FOUR SHORT STORIES ABOUT TOEPLITZ MATRIX CALCULATIONS

THOMAS STROHMER*

Abstract. The stories told in this paper are dealing with the solution of finite, infinite, and biinfinite Toeplitz-type systems. A crucial role plays the off-diagonal decay behavior of Toeplitz matrices and their inverses. Classical results of Gelfand et al. on commutative Banach algebras yield a general characterization of this decay behavior. We then derive estimates for the approximate solution of (bi)infinite Toeplitz systems by the finite section method, showing that the approximation rate depends only on the decay of the entries of the Toeplitz matrix and its condition number. Furthermore, we give error estimates for the solution of doubly infinite convolution systems by finite circulant systems. Finally, some quantitative results on the construction of preconditioners via circulant embedding are derived, which allow to provide a theoretical explanation for numerical observations made by some researchers in connection with deconvolution problems.

Key words. Toeplitz matrix, Laurent operator, decay of inverse matrix, preconditioner, circulant matrix, finite section method.

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1. Introduction. Toeplitz-type equations arise in many applications in mathematics, signal processing, communications engineering, and statistics. The excellent surveys [4, 17] describe a number of applications and contain a vast list of references. The stories told in this paper are dealing with the (approximate) solution of biinfinite, infinite, and finite hermitian positive definite Toeplitz-type systems. We pay special attention to Toeplitz-type systems with certain decay properties in the sense that the entries of the matrix enjoy a certain decay rate off the diagonal. In many theoretical and practical problems this decay is of exponential or polynomial type. Toeplitz equations arising in image deblurring are one example (since often the point spread function has exponential decay - or even stronger - compact support) [19]. Kernels of integral equations also frequently show fast decay, leading to Toeplitz systems inheriting this property (see e.g. [13]). Other examples include Weyl-Heisenberg frames with exponentially or polynomially decaying window functions [25] (yielding biinfinite block-Toeplitz systems with the same behavior when computing the so-called dual window), as well as channel estimation problems in digital communications [21].

Let $\boldsymbol{C}_{[-\frac{1}{2},\frac{1}{2})}$ be the set of all 1-periodic, continuous, real-valued functions defined on $[-\frac{1}{2},\frac{1}{2})$. For all $f \in \boldsymbol{C}_{[-\frac{1}{2},\frac{1}{2})}$, let

$$a_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega) e^{2\pi i \omega k}\, d\omega, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

* Department of Mathematics, University of California, Davis, CA 95616-8633, USA; E-mail: strohmer@math.ucdavis.edu. This work was partially supported by NSF grant 9973373.

be the Fourier coefficients of f. Since f is real-valued, we have $a_k = \overline{a}_{-k}$.

A *Laurent operator* or *multiplication operator* associated with its *defining function* f can be represented by the doubly infinite hermitian matrix $L = [L_{kl}]_{k,l=-\infty}^{\infty}$ with entries $L_{kl} = a_{k-l}$ for $k, l \in \mathbb{Z}$. For all $n \geq 1$ let $L_n = [(L_n)_{kl}]_{k,l=-n+1}^{n-1}$ be the Toeplitz matrix of size $(2n-1)\times(2n-1)$ with entries $(L_n)_{kl} = a_{k-l}$ for $k, l = -n+1, \dots, n-1$. L_n is a finite section of the biinfinite Toeplitz matrix L.

A *Toeplitz operator* with *symbol* f can be represented by the singly infinite matrix $T = [T_{kl}]_{k,l=0}^{\infty}$ with $T_{kl} = a_{k-l}$ for $k, l = 0, 1, \dots$. In this case we define $T_n = [(T_n)_{kl}]_{k,l=0}^{n-1}$ as the $n \times n$ matrix with entries $(T_n)_{kl} = a_{k-l}$ for $k, l = 0, \dots, n-1$. Of course $L_n = T_{2n-1}$, but in what follows it will sometimes be convenient to use the notations L_n and T_n.

As mentioned earlier, a crucial role throughout the paper plays the decay behavior of Toeplitz matrices and their inverses. Classical results of Gelfand et al. lead to a general characterization of this decay behavior for biinfinite Toeplitz matrices, see section 2. Section 3 is concerned with the approximate solution of (bi)infinite Toeplitz systems using the finite section method. Explicit error estimates are derived, showing that the approximation rate depends only on the condition number of the matrix and its decay properties. In section 4 we analyze the approximate solution of convolution equations via circulant matrices. Finally, in section 5, we derive some quantitative results for preconditioning of Toeplitz matrices by circulant embedding. Among others, we provide a theoretical explanation of numerical observations made by Nagy et al. in connection with (non)banded Toeplitz systems.

2. On the decay of inverses of Toeplitz-type matrices. It is helpful to review a few results on the decay of inverses of certain matrices. In what follows, if not otherwise mentioned, the 2-norm of a matrix or a vector will be denoted by $\|.\|$ without subscript.

The following theorem about the decay of the inverse of a band matrix is due to Demko, Moss, and Smith [8].

THEOREM 2.1. *Let A be a matrix acting on $\ell^2(\mathcal{I})$, where $\mathcal{I} = \{0, 1, \dots, N-1\}, \mathbb{Z}$, or $\mathbb{N}$, and assume A to be hermitian positive definite and s-banded (i.e., $A_{kl} = 0$ if $|k-l| > s$). Set $\kappa = \|A\|\|A^{-1}\|$, $q = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ and $\lambda = q^{\frac{1}{s}}$. Then*

$$|A^{-1}_{k,l}| \leq c\lambda^{|k-l|} \,,$$

where

$$c = \|A^{-1}\| \max\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\} \,.$$

One notes that the inverse of a banded matrix is in general not banded, the type of decay changes when we switch from A to A^{-1} (although exponential decay is "almost as good" as bandedness). This observation suggests to look at other classes of matrices, for which the type of decay is preserved under inversion. This leads naturally to the following

DEFINITION 2.2. *Let $A = [A_{k,l}]_{k,l\in\mathcal{I}}$ be a matrix, where the index set is $\mathcal{I} = \mathbb{Z}, \mathbb{N}$ or $\{0,\dots,N-1\}$.*
(i) A belongs to the space $\mathcal{E}_{\gamma,\lambda}$ if the coefficients A_{kl} satisfy

$$|A_{kl}| < ce^{-\gamma|k-l|^{\lambda}} \qquad \text{for } \gamma,\lambda > 0,$$

and some constant $c > 0$. If $\lambda = 1$ we simply write $\mathcal{E}_{\gamma}$.
(ii) A belongs to the space $\mathcal{Q}_s$ if the coefficients A_{kl} satisfy

$$|A_{kl}| < c(1+|k-l|)^{-s} \qquad \text{for } s > 1,$$

and some constant $c > 0$.

The following result is due to Jaffard [16].

THEOREM 2.3. *Let $A : \ell^2(\mathcal{I}) \to \ell^2(\mathcal{I})$ be an invertible matrix, where $\mathcal{I}$ is $\mathcal{I} = \mathbb{Z}, \mathbb{N}$ or $\{0,\dots,N-1\}$.*
(a) If $A \in \mathcal{E}_{\gamma}$, then $A^{-1} \in \mathcal{E}_{\gamma_1}$ for some $\gamma_1 < \gamma$.
(b) If $A \in \mathcal{Q}_s$, then $A^{-1} \in \mathcal{Q}_s$.

For finite-dimensional matrices these results (and in particular the involved constants) should be interpreted as follows. Think of the $n \times n$ matrix A_n as a finite section of an infinite-dimensional matrix A. If we increase the dimension of A_n (and thus consequently the dimension of $(A_n)^{-1}$) we can find uniform constants independent of n such the corresponding decay properties hold. This is of course not possible for arbitrary finite-dimensional invertible matrices.

Theorem 2.3(a) shows that the entries of A^{-1} still have exponential decay, however A^{-1} is in general not in the same algebra as A, since we may have to use a smaller exponent. However in Theorem 2.3(b) both, the matrix A and its inverse A^{-1} belong to the same algebra, the quality of decay does not change.

From this point of view Theorem 2.3(b) is the most striking result. The proof of Theorem 2.3(b) is rather delicate and lengthy. For biinfinite Toeplitz-type matrices this result can be proven much shorter (and extended to other types of decay) by using classical results of Gelfand et. al. on certain commutative Banach algebras. The following theorem is a weighted version of Wiener's Lemma. It is implicitly contained in [10], but since it may be of independent interest we state and prove it explicitly.

