OPTIMAL OFDM PULSE AND LATTICE DESIGN FOR DOUBLY DISPERSIVE CHANNELS

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ABSTRACT

We introduce a general framework for OFDM pulse shape design and demonstrate how to obtain a pulse optimally matched to given channel properties. Furthermore, standard OFDM systems are associated with a rectangular grid in the time-frequency plane, which is not optimal for pulse-shaping OFDM systems for doubly dispersive channels. We introduce Lattice OFDM systems which are OFDM systems using general lattices in the time-frequency plane.

1. INTRODUCTION

1.1. OFDM Structure

Orthogonal frequency division multiplexing (OFDM) is becoming increasingly popular for wireless applications and standards [1]. One of the assets of OFDM is its robustness against the vagaries of wireless channels such as multipath fading, while maintaining high spectral efficiency. In this paper we propose a strategy to optimize OFDM system design with respect to intersymbol interference (ISI) and interchannel intereference (ICI).

The principal idea of OFDM is to divide the available spectrum into several subchannels (subcarriers) in order to combat the frequency selectivity of the channel [1]. An OFDM system with N subcarriers, symbol length T, bandwidth W Hz, carrier separation $F := W/N$, and pulse shape ψ, uses the waveforms

$$\psi_{k,l}(t) = \psi(t-kT)e^{j2\pi lFt};\ k\in\mathbb{Z}, l = 0,\dots,N-1, \quad (1)$$

for data transmission, where the $\psi_{k,l}$ are mutually orthogonal. For our theoretical analysis we assume an infinite number of subcarriers, although most results carry over verbatim to the finite case.

Let $c_{k,l}, k \in \mathbb{Z}, l = 0,\dots N-1$ be the data symbols to be transmitted. The OFDM signal is then given by

$$s(t) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1} c_{k,l}\psi_{k,l}. \quad (2)$$

The OFDM receiver is a matched filter bank $\{\varphi_{k,l}\}$ with structure similar to the transmitter:

$$\varphi_{k,l}(t) = \varphi(t-kT)e^{j2\pi lFt};\ k\in\mathbb{Z}, l = 0,\dots,N-1. \quad (3)$$

The received signal is

$$r(t) = (\mathbf{H}s)(t)+\varepsilon(t) = \int_{-\infty}^{+\infty} h(t,\tau)s(t-\tau)d\tau+\varepsilon(t) \quad (4)$$

where $h(t,\tau)$ is the impulse response of the time-varying channel $\mathbf{H}$ [2] and $\varepsilon(t)$ is zero-mean stationary white noise. The transmitted data are recovered by projecting r onto the receiver functions, i.e.,

$$\tilde{c}_{k,l} = \langle r, \varphi_{k,l}\rangle = \sum_{k\in\mathbb{Z}}\sum_{l=0}^{N-1} q_{k,l,k',l'}c_{k',l'} + n_{k,l}, \quad (5)$$

where $q_{k,l,k',l'} = \langle \mathbf{H}\psi_{k',l'}, \varphi_{k,l}\rangle$ and $n_{k,l} = \langle \varepsilon, \varphi_{k,l}\rangle$. Here the right-hand side of (5) follows from (1) and (4). The coefficients $q_{k,l,k',l'}$ describe the ISI and ICI introduced by the channel.

1.2. Preliminaries and Notation

Unless specified otherwise, norms of functions and sequences will always be $\boldsymbol{L}^2$- and $\boldsymbol{\ell}^2$-norms respectively. For $f \in \boldsymbol{L}^2(\mathbb{R})$ we denote its Fourier transform by $\hat{f}$ or $\mathcal{F}f$ where $\mathcal{F}$ is the Fourier transform operator. The ambiguity function of $f \in \boldsymbol{L}^2(\mathbb{R})$ is defined to be

$$Af(x,\omega) = \int_{-\infty}^{+\infty} f(t+\tfrac{x}{2})\overline{f(t-\tfrac{x}{2})}e^{-2\pi i\omega t}dt. \quad (6)$$

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The cross-ambiguity function of f and $g \in \boldsymbol{L}^2(\mathbb{R})$ is

$$A(f,g)(x,\omega) = \int_{-\infty}^{+\infty} f(t+\tfrac{x}{2})\overline{g(t-\tfrac{x}{2})}e^{-2\pi i\omega t}dt. \quad (7)$$

