Approximation of dual Gabor frames, window decay, and wireless communications

Thomas Strohmer*

Abstract

We consider two problems involving Gabor frames that have recently received much attention. The first problem concerns the approximation of dual Gabor frames in $\boldsymbol{L}_2(\mathbb{R})$ by finite-dimensional methods. Utilizing the duality relations for Gabor frames we derive a method to approximate the dual Gabor frame, that is much simpler than previously proposed techniques. Furthermore it enables us to give estimates for the approximation rate when the dimension of the finite model approaches infinity. The second problem concerns the relation between the decay of the window function g and its canonical dual window $\gamma = S^{-1}g$ as well as its canonical tight window $h = S^{-\frac{1}{2}}g$. Based on results on commutative Banach algebras and Laurent operators we derive a general condition under which γ and h inherit the decay properties of g. These derivations are of relevance in the context of wireless communications. More precisely, our results provide a theoretical foundation for a recently proposed method for the design of time-frequency well localized pulse shapes for orthogonal frequency division multiplexing (OFDM) systems.

AMS Subject Classification: 42C15, 94A11, 94A12.
Key words: Gabor frame, Laurent operator, finite section method, tight frame, Wiener's algebra, orthogonal frequency division multiplexing.

*Department of Mathematics, University of California, Davis, CA 95616-8633, USA; Email: strohmer@math.ucdavis.edu. This work was supported by NSF grant 9973373.

1 Introduction and Preliminaries

1.1 Motivation and Announcement of Results

Gabor systems play an important role in signal processing and digital communication. In filter bank theory they are known under the name *oversampled modulated filter banks* [9], in wireline communications they correspond to the concept of *discrete multitone transmultiplexing*, and in wireless communications they are (implicitly) used in *orthogonal frequency division multiple access systems* [47, 33, 3].

A Gabor system consists of functions of the form

$$g_{na,mb}(t) = e^{2\pi imb} g(t - na), \quad n, m \in \mathbb{Z}, \; a, b \in \mathbb{R}, \tag{1}$$

where $g \in \boldsymbol{L}_2(\mathbb{R})$ is – depending on the context – called *window*, *atom*, or *pulse shape*. The parameters a and b represent the time-shift and frequency-shift, respectively.

We say that (g, a, b) generates a *Gabor frame* for $\boldsymbol{L}_2(\mathbb{R})$ for given shift parameters a, b if there exist constants (*frame bounds*) $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{n,m\in\mathbb{Z}} |\langle f, g_{na,mb}\rangle|^2 \le B\|f\|^2, \tag{2}$$

for any $f \in \boldsymbol{L}_2(\mathbb{R})$.

The Gabor frame operator is defined by

$$Sf = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle g_{na,mb}, \qquad f \in \boldsymbol{L}_2(\mathbb{R}). \tag{3}$$

A crucial role is played by the *canonical dual window*

$$\gamma = S^{-1} g, \tag{4}$$

since any f in $\boldsymbol{L}_2(\mathbb{R})$ can be represented as

$$f = \sum_{n,m\in\mathbb{Z}} \langle f, \gamma_{na,mb}\rangle g_{na,mb} = \sum_{n,m\in\mathbb{Z}} \langle f, g_{na,mb}\rangle \gamma_{na,mb}, \tag{5}$$

where the *dual frame* $\{\gamma_{na,mb}\}$ is given by

$$\gamma_{na,mb} = e^{2\pi imb}\gamma(t - na), \quad n, m \in \mathbb{Z}; \, a, b \in \mathbb{R}.$$

In general there are many functions generating dual frames that satisfy relation (5). The canonical dual window γ has several nice properties. One of them is that it has minimal $\boldsymbol{L}_2$-norm among all dual functions [29].

In applications formula (4) is not directly applicable for the computation of γ, since it involves the inversion of an infinite-dimensional operator. In practice one therefore often resorts to implementing (4) for sampled periodic signals, viz. for signals in $\mathbb{C}^N$, tacitly assuming that for N large enough this finite model will approximately represent the infinite-dimensional setting $\boldsymbol{L}_2(\mathbb{R})$ or $\boldsymbol{\ell}_2(\mathbb{Z})$ for which Gabor theory is usually developed.

The connection between Gabor frames on $\boldsymbol{\ell}_2(\mathbb{Z})$ and the associated periodized Gabor frames on $\mathbb{C}^N$ has been clarified in [32] and [40]. In [32] Janssen showed that if (g, a, b) with $a, 1/b \in \mathbb{Z}$ is a Gabor frame for $\boldsymbol{\ell}_2(\mathbb{Z})$ then (g^{per}, a, b) is a Gabor frame for $\mathbb{C}^N$, where $g^{per}(k)$ is the periodized function $g^{per} := \sum_{j=-\infty}^{\infty} g(k - jN)$. Furthermore the dual of g^{per} can be obtained by periodizing the dual of g. However it does not follow from the results in [32] that the dual of the periodized window converges to the dual of g when we increase the periodization interval N. In fact, in Janssen's approach N is fixed by $N := \mathrm{lcm}(a, b^{-1})$. Using a slightly different periodic model it is shown in [40] that the dual of the periodized window indeed converges to the dual of g with increasing length of the period (and thus increasing dimension of the finite-dimensional model).

For $g_{na,mb} \in \boldsymbol{L}_2(\mathbb{R})$ the finite-dimensional approximation of γ and of S^{-1} for poses more problems. Janssen has shown in [32] that the dual of the sampled function $g^D = \{g(k)\}_{k\in\mathbb{Z}}$ coincides with the sampled dual window $\gamma^D = \{\gamma(k)\}_{k\in\mathbb{Z}}$, i.e., $(S^D g^D)^{-1} = (S^{-1}g)^D$. Here D denotes the respective sampling operation. This result is very appealing, but it does not imply that the finite-dimensional approximation $(S^D g^D)^{-1}$ converges to γ with increasing sampling density.

The existing literature provides essentially two methods to approximate γ by finite-dimensional methods [27, 39, 6]. Both methods consider a finite subset $\{g_{na,mb}\}_{|n|,|m|\le N}$ of the frame $\{g_{na,mb}\}_{n,m\in\mathbb{Z}}$ in order to derive an approximation to γ. However both methods have serious drawbacks. They may lead either to considerable instability problems or require the estimation of auxiliary parameters that are difficult to obtain, see Section 2.1 for more details. Furthermore none of these methods is numerically efficient.

In Section 2 we propose an approach for the finite-dimensional approximation of γ that exploits the so-called duality conditions of Gabor frames. Our approach leads to a simple and stable approximation scheme. Furthermore we show how to derive explicit estimates for the rate with which the approximate dual converges to the actual dual γ. These estimates depend solely on the frame bounds and the decay properties of the window function g. For instance for windows with exponential decay in time and frequency our results predict an exponential rate of convergence.

Understanding the time-frequency localization properties of the window g and its dual γ is one of the core problems in Gabor analysis. The most famous result in this context is certainly the celebrated Balian-Low theorem [12], which precludes good joint time-frequency localization windows g for exact Gabor frames. It is folklore that in case of non-exact frames it is easy to find a g with excellent localization properties that generates a frame. It is clear that for a truly local time-frequency representation not only g but also its dual γ has to be well concentrated in the time-frequency plane. These considerations lead naturally to the problem of finding conditions under which γ inherits the localization properties from g. This problem is not only interesting from a mathematical view point (cf. e.g. [44, 29, 30, 31, 17, 42, 5] for recent results), but also of importance in digital communications – see e.g. [26, 33, 3] as well as Section 4 – and in quantum physics, see Chapter 16.1 in [1].

In Section 3 we will use Gelfand's theory of commutative Banach algebras to derive rather general condition on the decay properties of g such that the dual γ and the canonical tight window $\psi = S^{-\frac{1}{2}}g$ inherit these decay properties. Our result can be seen as an extension of results derived by Janssen [29, 30] and by Feichtinger and Gröchenig [17].