THEOREM 2.4. *Let $A = \{a_{kl}\}$ be a hermitian positive definite biinfinite Toeplitz matrix with inverse $A^{-1} = \{\alpha_{kl}\}$. Let $v(k)$ be a positive (weight) function with*

$$v(k+l) \le v(k)v(l)\,,$$

such that

$$\sum_{k=-\infty}^{\infty} |a_k| v(k) < \infty. \tag{2.5}$$

If

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{v(-n)}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sqrt[n]{v(n)} = 1, \tag{2.6}$$

then

$$\sum_{k=-\infty}^{\infty} |\alpha_k| v(k) < \infty. \tag{2.7}$$

In particular,

$$\textit{if} \quad A \in \mathcal{Q}_s \quad \textit{for } s > 1, \textit{ then } A^{-1} \in \mathcal{Q}_s; \tag{2.8}$$

$$\textit{if} \quad A \in \mathcal{E}_{\gamma,\lambda} \quad \textit{for } 0 < \lambda < 1, \textit{ then } A^{-1} \in \mathcal{E}_{\gamma,\lambda}. \tag{2.9}$$

Proof. Since A is positive definite we have

$$f(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k\omega} > 0 \tag{2.10}$$

and by the properties of Laurent operators [12]

$$1/f(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k e^{2\pi i k\omega}, \qquad \text{where} \quad (A^{-1})_{k,l} = \alpha_{k-l}.$$

We denote by $W[v]$ the set of all formal series $f = \sum_{k=-\infty}^{\infty} a_k X^k$ for which

$$\|f\| = \sum_{k=-\infty}^{\infty} |a_k| v(k) < \infty.$$

It follows from Chapter 19.4 of [10] that $W[v]$ is a Banach algebra with respect to the multiplication (discrete convolution)

$$fg = \sum_{l=-\infty}^{\infty} c_l X^l = \sum_{l=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{l-k} b_k \right) X^l,$$

where $f = \sum_k a_k X^k$ and $g = \sum_k b_k X^k$. By Theorem 2 on page 24 in [10] an element of $W[v]$ has an inverse in $W[v]$ if it is not contained in a maximal ideal of $W[v]$. Any maximal ideal of $W[v]$ consists of elements of the form (cf. Chapter 19.4 in [10])

$$\sum_{k=-\infty}^{\infty} a_k \xi^k = 0,$$

where $\xi = \rho e^{2\pi i\omega}$ with

$$\rho_1 \le \rho \le \rho_2,$$

and

$$\rho_1 = \lim_{n\to\infty} \frac{1}{\sqrt[n]{v_{-n}}} \quad \text{and} \quad \rho_2 = \lim_{n\to\infty} \sqrt[n]{v_n}.$$

Due to assumption (2.6) we get $\rho_1 = \rho_2 = 1$, hence $\rho = 1$. Thus a necessary and sufficient condition for an element in $W[v]$ to be not contained in a maximal ideal of $W[v]$ is $\sum_k a_k e^{2\pi i k\omega} \neq 0$ for all ω. By assumption A is positive definite, hence $f(\omega) = \sum_k a_k e^{2\pi i k\omega} > 0$ for all ω and (2.7) follows.

Statements (2.8) and (2.9) are now clear, since in both cases we can easily find a weight function such that (2.5) and (2.6) are satisfied. □

REMARK 2.5. (i) Theorem 2.11 in [9] by Domar and Theorem V B in [1] by Beurling are closely related to Theorem 2.4. Their results are concerned with (non)quasi-analytic functions, for which they have to impose the more restrictive condition

$$\sum_{k=1}^{\infty} \frac{\log[v(kx)]}{k^2} < \infty, \qquad \text{for all } x,$$

on the weight function (called Beurling-Domar condition in [22]). For instance the function $v(k) = \exp(\frac{|k|}{1+\log(|k|)}), k \neq 0$ satisfies condition (2.6), but not the Beurling-Domar condition.

(ii) Using Theorem 8.1 on page 830 in [12] we can extend Theorem 2.4 to biinfinite block-Toeplitz matrices with finite-dimensional non-Toeplitz blocks (i.e., Laurent operators with matrix-valued symbol). These matrices play an important role in filter bank theory [25].

(iii) Note that $v(n) = \exp(\gamma n)$ does not satisfy condition (2.6), that is why we have to introduce an exponent $\gamma_1 < \gamma$ in order to estimate the decay of A^{-1}, cf. also Theorem 2.3. However if $A \in \mathcal{E}_{\gamma,\lambda}$ with $\lambda < 1$, then condition 2.6 is satisfied and – as we have seen – the decay of the entries of A^{-1} can be bounded by using the *same* parameters γ, λ.

3. Approximation of infinite-dimensional Toeplitz-type systems. Infinite Toeplitz systems arise for instance in the discretization of Wiener-Hopf integral equations or, more generally, in one-sided infinite convolution equations, see [11]. Bi-infinite Toeplitz-type systems are encountered in doubly infinite (discrete) convolution equations, as well as e.g. in filter bank theory [25] or in the inverse heat problem [4]. In order to solve these problems we have to introduce a finite-dimensional model.

For let $A : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z})$ be a hermitian positive definite (*hpd* for short) biinfinite Toeplitz matrix given by $\{a_{k,l}\}_{k=-\infty}^{\infty}$. Let $y = \{y_k\}_{k=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$. We want to solve the system $Ax = y$.

For $n \in \mathbb{N}$ and $y \in \ell^2(\mathbb{Z})$ define the orthogonal projections P_n by

$$P_n y = (\ldots, 0, 0, y_{-n+1}, \ldots, y_{n-1}, 0, 0, \ldots). \tag{3.1}$$

By identifying the image of P_n with the $2n-1$-dimensional space $\mathbb{C}^{2n-1}$ we can express the $(2n-1) \times (2n-1)$ matrix A_n as

$$A_n = P_n A P_n,$$

where we have used that $P^* = P$. The n-th approximation $x^{(n)}$ to x is then given by the solution of the finite-dimensional system of equations

$$A_n x^{(n)} = y^{(n)}$$

where $y^{(n)} := P_n y$.

If A is a singly infinite Toeplitz matrix and $y \in \ell^2(\mathbb{N})$, we proceed analogously by defining P_n as

$$P_n y = (y_0, y_1, \dots, y_{n-1}, 0, 0, \dots).$$

This approach to approximate the solution of $Ax = y$ is usually called the *finite section method*, cf. [11].

The first question that arises when considering this method is "does $x^{(n)}$ converge to x?". For the case when A is not hpd this question has lead to deep mathematical results. See the book [11] and chapter 7 in [2] for more details. For the case when A is hpd the answer is easy and always positive. To see this, recall that since A is hpd it follows that A_n is also hermitian positive definite, see [15]. Furthermore, $\|A_n\| \le \|A\|$ and $\|(A_n)^{-1}\| \le \|A^{-1}\|$ for $n = 1, 2, \dots$. Applying the Lemma of Kantorovich [23] yields that $(A_n)^{-1}$ converges strongly to A^{-1} for $n \to \infty$, i.e., $x^{(n)}$ converges to x in the ℓ^2-norm for any $y \in \ell^2(\mathbb{Z})$ (or for any $y \in \ell^2(\mathbb{N})$ if A is singly infinite).

An important aspect for applications is if we can give an estimate on how fast $x^{(n)}$ converges to x. It will be shown that the rate of approximation depends on the decay behavior and the condition number of the matrix.

