OFDM, Laurent operators, and time-frequency localization

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ABSTRACT

Orthogonal frequency division multiplexing (OFDM) has gained considerable interest as an efficient technology for high-date-data transmission over wireless channels. The design of pulseshapes that are well-localized in the time-frequency plane is of great importance in order to combat intersymbol interference and interchannel interference caused by the mobile radio channel. Recently proposed methods to construct such well-localized (orthogonal) functions are utilizing the link between OFDM and Gabor systems. We derive a theoretical framework that shows why and under which conditions these methods will yield well-localized pulse shapes. In our analysis we exploit the connection between Gabor systems, Laurent operators and the classical work of Gelfand, Raikov, and Shilov on commutative Banach algebras. In the language of Gabor analysis we derive a general condition under which the dual window and the canonical tight window inherit the decay properties of the analysis window.

Keywords: OFDM, Laurent operator, Wiener's algebra, Gabor frame, tight frame, orthogonalization.

1. INTRODUCTION AND MOTIVATION

Orthogonal frequency division multiplexing (OFDM) has attracted a great deal of attention as an efficient technology for wireless data transmission.[1] It is currently used in the European digital audio broadcasting standard,[2] in digital terrestrial TV broadcasting[3] as well as for broadband indoor wireless systems.[4]

Propagation over the mobile radio channel is characterized by a spread of the signal in time due to multipath propagation and in frequency due to the Doppler effect. The time dispersion can lead to intersymbol interference and the frequency dispersion can cause interchannel interference.[1] The distortion resulting from channel dispersion depends crucially on the time-frequency localization of the OFDM pulse shapes. Robustness against doubly dispersive channels can be achieved by pulse shapes with good time-frequency localization. The design of well-localized OFDM pulse shapes has therefore attracted considerable interest.[5–8]

An attractive approach to construct such pulse shapes is based on the connection between OFDM and Gabor frames.[5,7,8] Orthogonal pulse shapes can be designed by computing an orthonormal incomplete Gabor system,[5,8] while biorthogonal pulse shapes can be derived from incomplete Gabor Riesz bases.[7,8]

The time-frequency localization property of the resulting pulse shape relies on the assumption that a time-frequency well-localized Gabor frame yields a dual frame and a canonical tight frame that inherit these localization properties. In this paper we show under which conditions such an assumption is justified. Our results provide a theoretical foundation for the time-frequency localization properties of the pulse shapes constructed in [5,7,8]. From a Gabor analysis viewpoint we derive a general condition under which the dual window and the canonical tight window inherit the decay properties of the analysis window.[9]

The rest of the paper is organized as follows. In Section 2 we briefly discuss some basics on Gabor analysis. Section 3 reviews the connection between OFDM and Gabor analysis, and describes the construction of OFDM pulse shapes via Gabor systems. In Section 4 we derive a general condition on the decay of the Gabor window function that guarantees that the dual window has the same type of decay. Finally, in Section 5 we show how the results of the previous sections yield a theoretical justification for the design of OFDM pulse shapes that are well-localized in the time-frequency plane.

Before we proceed we introduce a few notations used throughout the paper.

The Fourier transform of a function f is given by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi it\omega}dt.$$

A locally integrable function w is called *weight function*, if w is positive and submultiplicative, i.e., if $w(t) > 0$ and $w(t_1 + t_2) \leq w(t_1)w(t_2)$.

The space $\boldsymbol{L}_{1,w}(\mathbb{R})$ consists of all functions f with

$$\int_{-\infty}^{+\infty} |f(t)|w(t)dt < \infty,$$

where w is a weight function. Similarly $\boldsymbol{\ell}_{1,w}(\mathbb{Z})$ consists of all sequences $x = \{x_k\}_{k\in\mathbb{Z}}$ with

$$\sum_{k=\infty}^{\infty} |x_k|w(k) < \infty.$$

It is convenient to define following spaces:

$$\mathcal{C}_{[\alpha,\beta]} = \Big\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, \alpha, \beta \text{ such that } |f(t)| = 0 \ ,\forall |t| \notin [\alpha, \beta] \Big\}.$$

$$\mathcal{E}_\lambda = \Big\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, \lambda > 0, c > 0 \text{ such that } |f(t)| \leq ce^{-\lambda|t|} \ ,\forall |t| \Big\}.$$

$$\mathcal{Q}_s = \Big\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, s > 1, c > 0 \text{ such that } |f(t)| \leq c(1 + |t|)^{-s} \ ,\forall |t| \Big\}.$$