In Section 4 we will demonstrate that the theoretical framework developed in Section 3 is useful in wireless communications. We show how the results of Section 3 provide a theoretical foundation for a recently proposed algorithm for the construction of time-frequency well localized pulse shapes for OFDM systems.

1.2 Preliminaries and Notation

In this section we introduce a few basic concepts and notations used throughout the paper.

For a given Gabor frame $\{g_{na,mb}\}$ we define the analysis operator T by

$$T : f \in \boldsymbol{L}_2(\mathbb{R}) \to Tf = \{\langle f, g_{na,mb}\rangle\}_{n,m\in\mathbb{Z}}, \tag{6}$$

and the synthesis operator, which happens to be the adjoint of T, is

$$T^* : c \in \boldsymbol{\ell}_2(\mathbb{Z} \times \mathbb{Z}) \to T^*c = \sum_{n,m \in \mathbb{Z}} c_{nm} g_{na,mb}. \tag{7}$$

Of course the frame operator can be expressed as $S = T^*T$. Furthermore S satisfies

$$IA \leq S \leq IB,$$

where I is the identity operator on $\boldsymbol{L}_2(\mathbb{R})$. The ratio $B/A = \|S\|\|S^{-1}\|$ is the condition number of the frame operator.

A frame $\{g_{na,mb}\}$ is called tight if $A = B$. The frame (ψ, a, b) constructed via $\psi = S^{-\frac{1}{2}} g$ is called the canonical tight frame. In this paper we concentrate on canonical dual and tight windows. Henceforth we will simply talk about *the* dual window and *the* tight window. For more details about the properties of Gabor frames the reader is referred to [11] and [18] and the references therein.

The Fourier transform of a function f is given by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i t\omega} dt.$$

The *short time Fourier transform* (STFT) of f with respect to the (sufficiently nice) window g is

$$(\mathcal{V}_g f)(t, \omega) = \int_{-\infty}^{+\infty} f(x) \overline{g(x-t)} e^{-2\pi i x\omega} \, dx. \tag{8}$$

A locally integrable function w is called *weight function*, if w is positive and submultiplicative, i.e., if $w(t) > 0$ and $w(t_1 + t_2) \leq w(t_1)w(t_2)$.

The space $\boldsymbol{L}_{1,w}(\mathbb{R})$ consists of all functions f with $\int_{-\infty}^{+\infty} |f(t)|w(t)dt < \infty$, where w is a weight function. Similarly $\boldsymbol{\ell}_{1,w}(\mathbb{Z})$ consists of all sequences $x = \{x_k\}_{k \in \mathbb{Z}}$ with $\sum_{k=\infty}^{\infty} |x_k| w(k) < \infty$. The Schwartz space is denoted by $\boldsymbol{\mathcal{S}}$.

It is convenient to define following spaces:

$$\mathcal{C}_{[\alpha,\beta]} = \Big\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, \alpha, \beta \text{ such that } |f(t)| = 0 \ , \forall |t| \notin [\alpha, \beta] \Big\}.$$

$$\mathcal{E}_\lambda = \Big\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, \lambda > 0, c > 0 \text{ such that } |f(t)| \le c e^{-\lambda|t|}\ , \forall |t| \Big\}.$$

$$\mathcal{Q}_s = \Big\{ f \in \boldsymbol{L}_2(\mathbb{R}) : \exists\, s > 1, c > 0 \text{ such that } |f(t)| \le c(1+|t|)^{-s}\ , \forall |t| \Big\}.$$

Furthermore $\delta_{k,k'}$ denotes the Kronecker delta with its usual meaning, i.e., $\delta_{k,k'} = 0$ for $k \neq k'$ and 0 else. Finally, the Moore-Penrose inverse [15] of a bounded operator T is denoted by T^+.

2 Gabor frames, finite sections and the duality condition

The theoretical concepts for Gabor analysis are usually developed for infinite-dimensional function spaces, notably for $\boldsymbol{L}_2(\mathbb{R})$ or $\boldsymbol{\ell}_2(\mathbb{Z})$, whereas all numerical implementations have to be done within a finite-dimensional framework. The connection between Gabor systems on $\boldsymbol{\ell}_2(\mathbb{Z})$ and several finite-dimensional models has been clarified in [42, 40]. In this section we extend these results to Gabor frames for $\boldsymbol{L}_2(\mathbb{R})$. Different techniques than those used for $\boldsymbol{\ell}_2(\mathbb{Z})$ are required for $\boldsymbol{L}_2(\mathbb{R})$.

Our approach relies on a remarkable property of Gabor frames, whose discovery is usually associated with a paper by Wexler and Raz [45]. Their result was later made precise and extended by Janssen [29], Ron and Shen [38], and Daubechies, H. Landau, and Z. Landau [14]. It is interesting to note that Rieffel discovered a very general form of these duality relations already in 1988 in his work on projective modules over non-commutative tori [37].[1]

For given g, a, b we define the operator H by

$$Hf := \{\langle f, g_{k/b,l/a}\rangle\}_{k,l\in\mathbb{Z}}, \qquad f \in \boldsymbol{L}_2(\mathbb{R}),$$

where

$$g_{k/b,l/a}(t) = g(t-k/b)e^{2\pi i t l/a}.$$

The adjoint H^* of H is

$$H^* c = \sum_{k,l} c_{k,l}\, g_{k/b,l/a}, \qquad c = \{c_{k,l}\}_{k,l\in\mathbb{Z}} \in \boldsymbol{\ell}^2(\mathbb{Z}^2).$$

[1] Therefore it may be appropriate to replace the frequently used expression *Wexler-Raz duality condition* by *Rieffel duality condition*.

We identify HH^* with its matrix representation with respect to the basis $e_{k,l} = (\delta_{k,k'}\delta_{l,l'})_{k',l'}$ for $k, l \in \mathbb{Z}$. Hence

$$(HH^*)_{k,l,k',l'} = \langle g_{k'/b,l'/a}, g_{k/b,l/a}\rangle, \qquad k, l, k', l' \in \mathbb{Z}.$$

Hence HH^* can be interpreted as biinfinite block matrix with blocks of biinfinite size, the index (k, l) giving the position of the block and the index (k', l') giving the position of a specific entry within a block.

The duality relations for Gabor frames imply that the dual window satisfies [29, 14]:

$$\gamma = H^*(HH^*)^{-1}\sigma = ab \sum_{k,l\in\mathbb{Z}} [(HH^*)^{-1}]_{k,l,0,0}\; g_{k/b,l/a}, \tag{9}$$

where $\sigma = \{(ab)\delta_{k,0}\delta_{l,0}\}_{k,l\in\mathbb{Z}}$. Here the invertibility of HH^* follows from the following theorem (see [38], equivalent formulations are given in [29, 14]).

Theorem 2.1 *(a) $(g, 1/b, 1/a)$ is a Riesz basis for its closed linear span if and only if (g, a, b) is a frame for $\boldsymbol{L}_2(\mathbb{R})$.*
(b) Assume $\|g\| = 1$. $(g, 1/b, 1/a)$ is an orthonormal basis for its closed linear span if and only if (g, a, b) is a tight frame for $\boldsymbol{L}_2(\mathbb{R})$.

Formula (9) and Theorem 2.1 are the main ingredients for our approach to approximate the dual window γ using a finite-dimensional model.

For $x \in \boldsymbol{\ell}^2(\mathbb{Z}^2)$ and $n \in \mathbb{N}$ define the orthogonal projections P_n by

$$(P_n x)_{k,l} = \begin{cases} x_{k,l} & \text{if } \max\{|k|, |l|\} \le n, \\ 0 & \text{else.} \end{cases}$$

We identify the image of P_n with the $(2n+1)^2$-dimensional space $\mathbb{C}^{(2n+1)^2}$ and write

$$H_n f := P_n H f = \{\langle f, g_{k/b,l/a}\rangle\}_{|k|,|l|\le n}.$$

The matrix

$$H_n H_n^* = P_n HH^* P_n = \{\langle g_{k'/b,l'/a}, g_{k/b,l/a}\rangle\}_{|k|,|l|,|k'|,|l'|\le n}.$$

is a finite section of the infinite-dimensional matrix HH^*.