THEOREM 3.1. *Let $Lx = y$ be given, where $L = \{a_{k,l}\}$ is a hermitian positive definite biinfinite Toeplitz matrix and denote $x^{(n)} = L_n^{-1} y^{(n)}$.*
(a) If there exist constants c, c' such that

$$|a_k| \le c e^{-\gamma|k|} \;\; \textit{and} \;\; |y_k| \le c' e^{-\gamma|k|}, \;\; \gamma > 0 \tag{3.5}$$

then there exists a γ_1 with $0 < \gamma_1 < \gamma$ and a constant c_1 depending only on γ_1 and on the condition number of L such that

$$\|x - x^{(n)}\| \le c e^{-\gamma_1 n}. \tag{3.6}$$

(b) If there exist constants c, c' such that

$$|a_k| \le c(1+|k|)^{-s} \;\; \textit{and} \;\; |y_k| \le c'(1+|k|)^{-s}, \;\; s > 1, \tag{3.7}$$

then there exists a constant c_1 depending only on the condition number of L such that

$$\|x - x^{(n)}\| \le c_1 n^{(1-2s)/2}. \tag{3.8}$$

Proof. We have

$$\begin{aligned}\|x - x^{(n)}\| &= \|L^{-1}y - L_n^{-1}y^{(n)}\| \le \|L^{-1}\|\|y - LL_n^{-1}y^{(n)}\| \\ &\le \|L^{-1}\|\big(\|y - y^{(n)}\| + \|(L_n - L)L_n^{-1}y^{(n)}\|\big). \end{aligned} \tag{3.9}$$

To prove statement (a) we note that by Theorem 2.3(a) there exists a $\gamma_2 < \gamma$ such that $(L_n^{-1})_{kl} \le c_2 e^{-\gamma_2|k-l|}$ with a constant c_2 depending only on γ_2 and on the condition number of L_n. Since $\sigma(L_n) \subseteq [f_{\min}, f_{\max}]$ we get $\mathrm{cond}(L_n) \le \mathrm{cond}(L)$ for all n. That means we can choose c_2 independently of n. Write $z^{(n)} = (L_n - L)L_n^{-1}y^{(n)}$ and note that $z_k^{(n)} = 0$ for $|k| < n$. Since the non-zero entries of $(L_n - L)$ decay exponentially, it is easy to show that there exists a γ_1 with $0 < \gamma_1 < \gamma_2$ such that $\|z^{(n)}\| \le c_3 e^{-\gamma_1 n}$ for some constant c_3. It is trivial that $\|y - y^{(n)}\|$ also decays exponentially for $n \to \infty$. We absorb $\|L^{-1}\|$ and the other constants in the constant c_1 and get the desired result.

For the proof of part (b) we proceed analogously to above by applying Theorem 2.3(b) to conclude that

$$|(L_n^{-1}y^{(n)})_k| \le c_2(1 + k)^{-s}$$

for some constant c_2 depending only on $\mathrm{cond}(L)$ and on s. The norm $\|y - y^{(n)}\|$ can be estimated via

$$\|y - y^{(n)}\|^2 = \sum_{|k|\ge n} |y_k|^2 \le 2c\sum_{k=n}^{\infty}(1 + k)^{-2s} \le 2c\int_{n-1}^{\infty}(1 + x)^{-2s}dx \le 2c\frac{n^{1-2s}}{2s - 1}, \tag{3.10}$$

similarly for $\|z^{(n)}\|$ where $z^{(n)} := (L_n - L)L_n^{-1}y^{(n)}$. Since all arising constants - absorbed in one constant c_1 - depend only on $\mathrm{cond}(L)$ and on the exponent s, the proof is complete. □

Remark 3.2. Theorem 3.1 holds if we replace the system $Lx = y$ by a singly infinite Toeplitz system $Tx = y$ with corresponding decay conditions on T and y and approximate its solution by considering the finite system $T_n x^{(n)} = y^{(n)}$.

The proof of Theorem 3.1 is essentially based on the fact that under appropriate decay conditions on L and y (resp. L_n and $y^{(n)}$) L^{-1} and $L_n^{-1}y^{(n)}$ have similar decay properties. Thus, if one can show that L_n^{-1} has the same decay properties as L^{-1} one can use Theorem 2.4 in order to generalize Theorem 3.1 to various other decay conditions. This may however not always lead to simple and closed-form expressions for $\|(L_n - L)L_n^{-1}y^{(n)}\|$, therefore I have restricted myself to the most frequently encountered decay properties.

Example 1: We illustrate Theorem 3.1 by a numerical example. We consider $Lx = y$, where L is the biinfinite Toeplitz matrix with entries $a_k = (1+|k|)^{-s}, k \in \mathbb{Z}$ for $s = 2$ and y consists of random entries having the same polynomial decay rate as the

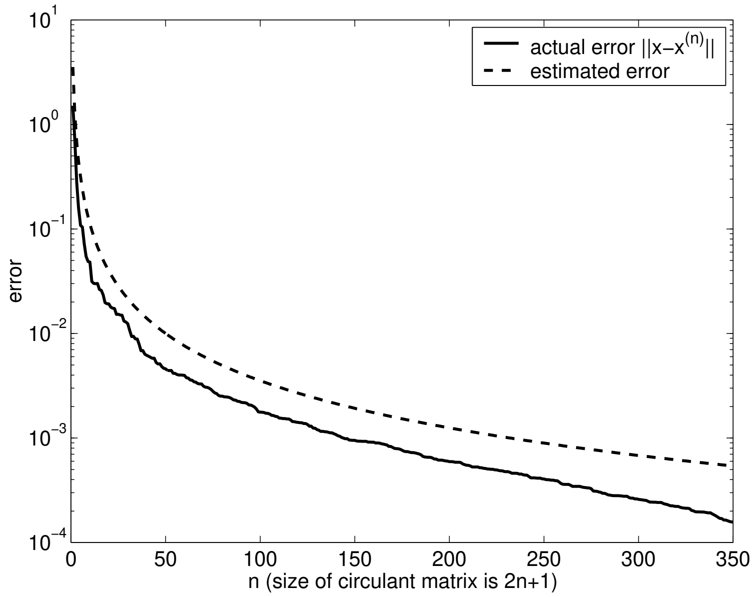

FIG. 3.1. *Actual approximation error and error estimate* (3.8) *from Theorem 3.1(b) for the system $Lx = y$, where L is a biinfinite hermitian Toeplitz matrix with polynomial decay.*

entries a_k. To compare the error $\|x-x^{(n)}\|$ with the error estimate (3.8) we would need the true solution x. Since the solution of this biinfinite system cannot be computed analytically we compute the "true" solution of $Lx = y$ by solving $L_{n_0}x^{(n_0)} = y^{(n_0)}$ for very large n_0 (we choose $n_0 = 32768$). Using (3.9) and (3.15) we can estimate that in the worst case $\|x - x^{(32768)}\| \approx 10^{-6}$, so that $x^{(32768)}$ can mimick the true solution with sufficiently high accuracy for this experiment.

Then we approximate this solution by the finite section method as in Theorem 3.1 for $n = 0, \dots, 350$ and compute for each n the error $\|x - x^{(n)}\|$ as well as the error estimate in (3.8). Note that an explicit expression for the constant c_1 in (3.8) is not known, we only know that it depends on the condition number of L. In this example we use $c_1 = \mathrm{cond}(L)$ (a different example may require a different choice). The result, illustrated in Figure 3.1, shows that the asymptotic behavior of the error rate is well estimated by the given error bound.

It is well-known that the product of two Laurent operators and the inverse of a Laurent operator (if it exists) is again a Laurent operator. This is of course not true for singly infinite or finite Toeplitz matrices (and this is one of the reasons which makes the "Toeplitz business" so interesting). Hence one may argue that the "canonical" finite-dimensional analogue of Laurent operators are not Toeplitz matrices but circulant matrices, since they also form an algebra. Thus for a given biinfinite hermitian Toeplitz matrix L with entries $L_{kl} = a_{k-l}$ we define the hermitian circulant matrix C_n of size $(2n-1) \times (2n-1)$ by

$$C_n = \begin{bmatrix} a_0 & \overline{a}_1 & \dots & \overline{a}_{n-2} & \overline{a}_{n-1} & a_{n-1} & a_{n-2} & \dots & a_1 \\ a_1 & a_0 & \overline{a}_1 & \dots & & & & & a_2 \\ \vdots & & \ddots & & & & & & \vdots \\ \overline{a}_1 & & \dots & & & & & a_1 & a_0 \end{bmatrix}. \tag{3.11}$$

We also say that C_n is generated by $\{a_k\}_{k=-n+1}^{n-1}$.

REMARK 3.3. C_n does not have to be positive definite if L is positive definite, e.g. see [5]. However - as pointed out in [5] - if L is at least in Wiener's algebra then one can always find an N such that C_n is invertible for all $n > N$. The faster the decay of the entries of L the smaller this N has to be.

We can do even a little better and estimate how well the extrema of the defining function of L are approximated by the extreme eigenvalues of C_n.