2. PRELIMINARIES ABOUT GABOR SYSTEMS

A Gabor system consists of functions of the form

$$g_{na,mb}(t) = e^{2\pi imb}g(t - na), \;\; n, m \in \mathbb{Z}, \;\; a, b \in \mathbb{R}, \tag{1}$$

where $g \in \boldsymbol{L}_2(\mathbb{R})$ is – depending on the context – called *window*, *atom*, or *pulse shape*. The parameters a and b represent the time-shift and frequency-shift, respectively.

We say that (g, a, b) generates a *Gabor frame* for $\boldsymbol{L}_2(\mathbb{R})$ for given shift parameters a, b if there exist constants (*frame bounds*) $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, g_{na,mb}\rangle|^2 \leq B\|f\|^2, \tag{2}$$

for any $f \in \boldsymbol{L}_2(\mathbb{R})$.

The analysis operator T is defined as

$$T : f \in \boldsymbol{L}_2(\mathbb{R}) \to Tf = \{\langle f, g_{na,mb}\rangle\}_{n,m\in\mathbb{Z}}, \tag{3}$$

and the synthesis operator, which happens to be the adjoint of T is

$$T^* : c \in \boldsymbol{\ell}_2(\mathbb{Z} \times \mathbb{Z}) \to T^*c = \sum_{n,m\in\mathbb{Z}} c_{nm}g_{na,mb}. \tag{4}$$

The Gabor frame operator is defined by

$$Sf = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle g_{na,mb}, \qquad f \in \boldsymbol{L}_2(\mathbb{R}), \tag{5}$$

and satisfies

$$IA \le S \le IB$$

where I is the identity operator on $\boldsymbol{L}_2(\mathbb{R})$. Of course $S = T^*T$.

If (g, a, b) establishes a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ then any f in $\boldsymbol{L}_2(\mathbb{R})$ can be represented as

$$f = \sum_{n,m\in\mathbb{Z}} \langle f, \gamma_{n,m}\rangle g_{na,mb} = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle \gamma_{n,m}, \tag{6}$$

where the *dual frame* $\{\gamma_{n,m}\}$ is given by

$$\gamma_{n,m} = e^{2\pi imb}\gamma(t - na), \;\; n,m \in \mathbb{Z}; a, b \in \mathbb{R},$$

with $\gamma = S^{-1}g$. In general there are many functions generating dual frames that satisfy relation (6). The "canonical" dual window γ has several nice properties. One of them is that it has minimal $\boldsymbol{L}_2$-norm among all dual functions. In this paper we concentrate on the canonical dual window and henceforth simply talk about *the* dual window and *the* dual frame. For more details about the properties of Gabor frames and their duals the reader is referred to [10,11].

For discrete-time signal processing and from the viewpoint of implementations it is useful to consider Gabor systems in $\boldsymbol{\ell}_2(\mathbb{Z})$. For, let $g \in \boldsymbol{\ell}_2(\mathbb{Z})$, then the functions $g_{na,mb}$ are defined as

$$g_{nm}(k) = g(k - nT)e^{2\pi ikm/M}, \tag{7}$$

for $T, M \in \mathbb{Z}, n \in \mathbb{Z}, m = 0, \ldots, M-1$.

3. OFDM AND GABOR SYSTEMS

The basic idea of OFDM is to divide the available spectrum into several subchannels (subcarriers). A baseband OFDM system is schematically represented in Figure 1.

Transmitter: Assuming N subcarriers, a bandwidth of W Hz, *symbol length* of T seconds, and *carrier separation* $F := W/N$, the transmitter of a general OFDM system uses the following waveforms

$$\psi_l(t) = \psi(t)e^{2\pi itlF}, \quad k \in \mathbb{Z}, l = 0, \ldots, N-1. \tag{8}$$

The transmitted baseband signal for OFDM symbol number k is

$$s_k(t) = \sum_{l=0}^{N-1} c_{kl}\psi_l(t - kT), \tag{9}$$

where $c_{k0}, c_{k1}, \ldots, c_{k,N-1}$ are the complex-valued information bearing coefficients (data symbols). Assuming an infinite sequence of OFDM symbols is transmitted, the output from the transmitter is a superposition of individual OFDM symbols