We will write $supp_\varepsilon Af$ for the subset of $\mathbb{R}^2$ for which $|Af| \geq \epsilon$. We define the dilation operator $\mathcal{D}_\alpha$ by

$$\mathcal{D}_\alpha f(t) = \sqrt{\alpha} f(\alpha t), \qquad \alpha \in \mathbb{R}, \quad (8)$$

and multiplication by a chirp is

$$\mathcal{C}_\beta f(t) = f(t)e^{-\pi i\beta t^2}, \qquad \beta \in \mathbb{R}. \quad (9)$$

Since $\mathcal{D}_\alpha$ and $\mathcal{C}_\beta$ are unitary operators we have $\mathcal{D}_\alpha^{-1} = \mathcal{D}_{\alpha^{-1}}$ and $\mathcal{C}_\beta^{-1} = \mathcal{C}_{-\beta}$. For $f,h \in \boldsymbol{L}^2(\mathbb{R})$ we have [13]

$$A(\mathcal{D}_\alpha f, \mathcal{D}_\alpha h)(x,\omega) = A(f,h)(\alpha x, \alpha^{-1}\omega) \quad (10)$$

$$A(\mathcal{C}_\beta f, \mathcal{C}_\beta h)(x,\omega) = A(f,h)(x, \omega + \beta x). \quad (11)$$

We define for $f \in \boldsymbol{L}^2(\mathbb{R})$ the Zak transform $Z_\lambda f$ by

$$(Z_\lambda f)(t,\nu) = \lambda^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} f(\lambda(t-k))e^{2\pi ik\nu} \quad (12)$$

Two properties of the Zak transform we require are

$$\lambda^{\frac{1}{2}} f(\lambda t) = \int_0^1 (Z_\lambda f)(t,\nu)d\nu \quad (13)$$

$$(Z_\lambda f)(t+1,\nu) = e^{2\pi i\nu}(Z_\lambda f)(t,\nu) \quad (14)$$

which hold almost everywhere for $t,\nu \in \mathbb{R}$ [7].

2. A GENERAL FRAMEWORK FOR PULSE SHAPE DESIGN

In the ideal (but unrealistic) case the channel $\mathbf{H}$ does not introduce any distortion and the data $c_{k,l}$ can be recovered exactly. In practice wireless channels introduce time dispersion and frequency dispersion (in addition to channel noise). Time dispersion, caused by multipath propagation, can lead to ISI. Frequency dispersion of the mobile radio channel is due to the Doppler effect and can cause ICI. The resultant ISI and ICI depends crucially on the time-frequency localization (TFL) of the pulse-shaping filter. OFDM systems with guard time interval or cyclic prefix combat ISI, but are not effective against ICI. The design of optimal time-frequency well-localized pulse-shaping OFDM filters is thus an active area of research [3, 4, 5, 6]. Some proposed pulse shape design methods require solving highly non-linear optimization problems [4, 5], and other approaches lead to pulse shapes that can be easily computed, but only a partial theoretical justification of the optimality of the so-obtainted pulse shapes has been given [3, 6]. In this section we present a general mathematical framework for optimal pulse shape design for time and/or frequency dispersive channels.

2.1. Natural Quantification of TFL

In the Heisenberg Uncertainty Principle time and frequency are treated separately. It seems more appropriate in our context to measure the TFL by considering time and frequency simultaneously. We are naturally interested in measuring the cross-interference between different transmitter functions $\varphi_{k,l}$ and $\varphi_{k',l'}$. Thus we consider the cross-ambiguity function $A(f,h)$, a joint time-frequency representation of two functions f and h. In light of the Heisenberg Uncertainty Principle it is not surprising that $A(f,h)$ cannot be arbitrarily well concentrated. We have the following Heisenberg-type inequality [14].

Theorem 2.1 For every $f,h \in \boldsymbol{L}^2(\mathbb{R})$ there holds

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (t-\tau)^2 |A(f,h)(t,\nu)|^2 dt\, d\nu \quad (15)$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\nu-\omega)^2 |A(f,h)(t,\nu)|^2 dt\, d\nu \geq \tfrac{\|f\|^4\|h\|^4}{4\pi^2}. \quad (16)$$

Equality holds if and only if

$$f(t) = (2\sigma)^{\frac{1}{4}} e^{2\pi i\mu t} e^{-\pi\sigma(t-s)^2},$$

$$h(t) = (2\sigma)^{\frac{1}{4}} e^{2\pi i(\mu+\omega)t} e^{-\pi\sigma(t-\tau-s)^2}, \quad (17)$$

with $\sigma > 0$ and $s,\mu \in \mathbb{R}$.