We say that the finite section method is *applicable* to HH^*, if, beginning with some $n \in \mathbb{N}$, for each $y \in \mathrm{range}(H)$ the equation

$$H_n^*[H_n H_n^*]^{-1} x^{(n)} = P_n y$$

has a unique solution $x^{(n)} \in \mathrm{Im}(P_n)$ and as $n \to \infty$ the vectors $x^{(n)}$ tend to the solution of $HH^*x = y$.

We set

$$\gamma^{(n)} := H_n^*[H_n H_n^*]^{-1} P_n \sigma, \quad \text{for } n = 0, 1, 2, \dots.$$

Note that the pseudo inverse of H and H_n, respectively, is given by

$$H^+ = H^*[HH^*]^{-1}, \quad H_n^+ = H_n^*[H_n H_n^*]^{-1}.$$

There holds:

Theorem 2.2 *Let (g, a, b) generate a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ and let γ be the dual window. Then*

$$\|\gamma - \gamma^{(n)}\| \to 0 \qquad \text{for } n \to \infty.$$

For the proof of Theorem 2.2 we need the following result, which is a modification of a lemma of Kantorovich [36].

Lemma 2.3 *Let $\dots \boldsymbol{H}_n \subseteq \boldsymbol{H}_{n+1} \subseteq \dots \subseteq \boldsymbol{H}, n \in \mathbb{N}$ be a sequence of Hilbert spaces. Let K be an operator invertible on $\boldsymbol{H}$ and let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of operators, invertible on $\boldsymbol{H}_n$ and converging strongly to K. If there exist constants $0 < C_1, C_2 < \infty$ with*

$$C_1 I_n \leq K_n \leq C_2 I_n \qquad \text{for all } n \in \mathbb{N}, \tag{10}$$

where I_n is the identity operator on $\boldsymbol{H}_n$, then $\{K_n^{-1}\}_{n\in\mathbb{N}}$ converges strongly to K^{-1} for $n \to \infty$.

Proof: Let P_n be the orthogonal projection from $\boldsymbol{H}$ onto $\boldsymbol{H}_n$. Given $Kx = y$ and $K_n x^{(n)} = P_n y$, we have to show that $x^{(n)} \to x$ as $n \to \infty$ for any $y \in \boldsymbol{H}$. Since $\|x - x^{(n)}\| \leq \|x - P_n x\| + \|P_n x - x^{(n)}\|$ and since $P_n x \to x$ pointwise as $n \to \infty$, we only have to consider the term $\|P_n x - x^{(n)}\|$. We compute

$$\|P_n x - x^{(n)}\| = \|P_n K^{-1} y - K_n^{-1} P_n y\| \leq \|K_n^{-1}\| \|K_n K^{-1} y - P_n y\| \tag{11}$$

$$\leq 1/C_1 \big(\|(K_n - K)K^{-1} y\| + \|y - P_n y\|\big). \tag{12}$$

Since $P_n y \to y$ pointwise and $K_n \to K$ strongly for $n \to \infty$ it follows that $\lim_{n\to\infty} \|P_n x - x^{(n)}\| = 0$ and the proof is complete. ❏

Proof of Theorem 2.2:

Theorem 2.1 implies that $(g, 1/b, 1/a)$ is a Riesz basis for its closed span with *Riesz bounds* A, B. Any finite subset of a Riesz basis is again a Riesz basis for its closed span, cf. Chapter 1 in [46]. Thus for $\{g_{k/b,l/a}\}_{|k|,|l|\leq n}, n \in \mathbb{N}$ there exist constants A_n, B_n such that the associated operator H_n satisfies

$$A \leq A_n \leq \frac{1}{ab} H_n H_n^* \leq B_n \leq B. \tag{13}$$

Relation (13) can be seen as follows. Take $x \in \operatorname{Im} P_n$ with $\|x\| = 1$, then

$$\langle H_n H_n^* x, x\rangle = \langle H^* P_n x, H^* P_n x\rangle = \langle HH^* x, x\rangle \leq \|HH^*\| = abB, \tag{14}$$

and similarly for the left-hand side of inequality (13). Now we apply Lemma 2.3 to conclude that the finite section method is applicable to HH^*.

Finally, note that H_n obviously converges to H pointwise for $n \to \infty$, thus $\|\gamma^{(n)} - \gamma\| \to 0$ for $n \to \infty$. ❑

It is interesting to ask if we can give some estimate on the rate of approximation of the finite section method in case the window g satisfies certain decay conditions in time and/or frequency. In the following theorem we concentrate on windows with exponential decay in time and frequency domain.

Theorem 2.4 *Let (g, a, b) be a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ and assume that there exist constants $C, D > 0$ such that*

$$|g(t)| \leq Ce^{-\lambda|t|}, \quad \text{and} \quad |\hat{g}(\omega)| \leq De^{-\lambda|\omega|}, \tag{15}$$

then there exists a $\lambda' < \lambda$ and a constant C' depending on the frame bounds and on λ', but independent of n such that

$$\|\gamma - \gamma^{(n)}\| \leq C' e^{-\lambda' n}.$$

Proof: First we show that (15) implies that the entries of HH^* satisfy

$$(HH^*)_{k,l,k',l'} \leq D_2 e^{-\lambda_1(|k-k'|+|l-l'|)} \tag{16}$$

for some $\lambda_1 < \lambda$ and some constant D_2.

It is clear that

$$|(HH^*)_{k,l,k',l'}| = |\langle g_{k'/b,l'/a}, g_{k/b,l/a}\rangle| = |\langle g, g_{(k-k')/b,(l-l')/a}\rangle| \tag{17}$$

and the STFT of g with respect to g fulfills

$$(\mathcal{V}_g g)(k/b, l/a) = \langle g, g_{k/b,l/a}\rangle, \quad k, l \in \mathbb{Z}. \tag{18}$$

Now

$$\begin{aligned} |(\mathcal{V}_g g)(t,\omega)| &= \Big|\int_{-\infty}^{+\infty} g(x)\overline{g(x-t)}e^{-2\pi i\omega x}\Big|dx \le \int_{-\infty}^{+\infty} |g(x)||\overline{g(x-t)}|dx \\ &\le C^2 \int_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda|x-t|}dx \le C^2 \int_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda_1|x-t|}dx \end{aligned} \tag{19}$$

for some $\lambda_1 < \lambda$. Set $\varepsilon = \lambda - \lambda_1$, then

$$\begin{aligned} C^2 \int_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda_1|x-t|}dx &\le C^2 \int_{-\infty}^{+\infty} e^{-\lambda|x|}e^{-\lambda_1(|t|-|x|)}dx \\ &\le C^2 \int_{-\infty}^{+\infty} e^{-\varepsilon|x|}e^{-\lambda_1|t|}dx = C^2 e^{-\lambda_1|t|}\int_{-\infty}^{+\infty} e^{-\varepsilon|x|}dx \\ &\Rightarrow |(\mathcal{V}_g g)(t,\omega)| \le C_1 e^{-\lambda_1|t|} \end{aligned} \tag{20}$$

for some constant C_1 depending on λ_1.

Similarly

$$|(\mathcal{V}_g g)(t,\omega)| = |(\mathcal{V}_{\hat{g}}\hat{g})(\omega,-t)| \le D_1 e^{-\lambda_1|\omega|}. \tag{21}$$

By combining (20) and (21) and taking the square root we get

$$|(\mathcal{V}_g g)(t,\omega)| \le \sqrt{C_1 D_1}e^{-\frac{\lambda_1}{2}(|t|+|\omega|)}. \tag{22}$$

This together with equations (17) and (18) yields (16).