LEMMA 3.4. *Let L be a biinfinite hermitian Toeplitz matrix with entries $L_{k,l} = a_{k-l}$ where $a = \{a_k\}_{k=-\infty}^{\infty}$ and set $f(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k\omega}$. Let C_m be the associated $(2m-1) \times (2m-1)$ circulant matrix with first row $(a_0, \overline{a}_1, \dots, \overline{a}_{m-1}, a_{m-1}, \dots, a_1)$. Denote the maximum and minimum eigenvalue resp. of C_m by $\lambda_{\max}^{(m)}$ and $\lambda_{\min}^{(m)}$.*
(a) If L is n-banded with $m > n$, then

$$\lambda_{\max}^{(m)} \le f_{\max} \le \lambda_{\max}^{(m)} + 2\sin\Big(\frac{\pi n}{2(2m-1)^2}\Big)\|a\|_1, \tag{3.12}$$

$$\lambda_{\min}^{(m)} \ge f_{\min} \ge \lambda_{\min}^{(m)} - 2\sin\Big(\frac{\pi n}{2(2m-1)^2}\Big)\|a\|_1. \tag{3.13}$$

(b) If $|a_k| \le ce^{-\gamma|k|}$ for $k \in \mathbb{Z}, c > 0$, then

$$|f_{\max} - \lambda_{\max}^{(m)}| \le \frac{2c}{1-e^{-\gamma}}\Big[2\sin\Big(\frac{\pi m}{2(2m-1)^2}\Big) + e^{-\gamma m}\Big], \tag{3.14}$$

a similar estimate holds for $|f_{\min} - \lambda_{\min}^{(m)}|$.
(c) If $|a_k| \le c(1+|k|)^{-s}$ for $k \in \mathbb{Z}, s > 1, c > 0$, then

$$|f_{\max} - \lambda_{\max}^{(m)}| \le \frac{2c}{s-1}\Big[2\sin\Big(\frac{\pi m}{2(2m-1)^2}\Big) + m^{1-s}\Big], \tag{3.15}$$

a similar estimate holds for $|f_{\min} - \lambda_{\min}^{(m)}|$.

Proof. (a): It is well-known [7] that the eigenvalues of C_m are given by

$$\sum_{k=-m+1}^{m-1} a_k e^{2\pi i k l/(2m-1)}, \qquad \text{for } l = -m+1, \dots, m-1. \tag{3.16}$$

For case (a) that means they are regularly spaced samples $f(\frac{l}{2m-1})$ of the function f, which immediately yields the left hand side of the inequalities (3.12) and (3.13). In order to prove the right hand side of (3.12) and (3.13) it is sufficient to estimate

$$\max_{\omega,l}\Big|f(\frac{l+\omega}{2m-1}) - f(\frac{l}{2m-1})\Big|,$$

where $\omega \in [-\frac{1}{2(2m-1)}, \frac{1}{2(2m-1)}]$ and $l = -m+1, \ldots, m-1$. Define the sequence $\{\tilde{a}_k\}_{k=-m}^{m}$ by $\tilde{a}_k = a_k$ if $|k| \le n$ and $\tilde{a}_k = 0$ if $|k| > n$. There holds

$$\begin{aligned}\max_{\omega,l}\Big|f(\frac{l}{2m-1}) - f(\frac{l+\omega}{2m-1})\Big| &= \max_{\omega,l}\Big|\sum_{k=-m}^{m} \tilde{a}_k e^{2\pi ikl/(2m-1)}(e^{2\pi ik\omega/(2m-1)} - 1)\Big| \\ &\le \max_{\omega,|k|\le n}\big|e^{2\pi ik\omega/(2m-1)} - 1\big| \sum_{k=-n}^{n} |a_k| \\ \le \max_{\omega,|k|\le n} 2|\sin(\pi k\omega/(2m-1))|\|a\|_1 &\le 2\sin\big(\frac{\pi m}{2(2m-1)^2}\big)\|a\|_1. \end{aligned} \tag{3.18}$$

Relations (3.12) and (3.13) follow now from this estimate.

Statements (b) and (c) can be proved similarly by using

$$|f(\frac{l+\omega}{2m-1}) - \lambda_l^{(m)}| \le \sum_{k=-m+1}^{m-1} |a_k||e^{2\pi ik\omega/(2m-1)} - 1| + \sum_{|k|\ge m} |a_k|, \tag{3.19}$$

and applying the corresponding decay properties to (3.19). □

REMARK 3.5. The left part of inequality (3.13) reads $\lambda_{\min}^{m} \ge f_{\min}$. This implies that Strang's preconditioner is always positive definite for n-banded hermitian Toeplitz matrices of size $> (2n \times 2n)$ with positive generating function. Hence the "sufficiently large n"-condition at the end of section 2 in [14] can be omitted.

It is obvious that decay properties for circulant matrices cannot be defined in the same way as for non-circulant matrices. Hence, by stating that C_m has, say, exponentially decaying entries, we mean that the generating sequence $\{a_k\}_{k=-m+1}^{m-1}$ satisfies $|a_k| \le ce^{-\gamma|k|}$, in which case the entries of C_m will decay exponentially off the corners of the matrix (instead of off the diagonal). In analogy to the theorems in section 2 it is natural to ask if the inverse of C_m also inherits these decay properties. The following theorem shows that at least for m sufficiently large this is the case. The entries of C_m^{-1} uniformly approximate the entries of L^{-1} with an error rate depends on the decay properties and the condition number of L.

THEOREM 3.6. *Let L be a hermitian positive definite Laurent operator with entries $L_{kl} = a_{k-l}$ and let C_m be the associated circulant $(2m-1)\times(2m-1)$ matrix as defined in (3.11). Denote the entries of L^{-1} by $(L^{-1})_{kl} = \{\alpha_{k-l}\}$ and let $(C_m)^{-1}$ (if it exists) be generated by $\{\beta_k\}_{k=-m+1}^{m-1}$.*
(i) If $L \in \mathcal{E}_\gamma$, then for sufficiently large m

$$|\alpha_k - \beta_k| \le ce^{-\gamma_1 m}, \tag{3.20}$$

with $0 < \gamma_1 \le \gamma$ and some constant c depending on $cond(L)$ and γ_1.
(ii) If $A \in \mathcal{Q}_s$ for $s > 1$, then for sufficiently large m

$$|\alpha_k - \beta_k| \le c\frac{m^{1-s}}{s-1} \tag{3.21}$$

with some constant c depending on $cond(L)$.

Proof. (i): Set $f(x)=\sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx}$ and $f_m(x)=\sum_{k=-m+1}^{m-1} a_k e^{2\pi ikx}$. By the properties of Laurent operators [12] $\{\alpha_k\}_{k\in\mathbb{Z}}$ is given by

$$\alpha_k=\int_{-1/2}^{1/2}\frac{1}{f(x)}e^{2\pi ikx}dx,\qquad k\in\mathbb{Z}, \tag{3.22}$$

By Remark 3.3 and Lemma 3.4 we can easily find an N such that C_m is invertible for all $m>N$, which implies that $f_m>0$. In this case by the properties of circulant matrices [7] the entries $\{\beta_k\}_{k=-m+1}^{m-1}$ of C_m^{-1} can be computed as

$$\beta_k=\frac{1}{2m-1}\sum_{l=-m+1}^{m-1}\frac{1}{f_m(\frac{l}{2m-1})}e^{2\pi ikl/(2m-1)},\qquad k=-m+1,\dots,m-1. \tag{3.23}$$

Now consider

$$|\alpha_k-\beta_k|=\Big|\int_{-1/2}^{1/2}\frac{1}{f(x)}e^{2\pi ikx}dx-\sum_{l=-m+1}^{m-1}\frac{1}{f_m(\frac{l}{2m-1})}e^{2\pi ikl/(2m-1)}\Big|$$

$$\le\Big|\int_{-1/2}^{1/2}\frac{1}{f(x)}e^{2\pi ikx}dx-\frac{1}{2m-1}\sum_{l=-m+1}^{m-1}\frac{1}{f(\frac{l}{2m-1})}e^{2\pi ikl/(2m-1)}\Big|+ \tag{3.24}$$

$$+\frac{1}{2m-1}\Big|\sum_{l=-m+1}^{m-1}\frac{1}{f(\frac{l}{2m-1})}e^{2\pi ikl/(2m-1)}-\sum_{l=-m+1}^{m-1}\frac{1}{f_m(\frac{l}{2m-1})}e^{2\pi ikl/(2m-1)}\Big| \tag{3.25}$$