Thus, as expected, translated, modulated, and dilated versions of the Gaussian lead to optimal TFL. From now on we denote $g^\sigma(t) := (2\sigma)^{1/4}e^{-\pi\sigma t^2}$; if $\sigma = 1$ we omit the superscript and simply write g instead.

2.2. Orthogonalization of Transmitter Functions

Clearly the functions $\{g_{k,l}\}$ do not constitute an orthogonal basis. We orthogonalize the set $\{g_{k,l}\}$ by the standard "inverse square root" approach (see e.g. [?]) and show that the so-obtained orthogonal pulses $\psi_{k,l}$ possess optimal TFL among all possible pulse shapes ψ in the sense that they minimize $\|Ag_{k,l} - A\varphi_{k,l}\|$ subject to the condition that $\{\varphi_{k,l}\}$ is an orthogonal set.

In general this procedure can be described as follows. Given a function f with "nice" decay, assume that (f,T,F) is linearly independent. We can construct an orthonormal system (φ,T,F) from (f,T,F) via[1]

$$\varphi = \sum_{k,l} R^{-\frac{1}{2}}_{k,l,0,0} f_{k,l}\,, \quad (18)$$

[1] The existence of $R^{-\frac{1}{2}}$ follows from the positive-definiteness of R, which in turn follows from the assumption that (f,T,F) is linearly independent.

where $R_{k,l,k',l'} = \langle f_{k',l'}, f_{k,l} \rangle$. We will write

$$\varphi = \mathrm{orth}(f, T, F) \tag{19}$$

for this procedure. For $f = g^\sigma$ the linear independence of (g^σ, T, F) follows from the fact that $(g^\sigma, 1/F, 1/T)$ is a frame when $TF > 1$ [13].

The following theorem shows that the orthogonalization procedure described above yields pulse shapes that are optimal in a certain sense.

Theorem 2.2 Let $\{f_{k,l}\}$ with $f_{k,l}(t) = f(t - kT)e^{2\pi i l F t}$ and $\|f\| = 1$ be a Riesz basis (see [13]) for the separable Hilbert space $\mathcal{H}$.
(i) The function $\varphi_{\mathrm{opt}} = \mathrm{orth}(f, T, F)$ is optimally close to f in the sense that φ_{opt} solves the optimization problem

$$\begin{aligned} &\arg\min_{\varphi} \|Af - A\varphi\|_2, \quad \text{subject to} \\ &(\varphi, T, F) \text{ is an orthonormal basis for } \mathcal{H}. \end{aligned} \tag{20}$$

(ii) Furthermore

$$\|Af - A\varphi_{\mathrm{opt}}\| \le \max\{|1 - \lambda_{\min}^{-\frac{1}{2}}|^2, |1 - \lambda_{\max}^{-\frac{1}{2}}|^2\}, \tag{21}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimal and maximal eigenvalues respectively of R.

Theorems 2.1 and 2.2 imply that the OFDM pulse shapes $\psi^\sigma = \mathrm{orth}(g^\sigma, T, F)$ have optimal TFL among all possible OFDM pulse shapes in the sense that they are closest to the Gaussian g^σ.

2.3. Adaptation of Transmitter Functions to Channel Properties

It has often been suggested that the time duration and frequency duration of the transmitter functions should be adapted to the channel properties, described by the channel scattering function $C_{\boldsymbol{H}}(x, \omega)$. The ambiguity function $A\psi_{k,l}(x, \omega)$ is the inner product of a receiver function centered at (kT, lF) and a channel-scattered transmitter function centered at $(kT \pm x, lF \pm \omega)$. Thus $A\psi_{k,l}$ describes the ability of an OFDM system to detect data symbols in the presence of time- and frequency-dispersion.

Consider Figure 1. In region Y, $|A\psi_{k,l}| \ge \epsilon$, but the channel scatters no input to Y so designing a receiver that can detect a Y-scattered pulse is wasteful. Conversely, if the receiver cannot detect a pulse scattered to the region X, data will be lost. Thus we should adapt the pulse so that $supp_\varepsilon A\psi_{k,l}$ coincides, at least roughly, to the support of the channel scattering function.

