2).

Recall that oversampled DFT filter banks are equivalent to Gabor frames (Weyl-Heisenberg frames) on $\ell^2(\mathbb{Z})$ [8, 4, 32]. A number of – at the first sight – different algorithms have been proposed for the computation of γ, see e.g. [48, 1, 37, 34, 52, 30, 15, 50, 38] and the referneces cited in [36, 41].

In the sequel we demonstrate that the use of the periodic model gives rise to additional properties of the Gabor frame operator S. These properties allow us to unitarily transform S into a block-diagonal matrix with blocks of small size. All other representations of the frame operator – such as the polyphase representation – and the associated numerical algorithms can be obtained from this factorization.

1.6.1 Matrix factorization of the frame operator

In Section 1.4 it was shown that if the synthesis functions $\gamma_{m,n}$ satisfy certain decay conditions they can be approximated arbitrarily well by functions $\gamma_{m,n}^{(N)}$ based on a periodic model. The periodic model has the advantage of preserving perfect symbol calculus of convolution operators. We will see in this section that the Gabor frame operator has a nice representation that goes beyond the well-known factorization by means of the Zak transform [53, 2].

In this section we do all our computations in the ring $\mathbb{Z}_N = \mathbb{Z} \bmod N$. In other words we consider discrete signals f of length N, extended periodically by $f(k+nN) = f(k)$ for $k = 0, \dots, N-1$ and $n \in \mathbb{Z}$. The choice of this model is justified by the theoretical results in Section 1.4.

Our guiding philosophy is to choose a language that best expresses the underlying structures arising in discrete Gabor expansions. Permutation matrices, Fourier matrices and Kronecker products constitute the vocabulary of this unifying language.

A few diagonal matrix notations are handy. If $d \in \mathbb{C}^N$, then $D = \operatorname{diag}(d) = \operatorname{diag}(d_0, \dots, d_{N-1})$ is the $N \times N$ diagonal matrix with diagonal entries $d_0, \dots, d_{N-1}$. If $D_0, \dots, D_{N-1}$ are $p \times p$ matrices, then $D = \operatorname{diag}(D_0, \dots, D_{N-1})$ is the block diagonal matrix defined by

$$D = \begin{bmatrix} D_0 & 0 & \dots & 0 \\ 0 & D_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & D_{N-1} \end{bmatrix}.$$

A *permutation matrix* is just the identity with its columns reordered. If P is a permutation and A is a matrix, then PA is a row permuted version of A and AP is a column permuted version of A. Note that $P^* = P^{-1}$. The permutation matrices are thus unitary, forming a subgroup of the unitary group. Among the permutation matrices, the mod p *perfect shuffle permutation* [46]

$$\begin{array}{rl}
p \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \\
P_{p,N} = & \left[\begin{array}{ccccccc} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ & & & & 1 & & \\ & 0 & & & & & \\ & 1 & & & & & \\ \vdots & & & & & & \vdots \\ & & & 1 & & & \\ 0 & & & & & & 1 \end{array}\right], \\
& \underbrace{\hphantom{1 \quad 0 \quad \dots \quad 0}}_{q \text{ columns}}
\end{array}$$

where $N = pq$, shall play a key role in the factorization of the Gabor frame operator. Its transpose $P_{p,N}^*$ is called the mod p *sort permutation* [46].

Observe that $P_{p,N} = P^*_{q,N}$. We illustrate the action of $P_{p,N}$ and $P^*_{p,N}$ by means of simple examples with $N = 15 = 3 \cdot 5 \equiv p \cdot q$:

$$\begin{aligned} P_{3,15}x &= [\, x_0\, x_5\, x_{10} \,|\, x_1\, x_6\, x_{11} \,|\, x_2\, x_7\, x_{12} \,|\, x_3\, x_8\, x_{13} \,|\, x_4\, x_9\, x_{14} \,]^* \\ P^*_{3,15}x &= [\, x_0\, x_3\, x_6\, x_9\, x_{12} \,|\, x_1\, x_4\, x_7\, x_{10}\, x_{13} \,|\, x_2\, x_5\, x_8\, x_{11}\, x_{14} \,]^* \,. \end{aligned}$$

It is not surprising that the *Fourier matrix* is of fundamental importance in the factorization of the Gabor frame operator. By the Fourier matrix of order N we shall mean the unitary matrix

$$F_N = \sqrt{\frac{1}{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

where $\omega = e^{-2\pi i/N}$.