We estimate the expression above in two steps:
1. We first consider (3.24). Note that $\frac{1}{f(x)}=\sum_{k=-\infty}^{\infty}\alpha_k e^{2\pi ikx}$ and

$$\frac{1}{f(\frac{l}{2m-1})}=\sum_{k=-\infty}^{\infty}\alpha_k e^{2\pi ikl/(2m-1)}=\sum_{p=-m+1}^{m-1}\left(\sum_{q=-\infty}^{\infty}\alpha_{p+(2m-1)q}\right)e^{2\pi ilp/(2m-1)}.$$

By setting $\delta_p=\sum_{q=-\infty}^{\infty}\alpha_{p+(2m-1)q}$ we get

$$\frac{1}{f(\frac{l}{2m-1})}=\sum_{p=-m+1}^{m-1}\delta_p e^{2\pi ilp/(2m-1)},\qquad l=-m+1,\dots,m-1,$$

and

$$\delta_k=\frac{1}{2m-1}\sum_{p=-m+1}^{m-1}\frac{1}{f(\frac{p}{2m-1})}e^{-2\pi ikp/(2m-1)}.$$

Hence

$$\Big|\int_{-1/2}^{1/2}\frac{1}{f(x)}e^{2\pi ikx}dx-\frac{1}{2m-1}\sum_{l=-m+1}^{m-1}\frac{1}{f(\frac{l}{2m-1})}e^{2\pi ikl/(2m-1)}\Big|$$

$$=|\alpha_k-\delta_k|\le\sum_{q\neq0}|\alpha_{k+(2m-1)q}|. \tag{3.27}$$

By Theorem 2.3(a) $A \in \mathcal{E}_\gamma$ implies $A^{-1} \in \mathcal{E}_{\gamma_1}$. Hence we get for $k = -m+1, \dots, m-1$

$$\sum_{q \neq 0} |\alpha_{k+(2m-1)q}| \le 2c_1 \sum_{q=1}^{\infty} e^{-\gamma_1 (k+(2m-1)q)}$$
$$\le 2c_1 \sum_{q=1}^{\infty} e^{-\gamma_1((2m-1)q-m+1)} = \frac{2c_1 e^{-\gamma_1 m}}{1-e^{-\gamma_1(2m-1)}}. \tag{3.28}$$

2. Now we estimate (3.25):

$$\frac{1}{2m-1}\Big|\sum_{l=-m+1}^{m-1} \frac{1}{f(\frac{l}{2m-1})} e^{2\pi ikl/(2m-1)} - \sum_{l=-m+1}^{m-1} \frac{1}{f_m(\frac{l}{2m-1})} e^{2\pi ikl/(2m-1)}\Big|$$
$$\le \frac{1}{2m-1} \sum_{l=-m+1}^{m-1} \Big|\frac{1}{f(\frac{l}{2m-1})} - \frac{1}{f_m(\frac{l}{2m-1})}\Big|$$
$$\le \frac{1}{2m-1} \sum_{l=-m+1}^{m-1} \Big|\frac{1}{f(\frac{l}{2m-1})}\Big|\Big|\frac{1}{f_m(\frac{l}{2m-1})}\Big|\Big|f(\frac{l}{2m-1}) - f_m(\frac{l}{2m-1})\Big|$$
$$\le \frac{1}{2m-1} \max_{|l|\le m} \Big|\frac{1}{f(\frac{l}{2m-1})}\Big|\Big|\frac{1}{f_m(\frac{l}{2m-1})}\Big| \sum_{l=-m+1}^{m-1} \sum_{|k|>m} |a_k|$$
$$\le \|A^{-1}\| \|C_m^{-1}\| \sum_{|k|>m} |a_k|. \tag{3.29}$$

By Lemma 3.4 we can easily find for any $\varepsilon > 0$ an N such that for all $m > N$ there holds $\|C_m^{-1}\| \le (1+\varepsilon)\|A^{-1}\|$. Thus

$$\|A^{-1}\| \|C_m^{-1}\| \sum_{|k|>m} |a_k| \le (1+\varepsilon)\|A^{-1}\|^2 2 \sum_{k=m}^{\infty} |a_k|$$
$$\le (1+\varepsilon)\|A^{-1}\|^2 2 \frac{e^{-\gamma m}}{1-e^{-\gamma}} \le (1+\varepsilon)\|A^{-1}\|^2 c_2 e^{-\gamma_1 m} \tag{3.30}$$

for some $\gamma_1 < \gamma$ and some constant c_2. By combining (3.27), (3.28), and (3.30) and hiding expressions as $(1+\varepsilon)\|A^{-1}\|$ in the constant c, we obtain estimate (3.20).

(ii): The proof of (3.21) is similar to the proof of (3.20). The only steps that require a modification are (3.28) and (3.30). By Theorem 2.3(a) $A \in \mathcal{Q}_s$ implies

$A^{-1} \in \mathcal{Q}_s$. Hence we can estimate $\sum_{q\neq 0} |\alpha_{k+(2m-1)q}|$ as follows.

$$\begin{aligned}
\sum_{q\neq 0} |\alpha_{k+(2m-1)q}| &\leq c' \sum_{q\neq 0}(1+|(2m-1)q+k|)^{-s} \\
&= 2c'(1+|2m-1+k|)^{-s} + 2c' \sum_{q=2}^{\infty}(1+|(2m-1)q+k|)^{-s} \\
&\leq 2c'(1+|2m-1-m+1|)^{-s} + 2c' \sum_{q=2}^{\infty}(1+|(2m-1)q-m+1|)^{-s} \\
&\leq 2c'(1+m)^{-s} + 2c' \int_1^{\infty}(1+|(2m-1)x-m+1|)^{-s}\, dx \\
&= 2c'(1+m)^{-s} + 2c' \frac{(1+m)^{-s}}{(2m-1)(s-1)} \leq \frac{2c(1+m)^{-s}}{s-1}.
\end{aligned} \tag{3.31}$$

By adapting (3.30) to the case of polynomial decay we can estimate (3.25) by

$$2c\|A^{-1}\|\|C_m^{-1}\|\frac{m^{1-s}}{s-1}. \tag{3.32}$$

Combining (3.31) with (3.32) yields the desired result. □

The main portion of the proof of statement (ii) above is due to Gabriele Steidl [24].

4. Error estimates for approximate solution of deconvolution problems. Consider the convolution of two sequences $a = \{a_k\}_{k=-\infty}^{\infty}, c = \{c_k\}_{k=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, given by $b = a * c$. Here a may represent an impulse response or a blurring function. Given a and b our goal is to compute c. This is known as deconvolution. In matrix notation the problem can be expressed as $Lc = b$ where L is a biinfinite Toeplitz matrix with entries $L_{kl} = a_{k-l}$.

Sometimes a and c have compact support, in which case b also has compact (although larger) support and c can be computed by solving a finite banded Toeplitz system, see [19]. It is well-known that this can be done efficiently via FFT by embedding the Toeplitz matrix into a circulant matrix. Of course, this approach is very attractive from a numerical viewpoint, at least if the system is well-conditioned (we discuss the ill-conditioned case in section 5).

However, if either a or c does not have compact support, the reduction of $Lc = b$ to a finite Toeplitz system obviously will introduce a truncation error and embedding the Toeplitz matrix into a circulant yields an additional (perturbation) error. We nevertheless can try to make use of the FFT-based approach with the hope to get a good approximation to the solution. According to [13] it has been shown in [18] that the solution of doubly infinite convolution systems can be approximated by solutions of finite circulant systems. (Note that the approximation by finite circulant systems does not apply to one-sided infinite convolution equations.)

Hence we are concerned with the problem of how good the approximation is obtained in that way and how fast the approximation converges to the true solution.

To answer these questions we proceed as follows. In the first step we study the approximate solution of a finite Toeplitz system by circulant embedding. In the second step we combine the obtained results with Theorem 3.1 to derive estimates for the approximate solution of doubly infinite convolution equations by finite circulant systems. The results that we will collect in the first step will also be very useful in section 5 for the analysis of circulant preconditioners.

Let $A_n x^{(n)} = y^{(n)}$ be given, where A_n is an hpd $n \times n$ Toeplitz matrix. Of course, we have in mind that A_n and $y^{(n)}$ are finite sections of the biinfinite Toeplitz matrix L and the right-hand side y, respectively.