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1} c_{k,l}\psi_l(t - kT). \tag{10}$$

Receiver: The OFDM receiver consists of a matched filter bank $\{\varphi_l\}$ of similar structure as the transmitter waveforms, i.e.,

$$\varphi_{kl}(t) = \varphi(t - kT)e^{2\pi itlF}, \quad k \in \mathbb{Z}, l = 0, \ldots, N-1. \tag{11}$$

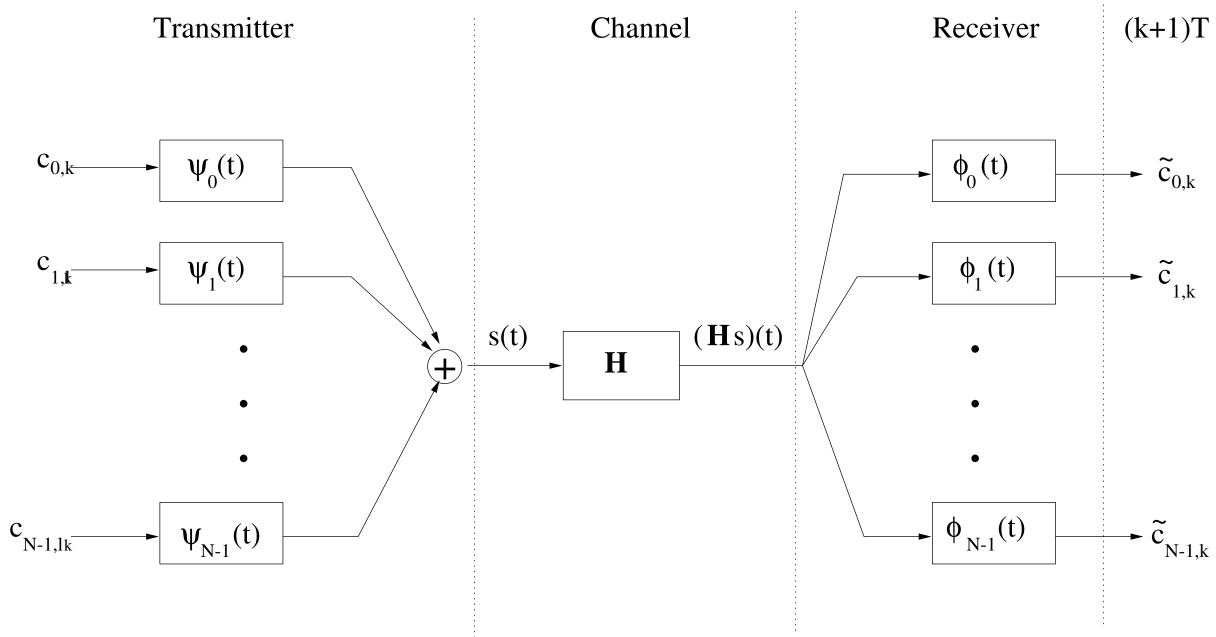

Figure 1. A general baseband OFDM system configuration

The transmitted data are recovered by projecting the received signal $r = \mathbf{H}s + \nu$ (where ν represents additive white Gaussian noise, AWGN for short) onto the functions φ_{kl}, i.e.,

$$\tilde{c}_{kl} = \langle r, \varphi_{kl} \rangle. \tag{12}$$

In the standard OFDM setup the functions ψ_{kl} are designed to be mutually orthogonal. In this case $\varphi \equiv \psi$. The situation where the sets $\{\psi_{kl}\}$ and $\{\varphi_{kl}\}$ are biorthogonal is referred to as *biorthogonal frequency division multiplexing* (BFDM).

It is useful to have the following two OFDM setups at hand.

Continuous-time model: Since $s(t)$ is a continuous signal, it is convenient to consider a continuous-time transmission set that is infinite in both indices k and l, which means we set $c_{kl} = 0$ if $l < 0$ or $l > N - 1$. Using the viewpoint it is clear that OFDM can be interpreted as an orthonormal Gabor system in $\boldsymbol{L}_2(\mathbb{R})$. For a theoretical analysis it is often advantageous to work with this time-continuous model.

Discrete-time model: In practice the OFDM system is digitally implemented. At the transmitter the signal s is passed through a digital-to-analog converter to obtain a continuous signal. At the receiver an analog-to-digital converter transforms the received signal back into a discrete-time signal. Therefore it is useful to consider the following discrete-time setup for OFDM.