Now consider

$$\begin{aligned} \|\gamma - \gamma^{(n)}\| &= \|H^+\sigma - H_n^+ P_n\sigma\| \\ &\le \|H^+\sigma - H^+H_nH_n^+P_n\sigma\| + \|H^+H_nH_n^+P_n\sigma - H^+HH_n^+P_n\sigma\| \\ &\le \underbrace{\|H^+(\sigma - P_n\sigma)\|}_{0} + \|H^+\|\|(H_nH_n^* - HH_n^*)(H_nH_n^*)^{-1}P_n\sigma\| \\ &\le A^{-1}\|(H_nH_n^* - HH_n^*)(H_nH_n^*)^{-1}P_n\sigma\|. \end{aligned}$$

Note that $(H_nH_n^* - HH_n^*)$ is a matrix that has a finite number of columns and a biinfinite number of rows, the rows of which with indices $|k| \le n, |l| \le n$ are zero.

By (16) the entries of $H_nH_n^*$ and the nonzero rows of $(H_nH_n^* - HH_n^*)$ decay exponentially off the diagonal. Relation (13) implies

$$\operatorname{cond}(H_nH_n^*) \le \operatorname{cond}(HH^*) \quad \text{for } n = 0, 1, \dots. \tag{23}$$

Using (23) and Proposition 2 in [28] it follows that there exists a λ_2 and a constant C_2 depending on λ_2 and on the condition number of HH^*, such that

$$\left|[(H_nH_n^*)^{-1}]_{k,l,0,0}\right| \le C_2 e^{-\lambda_2(|k|+|l|)} \tag{24}$$

and the same is true for $v^{(n)} := (H_nH_n^* - HH_n^*)[(H_nH_n^*)^{-1}]_{k,l,0,0}$. Now it is easy to see that there exist λ' and C' such that

$$\|v^{(n)}\| = \sum_{|k|,|l|>n} |v_{kl}^{(n)}|^2 \le C' e^{-\lambda' n}. \tag{25}$$

❑

Remark: The proof consists essentially of two main steps. In the first step we have shown that the STFT of g (and thus in turn the matrix $H_nH_n^*$) inherits the decay behavior of g. In the second step we have used the fact that the decay behavior of $H_nH_n^*$ is preserved under inversion. Other examples in which the decay behavior is preserved under matrix inversion (including polynomial decay) can be found in [28]. Results on the connection between the decay of a function and its short time Fourier transform can be found e.g. in [25]. With these results at hand it becomes conceivable that Theorem 2.4 can be extended to other types of decay.

2.1 Comparison to other approaches for approximating the dual window

To fully appreciate the simplicity and advantages of the proposed approach we compare it to other approaches proposed for the computation of dual frames, in particular for dual Gabor frames.

2.1.1 Finite section method for frames using the Gram matrix

The following method has been frequently proposed for the numerical computation of dual frames, see e.g. [2, 16, 43]. We describe it in the specific context of Gabor frames, but the mathematical difficulties that arise are similar for general frames. In this and the following subsection we assume that the frame is overcomplete, i.e., it is not a Riesz basis.

The definition of the operators T, T^* in (6) and (7) imply that we can identify TT^* with its Gram matrix representation (with respect to the basis $e_{k,l} = (\delta_{k,k'}\delta_{l,l'})$), hence $(TT^*)_{m,n,m',n'} = \langle g_{n'a,m'b}, g_{na,mb}\rangle$ for $m, n, m', n' \in \mathbb{Z}$. The dual window γ can now be written as

$$\gamma = \sum_{m,n\in\mathbb{Z}} c_{n,m} g_{na,mb}, \qquad \text{with } (TT^*)^+ d = c, \tag{26}$$

where $d = \{d_{n,m}\} = \{\langle g, g_{na,mb}\rangle\}$.

Setting $T_n T_n^* = P_n TT^* P_n$ and $d^{(n)} = P_n d$, we obtain the n-th approximation $c^{(n)}$ to c by solving

$$T_n T_n^* c^{(n)} = d^{(n)} \tag{27}$$

and the n-th approximation to γ by computing

$$\gamma^{(n)} = \sum_{m,n=-N}^{N} c^{(n)}_{m,n} g_{na,mb}. \tag{28}$$

The problem of this approach is that the generalized inverse of $T_n T_n^*$ is not bounded for $n \to \infty$, although $\|(TT^*)^+\| \le A^{-1}$. In fact, $\|(T_n T_n^*)^+\| \to \infty$ for $n \to \infty$. Hence $c^{(n)}$ does not converge to c, therefore $\gamma^{(n)}$ does not converge to γ. This is true even for tight frames. Although in this case the infinite-dimensional problem is perfectly conditioned (the frame operator is simply a multiple of the identity operator), the condition number of the associated truncated Gram matrix goes to infinity for $n \to \infty$, in some cases even exponentially fast, see [41]. Thus this approach turns a (very) well-posed problem into a severely ill-posed one.

If we nevertheless want to use this method an obvious idea would be to compute instead of $(T_n T_n^*)^+$ a regularized inverse via a truncated singular value decomposition by setting the singular values of $T_n T_n^*$ below a certain threshold τ_n to zero. Let $(T_n T_n^*)^{\tau_n,+}$ denote this regularized inverse. It is shown in [27] in the context of irregular sampling that $(T_n T_n^*)^{\tau_n,+}$ converges strongly to $(TT^*)^+$ if we allow the threshold parameter τ_n to vary for each n.

In order to use this approach for practical purposes we need good estimates for the optimal threshold τ_n. Assuming a numerical precision of δ of the data and setting $B_n = \|T_n T_n^*\|$, it is shown in [39] that τ_n can be estimated by

$$\tilde{\tau}_n \le B_n \left(\frac{\delta}{p}\right)^{\frac{1}{p+1}}.$$

Here – without going into details about regularization theory – p can be seen as "smoothness parameter" [15]; the standard setting for p in regularization theory is $p = 1$ or $p = 2$. Thus for large n we get

$$\tilde{\tau} \in [\min\{B(\delta/2)^{\frac{1}{3}}, B\delta^{\frac{1}{2}}\}, \max\{B(\delta/2)^{\frac{1}{3}}, B\delta^{\frac{1}{2}}\}],$$

where B is the upper frame bound. Good estimates for the upper frame bound are important to apply this method in practice. For certain windows and certain oversampling rates there exist explicit estimates for the frame bounds, see [11, 31]. In general however frame bound estimation can be a computationally expensive task.

2.1.2 Approximation by projections of finite subsets of a frame

A different finite section method has been proposed in [6]. The method of Christensen and Casazza applies for general frames, but again we discuss it in the specific context of Gabor analysis. Let $\boldsymbol{H}_N$ denote the space spanned by $\{g_{na,mb}\}_{m,n=-N}^{N}$ and let $S_N : \boldsymbol{H}_N \to \boldsymbol{H}_N$, $S_N f = \sum_{m,n=-N}^{N} \langle f, g_{na,mb}\rangle g_{na,mb}$ be the associated frame operator. In light of the discussion in Subsection 2.1.1 it is not surprising that S_N^{-1} does not even converge weakly to S^{-1} for $n \to \infty$ (cf. e.g. [6])). The authors propose the following modified finite section method.

Let Q_N denote the orthogonal projection of $\boldsymbol{L}_2(\mathbb{R})$ onto $\boldsymbol{H}_N$, given by

$$Q_N g_{na,mb} = \sum_{m,n=-N}^{N} \langle g_{na,mb}, S_N^{-1} g_{n'a,m'b}\rangle g_{n'a,m'b}.$$

Now select a properly chosen $N_0 > N$ and consider the family $\{Q_N g_{na,mb}\}_{m,n=-N_0}^{N_0}$ which is a frame for $\boldsymbol{H}_N$ with frame operator V_N. It is shown in [6] that the index N_0 can be chosen such that $V_N^{-1} Q_N$ converges strongly to S^{-1} for all

$M > N_0$, in particular $V_N^{-1}Q_N g \to S^{-1}g$. Hence we can approximate the dual window using this finite section method.