As usual, we embed A_n into a circulant matrix C_{2n} of size $2n \times 2n$ as follows:

$$C_{2n} = \begin{bmatrix} A_n & B_n^* \\ B_n & A_n \end{bmatrix}, \tag{4.1}$$

where B_n is the $n \times n$ Toeplitz matrix with first row given by $b_k = a_{n-k}$ for $k = 1, \dots, n-1$. If a_n is known we set $b_0 = a_n$, otherwise we define $b_0 = 0$. For the following considerations it does not matter if we embed A_n into a circulant matrix of size $(2n-1) \times (2n-1)$ or of size $2n \times 2n$ (or larger). Choosing B_n to be of the same size as A_n is just more convenient for the proofs below. Since C_{2n} is circulant we can find A_n again in the center of C_{2n} (and at any other position along the main diagonal of C_{2n}), and express the embedding of A_n into C_{2n} as follows

$$C_{2n} = \begin{bmatrix} \times & \times & \times \\ \times & A_n & \times \\ \times & \times & \times \end{bmatrix}.$$

In spite of the biinifite system $Lx = y$ in the background it is useful to embed $y^{(n)}$ symmetrically into a vector $\tilde{y}^{(n)}$ of length $2n$ as follows

$$\tilde{y}^{(n)} = [0, \dots, 0, y^{(n)}, 0, \dots, 0].$$

We assume for the moment that C_{2n} is invertible (and will justify this assumption later). An approximate solution $z^{(n)}$ to $A_n x^{(n)} = y^{(n)}$ is now obtained by solving $C_{2n} \tilde{z}^{(n)} = \tilde{y}^{(n)}$ and setting $z^{(n)} = \{\tilde{z}_k^{(n)}\}_{k=n/2+1}^{2n-n/2}$ (i.e., we take as approximation the central part of $\tilde{z}^{(n)}$ corresponding to the embedding of $y^{(n)}$ into $\tilde{y}^{(n)}$).

We partition C_{2n}^{-1} in the same way as C_{2n} as follows

$$C_{2n}^{-1} = \begin{bmatrix} M_n & T_n^* \\ T_n & M_n \end{bmatrix},$$

where M_n is a Toeplitz matrix and by Cauchy's interlace theorem (cf. [15]) invertible. Define $S_n := M_n^{-1}$, then it is easy to see that $z^{(n)}$ can be obtained as the solution of

$$S_n z^{(n)} = y^{(n)}.$$

The question is now how well does $z^{(n)}$ approximate $x^{(n)}$.

THEOREM 4.1. *Let $A_n x^{(n)} = y^{(n)}$ be given where A_n is an $n \times n$ hermitian positive definite Toeplitz matrix with $(A_n)_{kl} = a_{k-l}$ and $a_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega) e^{2\pi i \omega k}\, d\omega$. Assume that $|a_k| \le c e^{-\gamma|k|}$ and $|(y^{(n)})_k| \le c' e^{-\gamma|n/2-k|}$. Suppose that C_{2n} as defined in (4.1) is invertible and let $z^{(n)}$ be the solution of $S_n z^{(n)} = y^{(n)}$, where S_n is the $n \times n$ leading principal submatrix of C_{2n}^{-1}. Then there exists a γ_1 with $0 < \gamma_1 < \gamma$ and a constant c_1 depending only on $f_{\min}$ and $f_{\max}$ and on γ_1 such that*

$$\|x^{(n)} - z^{(n)}\| \le c_1 e^{-\gamma_1 n}. \tag{4.5}$$

Proof. Similar to Theorem 3.1 we write

$$\|x^{(n)} - z^{(n)}\| \le \|A_n^{-1}\| \|(A_n - S_n) S_n^{-1} y^{(n)}\|. \tag{4.6}$$

Using the Schur complement [15] we can write S_n as

$$S_n = A_n - B_n A_n^{-1} B_n^*. \tag{4.7}$$

We will first show that the entries of the matrix $A_n - S_n$ are exponentially decaying off the corners of the matrix. Note that equation (4.7) implies

$$A_n - S_n = E_n$$

where $E_n := B_n A_n^{-1} B_n^*$. We analyze the decay behavior of E_n in two steps by considering first $B_n A_n^{-1}$ and then $(B_n A_n^{-1}) B_n^*$.

Recall that $|(A_n)_{kl}| \le c e^{-\gamma|k-l|}$ and note that $|(B_n^*)_{kl}| \le c e^{-\gamma(n-|k-l|)}$. By Theorem 2.3 we know that $|(A_n^{-1})_{kl}| \le c_2 e^{-\gamma_2|k-l|}$ for some $0 < \gamma_2 < \gamma$, where c_2 depends on γ_2 and on $f_{\min}$ and $f_{\max}$, but is independent of n. We set $\delta = \gamma - \gamma_2$. There holds

$$|(B_n A_n^{-1})_{kl}| = |\sum_{j=0}^{n-1} (B_n)_{kj} (A_n^{-1})_{jl}| \le c c_2 \sum_{j=0}^{n-1} e^{-\gamma(n-|k-j|)} e^{-\gamma_2|j-l|}. \tag{4.9}$$

For simplicity we will absorb any constants arising throughout this proof that depend solely on γ (or γ_2) in the constant c. We analyze the sum (4.9) further by splitting it up into three parts and in addition consider first the entries $(B_n A_n^{-1})_{kl}$ with $k \ge l$.

(i) $0 \le j < l$:

$$c \sum_{j=0}^{l-1} e^{-\gamma(n-|k-j|)} e^{-\gamma_2|j-l|} = c e^{-\gamma(n-k)} e^{-\gamma_2 l} \sum_{j=0}^{l-1} e^{-(\gamma-\gamma_2)j} \le c e^{-\gamma_2(n-k+l)}.$$

(ii) $l \le j < k$:

$$c \sum_{j=l}^{k-1} e^{-\gamma(n-|k-j|)} e^{-\gamma_2|j-l|} \le c e^{-\gamma(n-k)} e^{\gamma_2 l} \frac{e^{-(\gamma+\gamma_2)l}}{1 - e^{-(\gamma+\gamma_2)}} \le c e^{-\gamma_2(n-k+l)}.$$

(iii) $k \le j < n$:

$$c\sum_{j=k}^{n-1} e^{-\gamma(n-|k-j|)}e^{-\gamma_2|j-l|} \le ce^{-\gamma(n+k)}e^{\gamma_2 l}\frac{e^{(\gamma-\gamma_2)(n-1)}}{1-e^{-(\gamma-\gamma_2)}} \tag{4.10}$$

$$= ce^{-\gamma_2(n+k-l)}e^{-\delta(n+k)}\frac{e^{\delta(n-1)}}{1-e^{-(\gamma-\gamma_2)}} \le ce^{-\gamma_2(n-l+k)} \le ce^{-\gamma_2(n-k+l)}.$$

Similar expressions can be obtained for the case $l \ge k$ by interchanging the roles of k and l in the derivations above. Thus

$$|(B_nA_n^{-1})_{kl}| \le ce^{-\gamma_2(n-|k-l|)}.$$

We now estimate the decay of the entries of $B_nA_n^{-1}B_n$. Since $B_nA_n^{-1}B_n^*$ is hermitian, it is sufficient to consider only the entries $(B_nA_n^{-1}B_n^*)_{kl}$ with $k \ge l$. There holds

$$|(B_nA_n^{-1}B_n^*)_{kl}| \le c\sum_{j=0}^{n-1} e^{-\gamma_2(n-|k-j|)}e^{-\gamma(n-|j-l|)}. \tag{4.12}$$

As before we proceed by splitting up this sum into three parts.
(i) $0 \le j < l$:

$$c\sum_{j=0}^{l-1} e^{-\gamma_2(n-|k-j|)}e^{-\gamma(n-|j-l|)} = ce^{-(\gamma+\gamma_2)n}e^{\gamma_2 k}e^{\gamma l}\frac{1}{1-e^{-(\gamma+\gamma_2)}} \le ce^{-\gamma_2(2n-k-l)}$$

(ii) $l \le j < k$:

$$c\sum_{j=l}^{k-1} e^{-\gamma_2(n-|k-j|)}e^{-\gamma(n-|j-l|)} \le ce^{-(\gamma_2+\delta)(n+l)}e^{-\gamma_2(n-k)}\frac{e^{\delta(k-1)}}{1-e^{-\delta}} \le ce^{-\gamma_2(2n-k+l)}$$