Let $\psi \in \boldsymbol{\ell}_2(\mathbb{Z})$ and set $\psi_{k,l}(n) = \psi(n - kT)e^{2\pi i l F}$ for $k, n \in \mathbb{Z}, l = 0, \dots, N-1$ and $F = W/N$. Then the OFDM signal becomes

$$s(n) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l} \psi_{k,l}. \tag{13}$$

It is clear that $\{\psi_{k,l}\}$ coincides with a discrete-time Gabor system.

3.1. OFDM, Gabor frames, and time-frequency localization

In the ideal case the wireless channel $\mathbf{H}$ does not introduce any distortion, and the data c_{kl} can be recovered exactly. However in practice wireless channels introduce time dispersion as well as frequency dispersion (in addition to the usual channel noise). The time dispersion is caused by multipath propagation and can lead to intersymbol interference (ISI). Frequency dispersion of the mobile radio channel is due to the Doppler effect and can cause interchannel interference (ICI). The distortion resulting from channel dispersion depends crucially on the time-frequency localization of the transmitter pulse shapes $\{\varphi_{kl}\}$. Robustness against doubly dispersive channels can be achieved by pulse shapes with good time-frequency localization.

An optimum OFDM system in case of doubly dispersive channels would consist of orthogonal basis functions with $TF = 1$, such that the $\psi_{k,l}$ are well localized in time and frequency. The condition $TF = 1$ (critical sampling) ensures maximal spectral efficiency of the transmission system. Unfortunately, such a system cannot exist due to the Balian-Low theorem.[12]

Therefore other approaches (cyclic prefix, pulse-shaping, BFDM) have been proposed for doubly dispersive channels.[1,6–8,5] These approaches can be interpreted in the time-frequency plane as using an undersampled grid, i.e., $TF > 1$, which results in a set of basis functions that is incomplete in $\boldsymbol{L}_2(\mathbb{R})$ (or $\boldsymbol{\ell}_2(\mathbb{Z})$, respectively). Although the choice $TF > 1$ leads to a loss in the capacity of the transmission system (determined by the undersampling rate), it is usually an acceptable price to pay to mitigate interference.

We know from Gabor theory that there exist Gabor systems with good time-frequency localization for $TF > 1$ (take for instance $TF = 2$ and a Gaussian window). In the present context of OFDM we are particularly interested in the construction of orthonormal incomplete Gabor systems with good time-frequency localization properties.

The following result ties tight Gabor frames for $\boldsymbol{L}_2(\mathbb{R})$ to orthonormal Gabor bases for subspaces of $\boldsymbol{L}_2(\mathbb{R})$.

Corollary 3.1 ([13]). *(i) $(g, 1/b, 1/a)$ is a Riesz basis for its closed linear span if and only if (g, a, b) is a frame for $\boldsymbol{L}_2(\mathbb{R})$.*
(ii) Assume $\|g\| = 1$. $(g, 1/b, 1/a)$ is an orthonormal basis for its closed linear span if and only if (g, a, b) is a tight frame for $\boldsymbol{L}_2(\mathbb{R})$. A similar theorem holds for Gabor frames in $\boldsymbol{\ell}_2(\mathbb{Z})$.

Now, let $\{g_{na,mb}\}$ be a frame for $\boldsymbol{L}_2(\mathbb{R})$ with frame operator S, then a standard way to construct a tight frame is the following. Compute

$$\tilde{g} = S^{-\frac{1}{2}} g \tag{14}$$

then the set $\{\tilde{g}_{na,mb}\}$ is a tight frame for $\boldsymbol{L}_2(\mathbb{R})$ and by Corollary 3.1 (after normalizing $\tilde{g}$) $\{\tilde{g}_{k/b,l/a}\}$ is an orthonormal basis for its closed linear span (equivalently we can apply this orthogonalization procedure directly to the set $\{g_{k/b,l/a}\}$). Our OFDM system is now given by setting $\psi := \tilde{g}$, $T = 1/b$, $F = 1/a$, or equivalently,

$$\psi_{kl} := \tilde{g}_{k/b,l/a}. \tag{15}$$

In order to obtain an OFDM system that is well localized in the time-frequency plane, it seems natural to start with a function g with good time-frequency localization such that (g, a, b) generates a frame for $\boldsymbol{L}_2(\mathbb{R})$, apply the orthogonalization procedure (14) and hope that $\tilde{g} = S^{-\frac{1}{2}} g$ and whence ψ inherit these localization properties. This approach is not new, it has already been considered in [7,8] and implicitly in [5], with different conclusions. The crucial question is if the orthonormal functions obtained by this procedure inherit the time-frequency localization properties of the original pulse shape g. Since no theoretical analysis of this important aspect is given in [7,8,5] we will provide an answer to this problem in the following. For this purpose we exploit the link to Gabor frames.