There are several obstacles when using this approach in practice. Firstly we have to determine the index N_0. The recipe for the estimation of N_0 given in [8] involves the lower frame bound A_N of the finite frame $\{g_{na,mb}\}_{m,n=-N}^{N}$. However since $A_N^{-1} = \|(T_N T_N^*)^+\|$, where $T_N T_N^*$ is as defined in Subsection 2.1.1, we get $A_N \to 0$ for $n \to \infty$. Thus these estimates require N_0 to be very large.

Assuming we have found a reasonable index N_0 we now have to compute the projections $Q_N g_{na,mb}$, which involves an inversion of S_N, before we can invert V_N in order to compute an approximation to γ.

The cumbersome calculations and estimations above and in Subsection 2.1.1 can be avoided when using the method proposed in Theorem 2.2. Moreover note that the condition number of the matrix to be inverted in Theorem 2.2 is bounded by the condition number of the infinite-dimensional frame operator S, see (13). Thus the stability of the infinite-dimensional inversion problem is preserved in the finite-dimensional model.

3 Laurent operators and decay of dual Gabor frames

A natural question for Gabor frames – also in spite of Theorem 2.2 is the following. Given a window g with certain decay properties in time and/or frequency, does its dual γ have the same decay properties? This question is not only of interest from a theoretical viewpoint, but has a number of practical implications, e.g., see Section 4.

We briefly summarize a few important results. We assume in the sequel that $\{g_{na,mb}\}$ constitutes a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$ or $\boldsymbol{\ell}_2(\mathbb{Z})$.
(i) If $g \in \mathcal{C}_{[\alpha,\beta]}$, then $\gamma \in \mathcal{C}_{[\alpha_1,\beta_1]}$ only in very special cases [4]. In general γ is no longer compactly supported, but has exponential decay (see [42] for a proof in $\boldsymbol{\ell}_2(\mathbb{Z})$, the result can be easily extended to $\boldsymbol{L}_2(\mathbb{R})$). Hence in this case g and γ do not belong to the same type of space.
(ii) If $g \in \mathcal{E}_\lambda$ then $\gamma \in \mathcal{E}_{\lambda_1}$, but in general with a different exponent $\gamma_1 < \gamma$ [42, 5]. Thus g and γ have the same type of decay, but do not belong to the same space, so we lose some quality of decay.[2]

[2]Note that there do exist Gabor frames with Gaussian decay, that have *non-canonical*

(iii) If $g \in \mathcal{Q}_s$, then $\gamma \in \mathcal{Q}_s$ (see [42] for a proof for $\boldsymbol{\ell}_2(\mathbb{Z})$, the result can be easily generalized to $\boldsymbol{L}_2(\mathbb{R})$ using the same approach as in [5]). In this case g and γ actually belong to the same space $\mathcal{Q}_s$.

From an algebraic point of view case (iii) is the most appealing one. Can we find a (simple) condition on the decay of g that implies that the dual γ belongs to the same space, similar to case (iii)? In this section we will give an exhaustive answer to this question.

In [29] Janssen has shown that if (g, a, b) is a frame and $g \in \boldsymbol{\mathcal{S}}(\mathbb{R})$ then $\gamma \in \boldsymbol{\mathcal{S}}(\mathbb{R})$ and $\psi \in \boldsymbol{\mathcal{S}}(\mathbb{R})$. To prove this result he chose an approach based on Fourier analysis (a "down-to-earth" approach in his words) and pointed out that one might obtain his results in a more economical way by using Gelfand's theory of normed rings. In fact, we will make heavily use of Gelfand's framework in order to show that the condition $g \in \boldsymbol{\mathcal{S}}$ can be extended to rather general decay conditions on g. Note however that in contrast to Janssen's result, our derivations "only" hold for rational oversampling.

We need some preparation before we proceed.

Definition 3.1 *Let $w(t)$ be a continuous weight function on $\mathbb{R}$. The (weighted) Wiener Algebra $\mathcal{A}_w$ is the Banach space of absolutely convergent Fourier series of period 1 (cf. [35]), i.e., $f \in \mathcal{A}_w$ if*

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k t} \tag{29}$$

with

$$\sum_{k=-\infty}^{\infty} |a_k| w(k) < \infty. \tag{30}$$

The norm on $\mathcal{A}_w$ is $\|f\|_{\mathcal{A}_w} = \sum_{k=-\infty}^{\infty} |a_k| w(k)$.

It follows from Chapter 19.4 of [22] that $\mathcal{A}_w$ is a Banach algebra under pointwise multiplication. $\mathcal{A}_w$ can be identified with the space of all sequences $a = \{a_k\}$ which are in $\boldsymbol{\ell}_{1,w}$.

Definition 3.1 can be extended to matrix-valued functions in a straightforward manner. Put

$$\Phi(e^{2\pi i t}) = \begin{bmatrix} f_{11}(e^{2\pi i t}) & \dots & f_{1m}(e^{2\pi i t}) \\ \vdots & & \vdots \\ f_{m1}(e^{2\pi i t}) & \dots & f_{mm}(e^{2\pi i t}) \end{bmatrix} \tag{31}$$

duals with Gaussian decay; see Example 3.10 in [5].

and set

$$a_k = \int_0^1 \Phi(e^{2\pi it})e^{-2\pi ikt}dt. \tag{32}$$

Let the weight function $w(t)$ act on $\mathbb{R}^m$. Then Φ belongs to the matrix Wiener algebra $\mathcal{A}_w^m$ if

$$\sum_{k=-\infty}^{\infty} \|a_k\|_2 w(k) < \infty, \tag{33}$$

where $\|a_k\|_2$ denotes the 2-norm of the matrix a_k.

Φ is unitarily equivalent to the block Laurent operator L whose matrix representation (with respect to the standard basis) is given by

$$\begin{bmatrix} \ddots & & & & \\ & a_0 & a_{-1} & a_{-2} & \\ & a_1 & \boxed{a_0} & a_{-1} & \\ & a_2 & a_1 & a_0 & \\ & & & & \ddots \end{bmatrix}.$$

Here $\boxed{a_0}$ denotes the $(0,0)$ entry which acts on the 0-th coordinate space. Φ is also called the defining function of the block Laurent operator L. By a slight abuse of notation we also write $L = [A_{k,l}]_{k,l=-\infty}^{\infty}$ where $A_{k,l} = a_{k-l}$.

We define $\mathcal{W}_w^m$ as the space consisting of all block Laurent operators whose $m \times m$ blocks a_k satisfy (33). If $m = 1$ we simply write $\mathcal{W}_w$ and L reduces to a scalar-valued Laurent operator.

Theorem 3.2 *Let $L \in \mathcal{W}_w$ be self-adjoint and positive definite. Assume that the weight function w satisfies*

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{w(-n)}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sqrt[n]{w(n)} = 1, \tag{34}$$

then $L^{-1} \in \mathcal{W}_w$ and $L^{-\frac{1}{2}} \in \mathcal{W}_w$.

Proof: Set $L = [A_{k,l}]_{k,l=-\infty}^{\infty}$ and $L^{-1} = [B_{k,l}]_{k,l=-\infty}^{\infty}$. Since L is positive definite we have

$$f(\omega) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ik\omega} > 0 \tag{35}$$

and by the properties of Laurent operators [23]

$$\frac{1}{f(\omega)} = \sum_{k=-\infty}^{\infty} B_{k,0} e^{2\pi i k\omega}.$$

The property $L \in \mathcal{W}_w$ is equivalent to $f \in \mathcal{A}_w$. By Theorem 2 on page 24 in [22] an element of $\mathcal{A}_w$ has an inverse in $\mathcal{A}_w$ if it is not contained in a maximal ideal. Any maximal ideal of $\mathcal{A}_w$ consists of elements of the form (cf. Chapter 19.4 in [22])

$$\sum_{k=-\infty}^{\infty} c_k \xi^k = 0,$$

where $\xi = \rho e^{2\pi i\omega}$ with $\rho_1 \le \rho \le \rho_2$ and

$$\rho_1 = \lim_{n\to\infty} \frac{1}{\sqrt[n]{w_{-n}}} \quad \text{and} \quad \rho_2 = \lim_{n\to\infty} \sqrt[n]{w_n}.$$

Due to assumption (34) we get $\rho_1 = \rho_2 = 1$, hence $\rho = 1$. Thus a necessary and sufficient condition for an element in $\mathcal{A}_w$ to be not contained in a maximal ideal is $\sum_k c_k e^{2\pi i k\omega} \neq 0$ for all ω. By assumption L is positive definite, hence $f(\omega) = \sum_k a_k e^{2\pi i k\omega} > 0$ for all ω, consequently $L^{-1} \in \mathcal{W}_w$.