(iii) $k \le j < n$:

$$c\sum_{j=k}^{n-1} e^{-\gamma_2(n-|k-j|)}e^{-\gamma(n-|j-l|)} \le e^{-(\gamma+\gamma_2)n}e^{-\gamma_2(k+l)}\frac{e^{(\gamma+\gamma_2)(n-1)}-e^{(\gamma+\gamma_2)(k-1)}}{1-e^{-(\gamma+\gamma_2)}}$$

$$\le ce^{-\gamma_2(k+l)}.$$

Hence, by combining (i), (ii), and (iii) we get

$$|(B_nA_n^{-1}B_n^*)_{kl}| \le c(e^{-2\gamma_2 n}e^{\gamma_2(k+l)} + e^{-2\gamma_2 n}e^{\gamma_2|k-l|} + e^{-\gamma_2(k+l)}). \tag{4.13}$$

The entries of C_{2n} satisfy

$$|(C_{2n})_{lk}| \le \begin{cases} ce^{-\gamma|k-l|} & \text{for } |k-l| = 0,\dots,n-1, \\ ce^{-\gamma(2n-|k-l|)} & \text{for } |k-l| = n,\dots,2n-1. \end{cases} \tag{4.14}$$

By Theorem 3.6 there exists a $\gamma_3 < \gamma$ and a constant c_3 depending on γ_3 and on $f_{\min}$ and $f_{\max}$ such that

$$|(C_{2n}^{-1})_{lk}| \le \begin{cases} c_3 e^{-\gamma_3|k-l|} & \text{for } |k-l| = 0,\dots,n-1, \\ c_3 e^{-\gamma_3(2n-|k-l|)} & \text{for } |k-l| = n,\dots,2n-1. \end{cases} \tag{4.15}$$

Hence $z^{(n)} := S_n^{-1}y^{(n)}$ satisfies $|z_k^{(n)}| \le c_3 e^{-\gamma_3|n/2-k|}$. Set $u^{(n)} = (A_n - S_n)z^{(n)}$. After some lengthy but straightforward computations we get

$$|(u^{(n)})_k| \le \begin{cases} c_1 e^{-\gamma_1(n/2+k)} & \text{for } k = 0,\dots,n/2, \\ c_1 e^{-\gamma_1(3n/2-k)} & \text{for } k = n/2,\dots,n \end{cases} \tag{4.16}$$

for some $\gamma_1 < \gamma_3$. Hence $\|u^{(n)}\| \le ce^{-\gamma_3 n/2}$, which together with (4.6) completes the proof. □

COROLLARY 4.2. *Let $Lx = y$ be given where L is a biinfinite hermitian positive definite Toeplitz matrix with entries $L_{kl} = a_{k-l}$ and let S_n be as defined in* (4.7). *Assume $L \in \mathcal{E}_\gamma$ and $|y_k| \le e^{-\gamma|k|}$ and let $z^{(n)}$ be the solution of $S_n z^{(n)} = y^{(n)}$. Then there exists an N such that for all $n > N$*

$$\|x - z^{(n)}\| \le ce^{-\gamma_1 n},$$

for some $0 < \gamma_1 < \gamma$ and a constant c independent of n.

Proof. First note that by Remark 3.3 we can always find an N such that S_n exists for all $n > N$. There holds

$$\|x - z^{(n)}\| = \|L^{-1}y - S_n^{-1}y^{(n)}\| \le \|L^{-1}y - L_n^{-1}y^{(n)}\| + \|L_n^{-1}y^{(n)} - S_n^{-1}y^{(n)}\| \tag{4.18}$$

where A_n is an $n \times n$ finite section of L. The result follows now by applying Theorem 3.1 and Theorem 4.1. □

Theorem 3.1 and Corollary 4.2 provide two different ways to approximate the solution of biinfinite Toeplitz systems. Which of the two is preferable? This depends on two criteria: (i) The accuracy of the approximation for given dimension n; (ii) the computational costs for solving each of the finite-dimensional systems.

The solution of the circulant system in Corollary 4.2 can be computed via 3 FFTs of size $2n$. The Toeplitz system in Theorem 3.1 can be solved by the conjugate gradient method in approximately $3k$ FFTs of size $2n$, where k is the number of iterations. Of course, additional preconditioning can significantly reduce this number at the cost of two additional FFTs per iteration (see also Section 5). For both, the circulant and the Toeplitz system, zeropadding can be used to extend the vectors to "power-of-two"-length.

EXAMPLE 2: We consider the same biinfinite Toeplitz system as in Example 1. We compare the error when approximating the solution by using the circulant system of Corollary 4.2 and by the Toeplitz system of Theorem 3.1. We compute for each

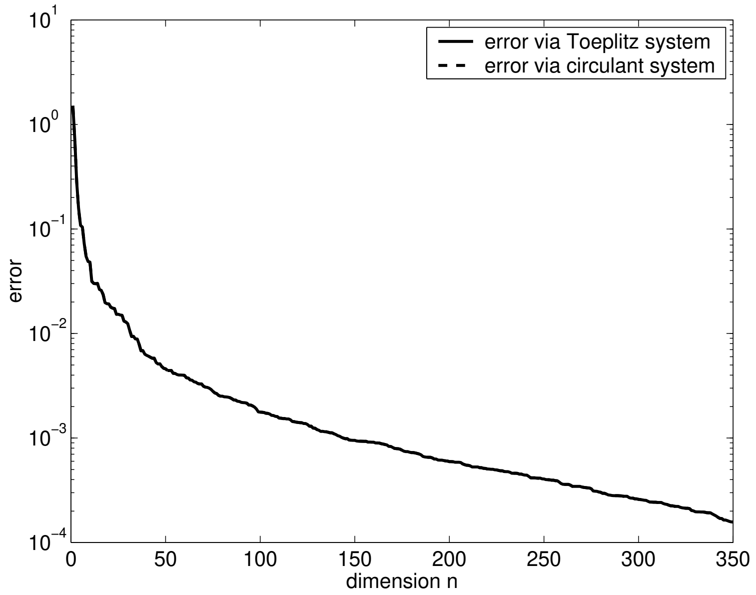

FIG. 4.1. *Comparison of error for the solution of a biinfinite Toeplitz system with polynomial decay. We compare the approximation error of the Toeplitz system described in Theorem 3.1 to that of the circulant system of Corollary 4.2 for increasing matrix dimension. The approximation error of both methods is almost identical, so that the difference between the two graphs is hardly visible.*

$n = 1, \dots, 350$ the approximation error $\|x - x^{(n)}\|$ and $\|x - z^{(n)}\|$ respectively. As can be seen from Figure 4.1 both methods give almost the same error, in fact the two lines showing the error are hardly distinguishable. A similar behavior can be observed for other examples involving biinfinite Toeplitz matrices with fast decay. Since solving a circulant system is cheaper than solving a Toeplitz system, the approximation scheme of Corollary 4.2 seems to be preferable in such situations.

Many variations of the theme are possible. For instance if A_n is an s-banded (biinfinite) Toeplitz matrix with $s < n/2$, we could use Strang's preconditioner as approximate inverse. Due to the explicit constants in Theorem 2.1 this approach allows us to give an error estimate with explicit constants (cf. also Theorem 5 in [25]).

THEOREM 4.3. *Let $A_n x^{(n)} = y^{(n)}$ be given where A_n is an $n \times n$ hermitian s-banded Toeplitz matrix with $s < n/3$ and positive generating function and let $y_k^{(n)} = 0$ for $|n/2 - k| > s$. Let C_n be the $n \times n$ circulant matrix with first row given by $(a_0, \overline{a}_1, \dots, \overline{a}_s, 0, \dots, 0, a_s, \dots, a_1)$ and let $z^{(n)}$ be the solution of $C_n z^{(n)} = y^{(n)}$. Then*

$$\|x^{(n)} - z^{(n)}\| \le 3\sqrt{2} c \lambda^{-\gamma n} (\lambda^{-\gamma s} - \lambda^{-\gamma(s+1)})^{-3}$$

where c and λ are as in Theorem 2.1.

Proof. The proof is similar to that of Theorem 4.1. To avoid unnecessary repetitions we only indicate the modifications, that are required.