If $A \in C^{p\times q}$ and $B \in C^{m\times n}$ then the *Kronecker product* $A \otimes B$ is the p-by-q block matrix

$$A \otimes B = \begin{bmatrix} a_{00}B & \dots & a_{0q-1}B \\ \vdots & & \vdots \\ a_{p-10}B & \dots & a_{p-1q-1}B \end{bmatrix} \in \mathbb{C}^{pm\times qn} \,.$$

Note that

$$I \otimes B = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & B \end{bmatrix}$$

is block diagonal, where I is the identity matrix. If $N = mn$ and $A \in \mathbb{C}^{m\times m}$, then [46]

$$P^*_{m,N}(I_n \otimes A) = (A \otimes I_n)P^*_{m,N} \,. \tag{1.53}$$

If A and B are unitary matrices, then $A \otimes B$ is a unitary matrix, since $(A \otimes B)^* = A^* \otimes B^*$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, see [46].

The set of Gabor analysis functions $\{g_{m,n}\}$ is given by

$$g_{m,n}(k) = g(k - na)e^{2\pi imb/N}, \tag{1.54}$$

for $m = 0, \dots, M-1$, $n = 0, \dots, K-1$ where $Ka = Mb = N$ with $a, b, K, M \in \mathbb{N}$.

Let G be the $MK \times N$ matrix having $g_{m,n}$ as its $(m + nM)$-th row for $m = 0, \dots, M-1$, $n = 0, \dots, K-1$. We write $c_{m,n} = \langle f, g_{m,n} \rangle$ and by slightly abusing our notation we treat the $K \times M$ array c also as column vector of length KM by stacking the columns of c. The Gabor transform of a function f can be written as

$$Gf = c\,.$$

The case $ab = N$ is usually referred to as critically sampled Gabor transform, whereas the choice $ab < N$ yields an oversampled Gabor transform, the ratio $\frac{N}{ab} = \frac{p}{q}$ giving the oversampling rate or redundancy. The case $p = 1$ is referred to as integer oversampling.

Recall that the Gabor frame operator S is defined as

$$Sf = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle f, g_{m,n} \rangle g_{m,n}\,.$$

S is a positive definite Hermitian $N \times N$ matrix which can be expressed as

$$S = G^* G\,.$$

It is obvious that the Laurent operator property of the filter bank frame operator in ltZ translates in $\mathbb{Z}_N$ to the property that S is block-circulant matrix. But for oversampled DFT filter banks the frame operator exhibits even more structural properties. These do not come as a big surprise due to the "symmetry" in the construction of the functions g_m and $g_{m,n}$.

Following representation of S is a consequence of the *Walnut representation* of S, cf. [24]

Lemma 1.6.1 *[43] S can be unitarily block-diagonalized via*

$$S = P_{M,N} D_S P_{M,N}^* \tag{1.55}$$

where $D_S = diag(B_0, \dots, B_{M-1})$ and B_k are $b \times b$ submatrix with entries

$$(B_k)_{mn} = S_{k+mM,k+nM} \tag{1.56}$$

for $m, n = 0, \dots, b-1,\ k = 0, \dots, M-1$.

A well-known factorization of S can be obtained by combining the sparsity of S as stated in Lemma 1.6.1 with the fact that S is of block-circulant type, see [53, 35]. This factorization of S can be further improved as demonstrated by the following theorem [41].

Theorem 1.6.2 (Fundamental Factorization) *Let $\{g_{m,n}\}$ be a Gabor frame for $\mathbb{Z}_N$ with time-frequency shift parameters a, b. Denote $\frac{N}{ab} = \frac{p}{q}$. Then S can be unitarily factorized into the block diagonal matrix*

$$I_p \otimes diag(C_0, \dots, C_{\frac{N}{pq}-1})\,,$$

where the $q \times q$ submatrices C_j are given by

$$diag(C_{\frac{bk}{q}}, \ldots, C_{\frac{b(k+1)}{q}-1}) = (F_{\frac{b}{q}} \otimes I_q)^* B_k (F_{\frac{b}{q}} \otimes I_q)$$

with B_k as in (1.56).

In other words the Gabor frame operator can be efficiently factorized by properly chosen Fourier transforms and permutations into $N/(pq)$ different submatrices of size $q \times q$ where each of these submatrices is repeated p times.

The "classical" result via Zak transform [53] states that S can be factorized into N/q different matrices of size $q \times q$. Hence we have "gained" a factor p.

In case of integer oversampling (i.e. $p = 1$), Theorem 1.6.2 reduces to the well-known diagonalization of the Gabor frame operator by the Zak transform, see for instance [2, 53].

Many fast algorithms for computing the dual Gabor function can be easily derived from Theorem 1.6.2, since they simply correspond to different factorizations of the frame operator S, see [41] for more details.

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