To show that $L^{-\frac{1}{2}} \in \mathcal{W}_w$ recall that L is self-adjoint and positive definite. This implies that $L^{\frac{1}{2}} \in \mathcal{W}_w$, see [21]. Thus it follows from the first part of this proof that $L^{-\frac{1}{2}} \in \mathcal{W}_w$. ❏

The proof of Theorem 3.2 is essentially based on results of Gelfand, Raikov, and Shilov [22]. A crucial role is played by condition (34), to which we will henceforth refer as the *Gelfand-Raikov-Shilov condition* (GRS-condition for short) .

Now let L be a block Laurent operator with defining function $\Phi = [f_{ij}]_{i,j=1}^m \in \mathcal{A}_w^m$. If L is hermitian, positive definite then

$$\det([f_{ij}(\lambda)]_{i,j=1}^m) > 0, \qquad |\lambda| = 1.$$

Hence we can apply Theorem 8.1 on page 830 in [23] to extend Theorem 3.2 to block Laurent operators and obtain the following

Corollary 3.3 *Let $L \in \mathcal{W}_w^m$ be self-adjoint and positive definite. Assume that the weight function w defined on $\mathbb{R}^m$ satisfies the GRS-condition, i.e.,*

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{w(-n)}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sqrt[n]{w(n)} = 1, \tag{38}$$

then $L^{-1} \in \mathcal{W}_w^m$ *and* $L^{-\frac{1}{2}} \in \mathcal{W}_w^m$.

It is clear that in the derivations above we can replace 1-periodic functions by α-periodic functions ($\alpha \in \mathbb{Q}$). Similarly, sequences with indices in $\mathbb{Z}$ can be replaced by sequences indexed by $\mathbb{Z}_\alpha$ (i.e. by indices of the form $k\alpha$).

Now we are ready to prove a general result on the decay properties of Gabor windows g and their duals γ and tight windows ψ.

Theorem 3.4 *(a) Let* (g, a, b) *generate a Gabor frame for* $\boldsymbol{L}_2(\mathbb{R})$ *with* $ab = \frac{p}{q} \in \mathbb{Q}$ *with relative prime integers* p *and* q*. Let* $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ *where* w *satisfies the GRS-condition. Then* $\gamma \in \boldsymbol{L}_{1,w}(\mathbb{R})$ *and* $\psi \in \boldsymbol{L}_{1,w}(\mathbb{R})$.

Proof: The assumption $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ implies $\sum_{k=-\infty}^{\infty} |g(k)|w(k) < \infty$ and $\sum_{k=-\infty}^{\infty} |g(k/b)|w(k/b) < \infty$ for $b \in \mathbb{N}$. Denote

$$G_{kl}(t) = \frac{1}{b} \sum_{r=-\infty}^{\infty} g(t - ra - \frac{k}{b}) g^*(t - ra - \frac{l}{b}), \qquad k, l \in \mathbb{Z}. \tag{39}$$

$G(t)$ is periodic in t with period a. Since $ab = \frac{p}{q}$ we also get $G_{kl}(t) = G_{k+p,l+p}(t)$, cf. [17], Theorem 3.2. In words, $G(t)$ is a block Laurent operator. The frame property implies that $G(t)$ is hermitian positive definite and

$$AI \le G(t) \le BI.$$

The submultiplicativity of the function w implies that the spaces $\boldsymbol{L}_{1,w}(\mathbb{R})$, $\boldsymbol{\ell}_{1,w}(\mathbb{Z})$, and $\boldsymbol{\ell}_{1,w}(\mathbb{Z}_{\frac{1}{b}})$ are Banach algebras under convolution (cf. [22]). Hence $\{G_{0k}(t)\}_{k \in \mathbb{Z}} \in \boldsymbol{\ell}_{1,w}(\mathbb{Z}^m_{\frac{1}{b}})$, and $G(t) \in \mathcal{W}_w^m$. By Corollary 3.3

$$G^{-1}(t) \in \mathcal{W}_w^m \quad \text{and} \quad G^{-\frac{1}{2}}(t) \in \mathcal{W}_w^m. \tag{40}$$

The dual γ can be expressed as (see e.g. [5])

$$\gamma(t) = \sum_{k=-\infty}^{\infty} [G^{-1}(t)]_{0,k}\, g(t - \frac{k}{b}) \tag{41}$$

and the tight window h satisfies

$$\psi(t) = \sum_{k=-\infty}^{\infty} [G^{-\frac{1}{2}}(t)]_{0,k}\, g(t - \frac{k}{b}). \tag{42}$$

Equations (41) and (42) together with (40) and the assumption $g \in \boldsymbol{L}_{1,w}(\mathbb{R})$ yield that $\gamma, \psi \in \boldsymbol{L}_{1,w}(\mathbb{R})$. ❏

Remark: (i) In [30] Janssen proved that a g generating a rationally oversampled frame and satisfying the Tolimieri-Orr condition A has a dual γ satisfying condition A as well. This is closely related but not identical to a non-weighted version of Theorem 3.4.
(ii) The idea to exploit the Laurent operator property of the Gabor frame operator in the context of window decay has been used by Feichtinger and Gröchenig [17] in connection with dual windows in modulation spaces with polynomial weights, see also [24]. Theorem 3.4 provides a way to extend their results to more general weights as well as to tight windows.
(iii) It is easy to reformulate Theorem 3.4 for windows whose decay properties are given in the frequency domain. We leave this modification to the reader.
(iv) It is well-known that Wilson bases can be constructed from twofold oversampled tight Gabor frames [13]. In light of this fact Theorem 3.4 provides an extension of existing results on the connection between the decay behavior of Gabor frames and Wilson bases. Furthermore due to the connection between Gabor frames and modulated filter banks [10], Theorem 3.4 also yields a characterization of the decay properties of paraunitary modulated filter banks.
(v) Theorem 3.4 can be extended to higher dimensions, as well as to the general setting of locally compact abelian groups (a special case of which is the setting $\ell_2(\mathbb{Z})$). We do not pursue this generalization here.

4 Wireless communications and time-frequency localization

Orthogonal frequency division multiplexing (OFDM) has attracted a great deal of attention as an efficient technology for wireless data transmission [47]. Among others it is currently used in the European digital audio broadcasting standard [7]. In a wireline environment OFDM is known under the name *discrete multitone transmultiplexing* (DMT).