4. DECAY OF DUAL GABOR FRAMES AND WIENER'S ALGEBRA

A natural question for Gabor frames – in particular in spite of the previous discussion is the following. Given a window g with certain decay properties in time and/or frequency, does its dual γ have the same decay properties? This question is not only of interest from a theoretical viewpoint, but has a number of practical implications, e.g., see Section 3.

We briefly summarize a few important results. We assume in the sequel that $\{g_{na,mb}\}$ constitutes a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ or $\boldsymbol{\ell}_2(\mathbb{Z})$.
(i) If $g \in \mathcal{C}_{[\alpha,\beta]}$, then $\gamma \in \mathcal{C}_{[\alpha_1,\beta_1]}$ only in very special cases.[14] In general γ is no longer compactly supported, but has exponential decay (see [15] for a proof in $\boldsymbol{\ell}_2(\mathbb{Z})$, the result can be easily extended to $\boldsymbol{L}_2(\mathbb{R})$). Hence in this case

g and γ do not belong to the same type of space.
(ii) If $g \in \mathcal{E}_\lambda$ then $\gamma \in \mathcal{E}_{\lambda_1}$, but in general with a different exponent $\gamma_1 < \gamma$.[15,16] Thus g and γ have the same type of decay, but do not belong to the same space, we loose some quality of decay.
(iii) If $g \in \mathcal{Q}_s$, then $\gamma \in \mathcal{Q}_s$ (see [15] for a proof for $\boldsymbol{\ell}_2(\mathbb{Z})$, the result can be easily generalized to $\boldsymbol{L}_2(\mathbb{R})$ using the same approach as in [16]). In this case g and γ actually belong to the same space $\mathcal{Q}_s$.

From an algebraic point of view case (iii) is the most appealing one. Can we find a (simple) condition on the decay of g that implies that the dual γ belongs to the same space, similar to case (iii)? In this section we will give an exhaustive answer to this question.

We need some preparation before we proceed.

Definition 4.1. *Let $w(t)$ be a continuous weight function on $\mathbb{R}$. The (weighted) Wiener Algebra $\mathcal{A}_w$ is the Banach space of absolutely convergent Fourier series of period 1 (cf. [17]), i.e., $f \in \mathcal{A}_w$ if*

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t} \tag{16}$$

with

$$\sum_{k=-\infty}^{\infty} |c_k| w(k) < \infty. \tag{17}$$

The norm on $\mathcal{A}_w$ is

$$\|f\|_{\mathcal{A}_w} = \sum_{k=-\infty}^{\infty} |c_k| w_k.$$

It follows from Chapter 19.4 of [18] that $\mathcal{A}_w$ is a Banach algebra under pointwise multiplication. $\mathcal{A}_w$ can be identified with the space of all sequences $c = \{c_k\}$ which are in $\boldsymbol{\ell}_{1,w}$.

Definition 4.1 can be extended to matrix-valued functions in a straightforward manner. Put

$$\Phi(e^{2\pi i t}) = \begin{bmatrix} f_{11}(e^{2\pi i t}) & \dots & f_{1m}(e^{2\pi i t}) \\ \vdots & & \vdots \\ f_{m1}(e^{2\pi i t}) & \dots & f_{mm}(e^{2\pi i t}) \end{bmatrix} \tag{18}$$

and set

$$A_k = \int_0^1 \Phi(e^{2\pi i t}) e^{-2\pi i t} dt. \tag{19}$$

Let the weight function $w(t)$ act on $\mathbb{R}^m$. Then Φ belongs to the matrix Wiener algebra $\mathcal{A}_w^m$ if

$$\sum_{k=-\infty}^{\infty} |A_k| w(k) < \infty. \tag{20}$$

Φ is unitarily equivalent to the block Laurent operator L whose matrix representation (with respect to the standard basis) is given by

$$\begin{bmatrix} \ddots & & & & \\ & A_0 & A_{-1} & A_{-2} & \\ & A_1 & \boxed{A_0} & A_{-1} & \\ & A_2 & A_1 & A_0 & \\ & & & & \ddots \end{bmatrix}.$$

Here $\boxed{A_0}$ denotes the $(0,0)$ entry which acts on the 0-th coordinate space. Φ is also called the defining function of the block Laurent operator L. By a slight abuse of notation we also write $L = [A_{kl}]_{k,l=-\infty}^{\infty}$ where $A_{kl} = A_{k-l}$.