In order to reduce ISI and ICI it often has been suggested to fix the time duration $\Delta\tau_f$ and the frequency duration $\Delta\nu_f$ (defined as the second-order moments of f in time and frequency, respectively) of the

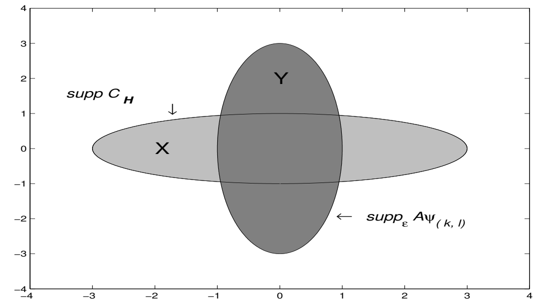


Fig. 1. Ambiguity function poorly matched to channel scattering function.

transmission pulse as well as the parameters F and T such that they are adapted to the channel properties; cf. e.g. [16, 5, 6] and for a more theoretical approach see [17]. That is, for a channel with maximal excess delay τ_0 and maximal Doppler spread ν_0 we choose $\Delta\tau_f$, $\Delta\nu_f$, T, and F such that

$$\frac{\tau_0}{\nu_0} \approx \frac{\Delta\tau_f}{\Delta\nu_f} \approx \frac{T}{F}. \tag{22}$$

For the Gaussian-based window ψ^σ this means we set $\sigma \approx \frac{\nu_0}{\tau_0}$ and $T = \sqrt{1/\sigma\rho}$, $F = \sqrt{\sigma/\rho}$ for given spectral efficiency ρ. Using basic properties of the dilation operator $\mathcal{D}_\alpha$ we obtain the relation

$$\begin{aligned} \psi^\sigma &= \mathrm{orth}(g^\sigma, 1/\sqrt{\sigma\rho}, \sqrt{\sigma/\rho}) \\ &= \mathcal{D}_{\sqrt{\sigma}}\big(\mathrm{orth}(g, 1/\sqrt{\rho}, 1/\sqrt{\rho})\big) \\ &= \mathcal{D}_{\sqrt{\sigma}}\psi. \end{aligned} \tag{23}$$

Hence it is very easy to adapt the pulse ψ to varying channel conditions. We need only know the ratio of the channel parameters ν_0 and τ_0, and dilate ψ accordingly. The pulse shape so obtained yields an OFDM system that is optimally matched to the given channel properties. This framework also applies to OQAM-OFDM pulse-shaping.

However, the adaptation rule (22) does not take into account practical constraints. For instance if we have a time-invariant channel ($\nu_0 = 0$) and fix ρ, the matching rule suggests using pulses with extremely long time duration, while $F \to 0$. However, in any DSP-based implementation there is a fixed minimum carrier separation, and the possible window length is limited by the maximum allowable transmission delay, oscillator drift, available memory and other side constraints. Thus in a more realistic scenario we must restrict the range of F and T to $F_{\min} \le F \le F_{\max}$ and $T_{\min} \le T \le T_{\max}$, where $T_{\min}, T_{\max}, F_{\min}, F_{\max}$ depend on the aforementioned constraints.

One might hope that it is still useful to match the pulse shape to the channel via setting $\frac{\tau_0}{\nu_0} = \frac{\Delta\tau_f}{\Delta\nu_f}$, even if $\frac{T}{F}$ has already reached its maximal or minimal admissible value. In particular we want to analyze the extreme cases when $\frac{\tau_0}{\nu_0} \approx \frac{1}{\sigma}$ is very large or very small. As long as the ratio $\frac{\tau_0}{\nu_0}$ is such that T and F are in the admissible range the situation is completely described by relation (23). Now we let $\frac{\nu_0}{\tau_0} = \frac{1}{\sigma} \to \infty$ and $\frac{\nu_0}{\tau_0} = \frac{1}{\sigma} \to 0$ respectively. If we fix T and F and let $\sigma \to \infty$ or $\sigma \to 0$ we get $\mathrm{cond}(R(g^\sigma, T, F)) \to \infty$ (this follows from the results in Section 6 of [8]). Thus we can no longer claim that ψ^σ is close to g^σ nor that ψ^σ has good time-frequency localization, since the bound in (21) becomes useless.