The basic idea of OFDM is to divide the available spectrum into several subchannels (subcarriers). By making all subchannels narrowband, they experience almost flat fading, which makes equalization very simple [34]. A baseband OFDM system is schematically represented in Figure 1. Assuming N subcarriers, a bandwidth of W Hz, *symbol length* of T seconds, and *carrier*

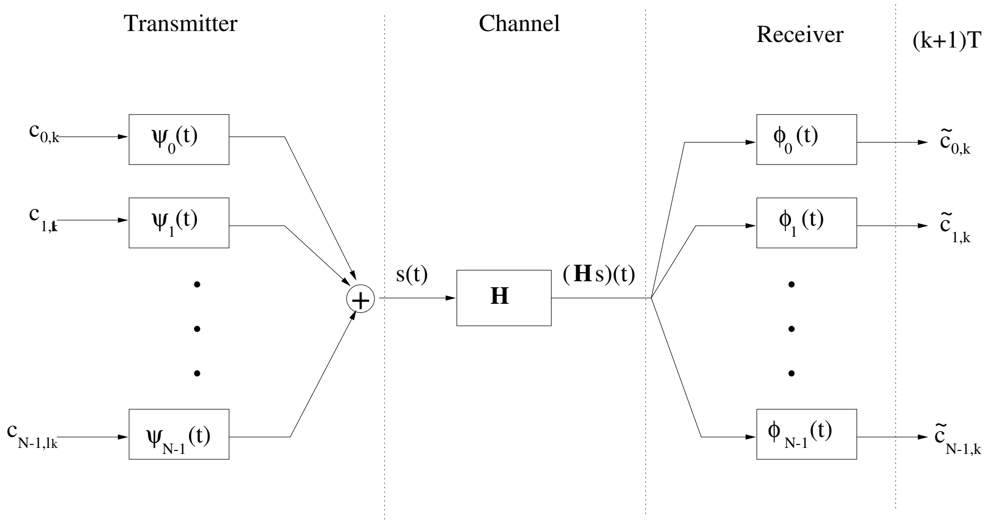

Figure 1: A general baseband OFDM system configuration

separation $F := W/N$, the transmitter of a general OFDM system uses the following waveforms

$$\psi_l(t) = \psi(t)e^{2\pi itlF}, \quad k \in \mathbb{Z}, l = 0, \dots, N-1. \tag{43}$$

Assuming an infinite sequence of data symbols $\{c_{k0}, c_{k1}, \dots, c_{k,N-1}\}_{k\in\mathbb{Z}}$ is transmitted, the output from the transmitter is

$$s(t) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l}\psi_l(t-kT). \tag{44}$$

The OFDM receiver consists of a matched filter bank $\{\phi_l\}$ of similar structure as the transmitter waveforms, i.e.,

$$\phi_{kl}(t) = \phi(t-kT)e^{2\pi itlF}, \quad k \in \mathbb{Z}, l = 0, \dots, N-1. \tag{45}$$

The transmitted data are recovered via $\tilde{c}_{kl} = \langle r, \phi_{kl}\rangle$, where $r = \mathbf{H}s + \nu$ is the received signal. Here $\mathbf{H}$ represents the time-varying channel and ν is additive white Gaussian noise (AWGN for short). In the standard OFDM setup the functions ψ_{kl} are designed to be mutually orthogonal. In this case $\phi \equiv \psi$.

An obvious requirement for an OFDM system with perfect reconstruction is that the $\psi_{k,l}$ have to be linear independent (but not necessarily complete in $\boldsymbol{L}_2(\mathbb{R})$), which implies $TF \geq 1$. This is dual to the frame setting and filter bank setting, where (over)completeness (but necessarily linear independence) is a requirement for perfect reconstruction. Hence the functions $\psi_{k,l}$ are of similar type as the functions of a Gabor frame, expect that an OFDM system corresponds to an undersampled Gabor system.

Wireless channels introduce time dispersion as well as frequency dispersion (in addition to the usual channel noise). The time dispersion is caused by multipath propagation and can lead to intersymbol interference (ISI). Frequency dispersion of the mobile radio channel is due to the Doppler effect and can cause interchannel interference (ICI). Robustness against doubly dispersive channels can be achieved by pulse shapes with good time-frequency localization [26, 33, 3, 20]. An optimum OFDM system in case of doubly dispersive channels would consist of orthogonal basis functions with $TF = 1$, such that the $\psi_{k,l}$ are well localized in time and frequency. The condition $TF = 1$ ensures maximal spectral efficiency of the transmission system. Unfortunately, such a system cannot exist due to the Balian-Low theorem [12]. Thus in order to mitigate interference we give up to condition of optimal spectral efficiency and consider the construction of OFDM systems for $TF > 1$.

As pointed out in [3] the duality conditions of Theorem 2.1 provide a nice connection between OFDM systems and Gabor frames. For if (g, a, b) is a Gabor frame, then (ψ, T, F) is an OFDM system, where

$$\psi := S^{-\frac{1}{2}} g, \tag{46}$$

and $T = 1/b, F = 1/a$. The orthogonality of the set $\psi_{k,l}$ follows from the fact that $(S^{-\frac{1}{2}} g, a, b)$ generates a tight frame.

Hence it seems natural to choose a function g with good time-frequency localization such that (g, a, b) generates a Gabor frame for $\boldsymbol{L}_2(\mathbb{R})$, apply the orthogonalization procedure (46) and use the resulting pulse shape $\psi = S^{-\frac{1}{2}} g$ as OFDM pulse shape, with the hope that ψ inherits the localization properties from g. This approach has been considered in [3, 33] and implicitly also in [20], however with very different conclusions.

It is stated in [33] that "*such an orthogonalization of the pulses, however, is not desirable ... because the good time-frequency localization of the pulses is destroyed by such a transformation*". In contrast in [3] it is claimed that the orthogonalization procedure (46) "*in practice ... starting from a well-localized* *initial filter ... yields well-localized orthogonal filters*". But no theoretical explanation has been given in these papers to support these claims.

So, who is right and who is wrong? The answer is:

Both and nobody – it depends!

The first step toward this answer is Theorem 3.4, which gives a general condition on the decay of g such that the orthogonalization procedure (46) yields a pulse shape ψ that is in the same space (in the same *algebra*) as g, thus the quality of decay does not change. Observe that condition (34) in Theorem 3.4 is satisfied e.g. for $w(x) = (1 + |x|)^s$ and $w(x) = e^{\lambda|x|^\gamma}, \gamma < 1$, but not for $w(x) = e^{\lambda|x|}$. For a window g with exponential decay (in time and frequency) it has been shown that ψ also has exponential decay (see [42] for a proof for $\boldsymbol{\ell}_2(\mathbb{Z})$, this result has later been extended to $\boldsymbol{L}_2(\mathbb{R})$, cf. [5]), however with a smaller exponent, thus g and ψ are in general not in the same space $\mathcal{E}_\lambda$.

Since the Gaussian is optimally localized in the time-frequency plane, in the sense that it minimizes the uncertainty principle, it is interesting to note that for $g(t) = e^{-\pi t^2}$ the function ψ has only exponential decay in time and frequency, cf. [5]. Thus we certainly lose some time-frequency localization in this case, although the resulting function ψ still has exponential decay.

Thus it seems there exists a large class of windows whose time-frequency localization properties are not affected by applying (46). This is however only half of the truth, since in the considerations above we have ignored any constants that come into play. This is acceptable from an asymptotic-analysis viewpoint, but not for applications.

Consider for example the following situation. Take a window g with exponential decay in time and frequency and assume that $(g, 1/F, 1/T)$ constitutes a frame for $TF > 1$. Then for any $\varepsilon > 0$ there exists an N such that for $T > N, F > N$

$$\left\| \frac{g}{\|g\|} - \frac{\psi}{\|\psi\|} \right\| \leq \varepsilon$$

since $(TF)\gamma \to g$ for $T \to \infty, F \to \infty$ (see [19] for a mathematically precise formulation). But of course $TF \gg 1$ leads to an unacceptable large loss of spectral efficiency for OFDM, already $TF > 2$ seems to be prohibitive in this context.

On the other hand, if we let $TF \to 1$, then ψ and γ will become increasingly "ill-localized" in the time-frequency plane (since in the limit case

$TF = 1$ we are confronted with the Balian-Low theorem), although from theory we know that ψ has exponential decay as long as $TF > 1$. However, as pointed out earlier, this decay involves a constant C that depends on the ratio of the frame bounds. Since $A \to 0$ for $TF \to 1$ it follows that $C \to \infty$. Thus for applications such as OFDM the exponential-decay property of γ and ψ quickly becomes meaningless for TF close to 1. If we choose a well-localized window, there is a trade-off between increasing TF and increasing the frame bound ratio B/A.