We define $\mathcal{W}_w^m$ as the space consisting of all block Laurent operators whose $m \times m$ blocks A_k satisfy (20). If $m = 1$ we simply write $\mathcal{W}_w$ and L reduces to a scalar-valued Laurent operator.

THEOREM 4.2 ([9]). *Let $L \in \mathcal{W}_w$ be self-adjoint and positive definite. Assume that the weight function w satisfies*

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{w(-n)}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sqrt[n]{w(n)} = 1, \tag{21}$$

then $L^{-1} \in \mathcal{W}_w$.

PROOF: Set $L = [A_{kl}]_{k,l=-\infty}^{\infty}$ and $L^{-1} = [B_{kl}]_{k,l=-\infty}^{\infty}$. Since L is positive definite we have

$$f(\omega) = \sum_{k=-\infty}^{\infty} A_k e^{2\pi i k\omega} > 0 \tag{22}$$

and by the properties of Laurent operators[19]

$$\frac{1}{f(\omega)} = \sum_{k=-\infty}^{\infty} B_k e^{2\pi i k\omega}.$$

The property $L \in \mathcal{W}_w$ is equivalent to $f \in \mathcal{A}_w$. By Theorem 2 on page 24 in [18] an element of $\mathcal{A}_w$ has an inverse in $\mathcal{A}_w$ if it is not contained in a maximal ideal. Any maximal ideal of $\mathcal{A}_w$ consists of elements of the form (cf. Chapter 19.4 in [18])

$$\sum_{k=-\infty}^{\infty} a_k \xi^k = 0,$$

where $\xi = \rho e^{2\pi i\omega}$ with

$$\rho_1 \le \rho \le \rho_2$$

and

$$\rho_1 = \lim_{n\to\infty} \frac{1}{\sqrt[n]{w_{-n}}} \quad \text{and} \quad \rho_2 = \lim_{n\to\infty} \sqrt[n]{w_n}.$$

Due to assumption (21) we get $\rho_1 = \rho_2 = 1$, hence $\rho = 1$. Thus a necessary and sufficient condition for an element in $\mathcal{A}_w$ to be not contained in a maximal ideal is $\sum_k a_k e^{2\pi i k\omega} \neq 0$ for all ω. By assumption L is positive definite, hence $f(\omega) = \sum_k A_k e^{2\pi i k\omega} > 0$ for all ω, consequently $L^{-1} \in \mathcal{W}_w$. ❑

The proof of Theorem 4.2 is essentially based on results of Gelfand, Raikov, and Shilov.[18] A crucial role plays condition (21), to which we will henceforth refer as the *Gelfand-Raikov-Shilov condition* (GRS-condition for short) .

Now let L be a block Laurent operator with defining function $\Phi = [f_{ij}]_{i,j=1}^m \in \mathcal{A}_w^m$. If L is hermitian, positive definite then

$$\det([f_{ij}(\lambda)]_{i,j=1}^m) > 0, \qquad |\lambda| = 1.$$

Hence we can apply Theorem 8.1 on page 830 in [19] to extend Theorem 4.2 to block Laurent operators and obtain the following

COROLLARY 4.3. *Let $L \in \mathcal{W}_w^m$ be self-adjoint and positive definite. Assume that the weight function w defined on $\mathbb{R}^m$ satisfies the GRS-condition, i.e.,*

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{w(-n)}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sqrt[n]{w(n)} = 1, \tag{23}$$

then $L^{-1} \in \mathcal{W}_w^m$.

It is clear that in the derivations above we can replace 1-periodic functions by α-periodic functions ($\alpha \in \mathbb{Q}$). Similarly, sequences with indices in $\mathbb{Z}$ can be replaced by sequences indexed by $\mathbb{Z}_\alpha$ (i.e. by indices of the form $k\alpha$).