By Remark 3.3 C_n is invertible. Note that C_n is a matrix with three bands, one band is centered at the main diagonal, and the two other bands of width $2s$ are located at the lower left and upper right corner of the matrix. It follows from Proposition 5.1 in [8] that the entries of C_n^{-1} decay exponentially off the diagonal and off the lower right and upper left corner. More precisely,

$$|(C_n^{-1})_{k,l}| \leq \begin{cases} c\lambda^{|k-l|} & \text{if } 0 \leq |k-l| \leq n \\ c\lambda^{2n+1-|k-l|} & \text{if } n+1 \leq |k-l| \leq 2n\,, \end{cases}$$

where c is as in Theorem 2.1 with $\lambda = q^{\frac{1}{2s}}$. With this result at hand it is easy to show that the entries of $C_n^{-1}y^{(n)}$ decay exponentially.

$A_n - C_n$ has a simple form, it is a Toeplitz matrix with first row

$$(0, \ldots, 0, a_s a_{s-1}, \ldots, a_1).$$

When we compute $u^{(n)} := (A_n - C_n)(C_n^{-1}y^{(n)})$ the non-zero entries of $(A_n - C_n)$ are multiplied by exponentially decaying entries due to the exponential decay of $((C_n^{-1}y^{(n)})$. This leads to the estimate $\|u^{(n)}\| \leq 2\sqrt{2}c(\lambda^s - \lambda^{s+1})^{-3}\lambda^n$. □

An interesting alternative to periodic boundary conditions is the use of Neumann boundary conditions considered in [20]. This modification will be discussed elsewhere.

5. Preconditioning by embedding and exponentially decaying Toeplitz matrices. The accuracy of the solution of the Toeplitz system $A_n x^{(n)} = y^{(n)}$ by using S_n^{-1} as approximate inverse depends crucially on the decay properties of the right hand side $y^{(n)}$. If $y^{(n)}$ does not have appropriate decay conditions the approach in section 4 may not yield an approximation with sufficient accuracy. But we can still use S_n^{-1} as preconditioner and solve $A_n x^{(n)} = y^{(n)}$ by the preconditioned conjugate gradient method [4]. The construction of preconditioners via circulant embedding is well known, it has been thoroughly investigated in [6] and for the special case of band Toeplitz matrices in [19, 14].

Compared to the certainly more elegant and more general approach in [6], the approach undertaken in this section has the advantage that it yields some quantitative results. It shows that the clustering behavior of the preconditioned matrix $S_n^{-1}A_n$ is the stronger the faster the decay of A_n is. Moreover, our approach allows us to prove a conjecture by Nagy et al., cf. [19], and will provide a theoretical explanation for some numerical results presented in [19] and [14].

The theoretical analysis of the clustering behavior of the eigenvalues of the preconditioned matrix $S_n^{-1}A_n$ is inspired by the work of Raymond Chan [3]. We will show that $S_n^{-1}A_n$ can be written as $S_n^{-1}A_n = I_n + R_n + K_n$, where I_n is the identity matrix, R_n is a matrix of small rank, and K_n is a matrix of small 2-norm.

THEOREM 5.1. *Let A_n be a hermitian Toeplitz matrix whose entries a_k decay exponentially, i.e., $|a_k| \leq Ce^{\gamma|k|}, k = 0, \ldots, n-1$. Set $S_n = A_n - B_n A_n^{-1} B_n^*$, where B_n is as defined in* (4.1)*. Then for all $\varepsilon > 0$, there exist N and M such that for all $n > N$ at most M eigenvalues of $A_n - S_n$ have absolute value exceeding ε.*

Proof. By definition of S_n we have

$$A_n - S_n = E_n,$$

where $E_n = B_n A_n^{-1} B_n^*$. We know from equation (4.13) in the proof of Theorem 4.1 that the entries of E_n can be bounded by

$$|(E_n)_{kl}| \le c(e^{-2\gamma_1 n} e^{\gamma_1 (k+l)} + e^{-2\gamma_1 n} e^{\gamma_1 |k-l|} + e^{-\gamma_1 (k+l)}). \tag{5.2}$$

For $x = [x_0, x_1, \dots, x_{n-2}, x_{n-1}] \in \mathbb{C}^n$ and $N < n/2$ we define the orthogonal projection P_N by

$$P_N x = [0, \dots, 0, x_N, x_{N+1}, \dots, x_{n-N-1}, 0, \dots, 0],$$

and identify the image of P_N with $\mathbb{C}^{n-2N}$. We set $E_n^{(N)} = P_N E_n P_N$. In words, $E_n^{(N)}$ is obtained from E_n by taking only the central $(n-2N) \times (n-2N)$ submatrix of E_n and setting the other entries surrounding this block equal to zero. Then $E_n - E_n^{(N)}$ has $2N$ "full" rows and $n - 2N$ "sparse" rows, where each of the latter rows has non-zero entries only at the first N and last N coordinates. Thus the dimension of the space spanned by the sparse rows is at most $2N$. Hence $\operatorname{rank}(E_n - E_n^{(N)}) \le 4N$. Due to the decay properties of E_n it is easy to see that

$$\|E_n^{(N)}\|_1 = \sum_{l=N}^{n-N-1} |(E_n^{(N)})_{Nl}|.$$

Using (5.2) we get after some straightforward calculations

$$\|E_n^{(N)}\|_1 \le c(e^{-\gamma_1 n} + e^{-\gamma_1 (n-2N)} + e^{-2\gamma_1 N}). \tag{5.3}$$

It is obvious that for each given $\varepsilon > 0$ we can find an N such that for all $n > 2N$ $\|E_n^{(N)}\|_1 \le \varepsilon$. Since $E_n^{(N)}$ is hermitian, we have $\|E_n^{(N)}\|_1 = \|E_n^{(N)}\|_\infty$. Thus

$$\|E_n^{(N)}\|_2 \le (\|E_n^{(N)}\|_1 \|E_n^{(N)}\|_\infty)^{1/2} \le \varepsilon.$$

Hence for large n the spectrum of $E_n^{(N)}$ lies in $(-\varepsilon, \varepsilon)$. By the Cauchy interlace theorem we conclude that at most $4N$ eigenvalues of $A_n - S_n$ have absolute value exceeding ε. □

Lemma 3.4 implies that for any $\varepsilon > 0$ we can find an M such that for all $n > M$ S_n and S_n^{-1} exist and $\|S_n^{-1}\|$ is bounded by $f_{\min} - \varepsilon > 0$. Proceeding as in [3], Chapter 2, we express $S_n^{-1} A_n$ as

$$S_n^{-1} A_n = I_n + S_n^{-1}(A_n - S_n)$$

and arrive at the following

COROLLARY 5.2. *Let $A = \{a_{k,l}\}$ be a hermitian positive definite Toeplitz matrix with $|a_k| \le ce^{-\gamma |k|}$ for $\gamma > 0$. Then for all $\varepsilon > 0$ there exist N and $M > 0$ such that for all $n > M$ at most N eigenvalues of $S_n^{-1} A_n - I_n$ have absolute values larger than ε.*

With the results presented in this paper it should not be difficult for the reader to derive Theorem 5.1 and Corollary 5.2 for Toeplitz matrices with polynomial decay.

Theorem 5.1 (in particular (5.3)) and Corollary 5.2 show that the clustering behavior of $S_n^{-1}A_n$ is the stronger the faster the decay of the entries of the Toeplitz matrix is.

In [19] and in [14] Nagy et al. consider the solution of convolution equations and banded Toeplitz systems by the preconditioned conjugate gradient method using a preconditioner similar to the one in this section. In [19] the Toeplitz matrix is rectangular, but the embedding is in principle the same. The theoretical results presented in [19, 14] only hold for banded Toeplitz systems, but in the numerical experiments the authors consider also non-banded Toeplitz systems, where the Toeplitz matrix has exponential decay (see [19]) or polynomial decay (see Example 4 in [14]). It is noted in [19] that "it is surprising that the number of iterations is still quite small". The authors also point out that the numerical experiments indicate additional clustering around one of the spectrum of the preconditioner matrix, which is not covered by their theoretical results. With Theorem 5.1 and Corollary 5.2 at hand we can provide a theoretical explanation for the numerical observations in [19] and [14], at least for the 1-D case. The key lies in the fast decay of the inverse of the Toeplitz matrix.

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