Hence a correct formulation from a practical viewpoint of the two contradicting statements above is:

> *If g is well localized in time and frequency and if the frame bounds satisfy $B/A \approx 1$, then ψ is also well localized in time and frequency.*

Numerical experiments indicate that for $TF = 1.3$ or $TF = 1.4$ it is possible to construct OFDM basis functions with good time-frequency localization, e.g. choose the Gaussian as initial window g. A definite answer if the time-frequency properties of the resulting OFDM basis functions obtained in that way are sufficient for practical purposes is difficult, since it depends on the actual ISI and ICI, as well as the AWGN behavior of the channel.

References

[1] S.T. Ali, J.P. Antoine, and J.P. Gazeau. *Coherent states, wavelets and their generalizations*. Springer-Verlag, New York, 2000.

[2] J. Benedetto and W. Heller. Irregular sampling and the theory of frames, I. *Mat. Notes*, 10 (Suppl.1):103–125, 1990.

[3] H. Bölcskei. Efficient design of pulse shaping filters for OFDM systems. In *SPIE Proc., "Wavelet Applications in Signal and Image Processing VII*, volume 3813, pages 625–636, Denver, 1999.

[4] H. Bölcskei. A necessary and sufficient condition for dual Weyl-Heisenberg frames to be compactly supported. *J. Fourier Anal. Appl.*, 5(5):409–419, 1999.

[5] H. Bölcskei and A.J.E.M. Janssen. Gabor frames, unimodularity, and window decay. *J. Four. Anal. Appl.*, 6(3):255–276, 2000.

[6] P.G. Casazza and O. Christensen. Approximation of the inverse frame operator and applications to Gabor frames. *J. Approx. Theory*, 103(2):338–356, 2000.

[7] D. Castelain, B. Le Floch, and R. Halbert-Lassalle. Digital sound broadcasting to mobile receivers. *IEEE Trans. Consumer Electron.*, 73:30–34, 1989.

[8] O. Christensen. Finite-dimensional approximation of the inverse frame operator. *J. Four. Anal. Appl.*, 6(1):79–91, 2000.

[9] R.E. Crochiere and L.R. Rabiner. *Multirate Digital Signal Processing.* Prentice–Hall, New Jersey, 1983.

[10] Z. Cvetkovic and M. Vetterli. Tight Weyl-Heisenberg frames in $\boldsymbol{\ell}_2(\mathbb{Z})$. *IEEE Trans. Signal Proc.*, 46(5):1256–1259, 1998.

[11] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Info. Theory*, 36:961–1005, 1990.

[12] I. Daubechies. *Ten Lectures on Wavelets.* CBMS-NSF Reg. Conf. Series in Applied Math. SIAM, 1992.

[13] I. Daubechies, S. Jaffard, and J.L. Journé. A simple Wilson orthonormal basis with exponential decay. *SIAM J. Math. Anal.*, 22(2):554–572, 1991.

[14] I. Daubechies, H. Landau, and Z. Landau. Gabor time-frequency lattices and the Wexler-Raz identity. *J. Four. Anal. Appl.*, 1(4):437–478, 1995.

[15] H.W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems.* Kluwer Academic Publishers Group, Dordrecht, 1996.

[16] H.G. Feichtinger and K. Gröchenig. Non-orthogonal wavelet and Gabor expansions, and group representations. In G. Beylkin, R. Coifman, I. Daubechies, and et al., editors, *Wavelets and their applications*, pages 353–376. Jones and Bartlett, Boston, 1992.

[17] H.G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *J. Funct. Anal.*, 146(2):464–495, 1996.

[18] H.G. Feichtinger and T. Strohmer, editors. *Gabor Analysis and Algorithms: Theory and Applications.* Birkhäuser, Boston, 1998.

[19] H.G. Feichtinger and G. Zimmermann. A space of test functions for Gabor analysis. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, chapter 3, pages 123–170. Birkhäuser, Boston, 1998.

[20] B. Le Floch, M Alard, and C. Berrou. Coded orthogonal frequency division multiplex. *Proc. of IEEE*, 83(6):982–996, June 1995.

[21] L.T. Gardner. Square roots in Banach algebras. *Proc. Amer. Math. Soc.*, 17:132–134, 1966.

[22] I. Gelfand, D. Raikov, and G. Shilov. *Commutative normed rings.* Chelsea Publishing Co., New York, 1964. Translated from the Russian.

[23] I. Gohberg, S. Goldberg, and M. A. Kaashoek. *Classes of linear operators. Vol. II*, volume 63 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1993.

[24] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Birkhäuser, Boston, 2000.

[25] K. Gröchenig and G. Zimmermann. Hardy's theorem and the time-frequency analysis of test functions and ultra-distributions. *J. London Math. Soc.*, to appear.

[26] R. Haas and J.C. Belfiore. A time-frequency well-localized pulse for multiple carrier transmission. *Wireless Personal Communications*, 5:1–18, 1997.

[27] M.L. Harrison. *Frames and irregular sampling from a computational perspective.* PhD thesis, University of Maryland – College Park, 1998.

[28] S. Jaffard. Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[29] A.J.E.M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *J. Four. Anal. Appl.*, 1(4):403–436, 1995.

[30] A.J.E.M. Janssen. On rationally oversampled Weyl-Heisenberg frames. *Signal Proc.*, 47:239–245, 1995.

[31] A.J.E.M. Janssen. Some Weyl-Heisenberg frame bound calculations. *Indag. Mathem.*, 7(2):165–182, 1996.

[32] A.J.E.M. Janssen. From continuous to discrete Weyl-Heisenberg frames through sampling. *J. Fourier Anal. Appl.*, 3(5):583–596, 1997.

[33] W. Kozek and A. Molisch. Nonorthogonal pulseshapes for multicarrier communications in doubly dispersive channels. *IEEE J. Sel. Areas Comm.*, 16(8):1579–1589, 1998.

[34] T. S. Rappaport. *Wireless communications: Principles & Practice*. Prentice Hall, New Jersey, 1996.

[35] H. Reiter. *Classical Harmonic Analysis and Locally Compact Abelian Groups*. Oxford University Press, 1968.

[36] R.D. Richtmyer and K.W. Morton. *Difference Methods for Initial-Value Problems*. Krieger Publishing Company, Malabar, Florida, 1994.

[37] M.A. Rieffel. Projective modules over higher-dimensional noncommutative tori. *Canad. J. Math.*, 40(2):257–338, 1988.

[38] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbf{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.

[39] T. Strohmer. Painless approximation of dual frames, with applications to shift-invariant systems. In *SPIE Proc., "Wavelets Applications in Signal and Image Processing VII"*, volume 3813, pages 637–647, Denver, 1999.

[40] T. Strohmer. Finite and infinite-dimensional models for oversampled filter banks. In J.J. Benedetto and P.J.S.G Ferreira, editors, *Modern Sampling Theory: Mathematics and Applications*, pages 297–320. Birkhäuser, Boston, 2000.

[41] T. Strohmer. Numerical analysis of the non-uniform sampling problem. *J. Comp. Appl. Math.*, 122(1–2):297–316, 2000.

[42] T. Strohmer. Rates of convergence for the approximation of dual shift-invariant systems in $\ell_2(\mathbb{Z})$. *J. Four. Anal. Appl.*, 5(6):599–615, 2000.

[43] A. Teolis and J.J. Benedetto. Local frames and noise reduction. *Signal Process.*, 45:369–387, 1995.

[44] D.F. Walnut. Continuity properties of the Gabor frame operator. *J. Math. Anal. Appl.*, 165(2):479–504, 1992.

[45] J. Wexler and S. Raz. Discrete Gabor expansions. *Signal Processing*, 21(3):207–221, November 1990.

[46] R. Young. *An Introduction to Nonharmonic Fourier Series.* Academic Press, New York, 1980.

[47] W. Y. Zou and Y. Wu. COFDM: An overview. *IEEE Trans. Broadc.*, 41(1):1–8, March 1995.