Now we are ready to prove a general result on the decay properties of Gabor windows g and their duals γ.

THEOREM 4.4 ([9]). *Let (g, a, b) generate a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ with $ab = \frac{p}{q} \in \mathbb{Q}$ with relative prime integers p and q. Let $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ where w satisfies the GRS-condition. Then $\gamma \in \boldsymbol{L}_{1,w}(\mathbb{R})$.*

PROOF: The assumption $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ implies $\sum_{k=-\infty}^{\infty} |g(k)|w(k) < \infty$ and $\sum_{k=-\infty}^{\infty} |g(k/b)|w(k/b) < \infty$ for $b \in \mathbb{N}$. Denote

$$G_{kl}(t) = \frac{1}{b} \sum_{r=-\infty}^{\infty} g(t - ra - \frac{k}{b}) g(t - ra - \frac{l}{b}), \qquad k, l \in \mathbb{Z}. \tag{24}$$

$G(t)$ is periodic in t with period a. Since $ab = \frac{p}{q}$ we also get $G_{kl}(t) = G_{k+p,l+p}(t)$, cf. [20], Theorem 3.2. In words, $G(t)$ is a block Laurent operator. The frame property implies that $G(t)$ is hermitian positive definite and

$$AI \le G(t) \le BI.$$

The submultiplicativity of the function w implies that the spaces $\boldsymbol{L}_{1,w}(\mathbb{R})$, $\boldsymbol{\ell}_{1,w}(\mathbb{Z})$, and $\boldsymbol{\ell}_{1,w}(\mathbb{Z}_{\frac{1}{b}})$ are Banach algebras under convolution (cf. [18]). Hence $\{G_{0k}(t)\}_{k\in\mathbb{Z}} \in \boldsymbol{\ell}_{1,w}(\mathbb{Z}^m_{\frac{1}{b}})$, and $G(t) \in \mathcal{W}_w^m$. By Corollary 4.3

$$G^{-1}(t) \in \mathcal{W}_w^m. \tag{25}$$

The dual γ can be expressed as (see e.g. [16])

$$\gamma(t) = \sum_{k=-\infty}^{\infty} [G^{-1}(t)]_{0k} g(t - \frac{k}{b}). \tag{26}$$

This equation together with (25) and the assumption $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ yields that $\gamma \in \boldsymbol{L}_{1,w}(\mathbb{R})$. ❏

REMARK: (i) The idea to exploit the Laurent operator property of the Gabor frame operator in the context of window decay is not new. It has been used by Feichtinger and Gröchenig[20] in connection with modulation spaces and polynomial weights.
(ii) It is easy to reformulate Theorem 4.4 for windows whose decay properties are given in the frequency domain. We leave this modification to the reader.

Observe that condition (21) is satisfied e.g. for $w(x) = (1+|x|)^s$ and $w(x) = e^{\lambda|x|^\gamma}, \gamma < 1$, but not for $w(x) = e^{\lambda|x|}$. This is why we have to use a smaller exponent $\lambda_1 < \lambda$ for the dual to bound the decay in case of an exponentially decaying window.

5. TIME-FREQUENCY LOCALIZATION OF OFDM PULSE SHAPES

Now we are ready to give a general condition on the decay of the window g such that the resulting tight window $\tilde{g}$ and thus in turn the pulse shape ψ as defined in (15) is in the same algebra as g.

COROLLARY 5.1 ([9]). *(i) Let $g, \hat{g} \in \boldsymbol{L}_{1,w}$ and let (g, a, b) generate a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$. Set $\tilde{g} = S^{-\frac{1}{2}} g$. If the weight function w satisfies the GRS-condition, then $\tilde{g}, \hat{\tilde{g}} \in \boldsymbol{L}_{1,w}$.*
(ii) Let $g, \hat{g} \in \boldsymbol{\ell}_{1,w}$ and let (g, a, b) generate a Gabor frame for $\boldsymbol{\ell}_2(\mathbb{Z})$. Set $\gamma = S^{-1} g$ and $\tilde{g} = S^{-\frac{1}{2}} g$. If the weight function w satisfies the GRS-condition, then $\gamma, \hat{\gamma} \in \boldsymbol{\ell}_{1,w}$ and $\tilde{g}, \hat{\tilde{g}} \in \boldsymbol{\ell}_{1,w}$.

PROOF: (i) The proof of this result is similar to the proof of Theorem 4.4. We only indicate the necessary modifications. Let Φ be the matrix-valued defining function of a hermitian positive definite Laurent operator L and denote $L^{-\frac{1}{2}} = [C_k]_{k,l=-\infty}^{\infty}$. It follows from the basic properties of block Laurent operators that $\Phi(\omega) > 0$ for all ω and

$$C_k = \int_0^1 \frac{1}{\sqrt{\Phi(\omega)}}\, e^{-2\pi i k\omega}\, d\omega, \qquad k \in \mathbb{Z}.$$

Thus if $L \in \mathcal{W}_w^m$ for weights satisfying (23), then $L^{-\frac{1}{2}} \in \mathcal{W}_w^m$. The rest is now by repeating the steps in Theorem 4.4 and using the remark following Theorem 4.4.
(ii) In this case the frame operator S is a block Laurent operator (see [15]). The assumption $g \in \boldsymbol{\ell}_{1,w}$ implies that $S \in \mathcal{W}_w^m$. Along the same lines as above and in Theorem 4.4 it follows that S^{-1} and $S^{-\frac{1}{2}} \in \mathcal{W}_w^m$ and consequently $\gamma, \hat{\gamma} \in \boldsymbol{\ell}_{1,w}$, similar for $\tilde{g}, \hat{\tilde{g}}$. ❑

Thus if the window (i.e. pulse shape) g satisfies the properties of Corollary 5.1 the orthogonalization procedure (14) yields a function that is in the same space (in the same *algebra*) as the function g, and the quality of decay does not change.

REMARK: Due to the connection between Gabor frames and modulated filter banks,[21] Corollary 5.1 also yields a characterization of the decay properties of paraunitary modulated filter banks.

For a window g with exponential decay (in time and frequency) it has been shown that $\tilde{g}$ also has exponential decay (see [15] for a proof for $\boldsymbol{\ell}_2(\mathbb{Z})$, this result has later been extended to $\boldsymbol{L}_2(\mathbb{R})$, cf. [16]). However, similar to the dual window, in general the exponent for the decay of $\tilde{g}$ will be smaller than for g.

Since the Gaussian is optimally localized in the time-frequency plane, in the sense that it minimizes the uncertainty principle, it is interesting to note that for $g(t) = e^{-\pi t^2}$ the function $\tilde{g}$ has only exponential decay in time and frequency, cf. [16]. Thus we certainly loose some time-frequency localization in this case, although the resulting function $\tilde{g}$ still has exponential decay.

Thus there exists a large class of windows for which the time-frequency localization properties are not affected by applying (14). This is however only half of the truth, since in the considerations above we have ignored any constants that come into play.

Consider for example the following situation. Take a window g with exponential decay in time and frequency and assume that (g, T, F) constitutes a frame for $TF > 1$. Then for any $\varepsilon > 0$ there exists an N such that for $T > N, F > N$

$$\left\| \frac{g}{\|g\|} - \frac{\tilde{g}}{\|\tilde{g}\|} \right\| \le \varepsilon$$

since $(TF)\gamma \to g$ for $T \to \infty, F \to \infty$ (see [22] for a mathematically precise formulation). But of course $TF \gg 1$ leads to an unacceptable large loss of capacity for OFDM, already $TF > 2$ seems to be prohibitive in this context.

On the other hand, if we let $TF \to 1$, then $\tilde{g}$ and γ will become increasingly "ill-localized" in the time-frequency plane (since in the limit case $TF = 1$ we are confronted with the Balian-Low theorem), although from theory we know that $\tilde{g}$ has exponential decay as long as $TF > 1$. However, as pointed out earlier, this decay involves a constant C that depends on the ratio of the frame bounds. Since $A \to 0$ for $TF \to 1$ it follows that $C \to \infty$. Thus for applications such as OFDM the exponential-decay property of γ and $\tilde{g}$ quickly becomes meaningless for TF close to 1. If we choose a well-localized window, there is a trade-off between increasing TF and increasing the frame bound ratio B/A. As long as B/A is small the time-frequency localization is preserved in the transition from g to $\tilde{g} = S^{-\frac{1}{2}}g$. The problem of estimating the constants involved in the orthogonalization procedure remains a challenging problem for future research.

ACKNOWLEDGEMENTS

I acknowledge partial support by NSF grant 9973373.

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