A more precise analysis of this situation is given by the following theorem. The theorem itself is a simple, but in the context of OFDM quite useful, generalization of a remarkable result of A.J.E.M. Janssen (see Theorem 8.1 in [12]).

Theorem 2.3 Set $g^\sigma(t) = (2\sigma)^{\frac{1}{4}} e^{-\pi\sigma t^2}$ and let T, F be fixed with $TF \in \mathbb{N}$. Let $\psi^\sigma = \mathrm{orth}(g^\sigma, T, F)$ be the OFDM-generating pulse shape canonically associated to (g^σ, T, F). Then there holds in the $\boldsymbol{L}^2$-sense

$$\lim_{\sigma\to\infty} \psi^\sigma(t) = \frac{1}{\sqrt{T}} \chi_{[-\frac{1}{2F}, \frac{1}{2F}]}, \tag{24}$$

$$\lim_{\sigma\downarrow 0} \psi^\sigma(t) = \frac{1}{\sqrt{F}} \frac{\sin \pi t/T}{\pi t}. \tag{25}$$

Proof : For $TF = 1$ the result has been proven by Janssen in [12]. Hence we assume $TF \in \mathbb{N}$ with $TF > 1$. The proof of (24) is only a simple modification of Janssen's proof. The proof of (25) is simpler than Janssen's proof, since we can exploit the fact that $(g^\sigma, 1/F, 1/T)$ is a frame for $TF > 1$ (but not if $TF = 1$), see e.g. [13], Chapter 7. We will prove the results by using the link between OFDM systems and Gabor frames; see [11]. Hence we have that the OFDM pulse shape canonically associated with the system (g^σ, T, F) coincides with the tight window canonically associated to the Gabor frame (g^σ, a, b) where $a = 1/F, b = 1/T$.

We first show (24). Set $q := TF = (ab)^{-1}$. It is known [?] that ψ^σ can be expressed via the Zak transform of g^σ as

$$(Z_{b^{-1}}\psi^\sigma)(t,\nu) = \frac{Z_{b^{-1}} g^\sigma(t,\nu)}{\left(\sum_{\ell=0}^{q-1} |(Z_{b^{-1}} g^\sigma(t - \frac{\ell}{q}, \nu)|^2\right)^{\frac{1}{2}}}. \tag{26}$$

Thus

$$(Z_{b^{-1}}\psi^\sigma)(t,\nu) = \frac{\sum_{k=-\infty}^{\infty} e^{-\pi\sigma a^2 q^2 (t-k)^2} e^{2\pi i k\nu}}{\left(\sum_{\ell=0}^{q-1} \left| \sum_{k=-\infty}^{\infty} e^{-\pi\sigma a^2 q^2 (t-k+\frac{\ell}{q})^2} e^{2\pi i k\nu} \right|^2 \right)^{\frac{1}{2}}} \tag{27}$$

for $(t,\nu) \in [0,1)^2$. When $t \in [0, \frac{1}{2q})$, in the numerator the term with $k=0$ is dominant when $\sigma \to \infty$; similarly the term in the denominator with $k=0, \ell=0$ is dominant as $\sigma \to \infty$. When $t \in (1 - \frac{1}{2q})$, the term with $k=1$ is dominant in the numerator, while in the denominator the term with $k=1, \ell=0$ is dominant. Finally, when $t \in (\frac{n}{2q}, \frac{n+1}{2q})$, $n = 1, \ldots, 2q-2$ the term with $k=1, \ell=n$ is dominant in both the numerator and denominator. Gathering these results, we have, for $\nu \in [0,1)$,

$$\lim_{\sigma\to\infty} (Z_{b^{-1}}\psi^\sigma)(t,\nu) = \begin{cases} 1, & t \in [0, \frac{1}{2q}), \\ 0, & t \in [\frac{1}{2q}, \frac{2q-1}{2q}), \\ e^{2\pi i\nu}, & t \in [\frac{2q-1}{2q}, 1), \end{cases} \tag{28}$$

where the convergence is in the $\boldsymbol{L}^2$-sense. Using (13) we compute

$$\psi^\sigma(b^{-1}t) = \sqrt{b} \int_0^1 (Z_{b^{-1}}\psi^\sigma)(t,\nu) d\nu. \tag{29}$$

Integrating $Z_{b^{-1}}\psi^\sigma$ over ν, applying (14), and using $a = 1/T, b = 1/F$ yields formula (24).

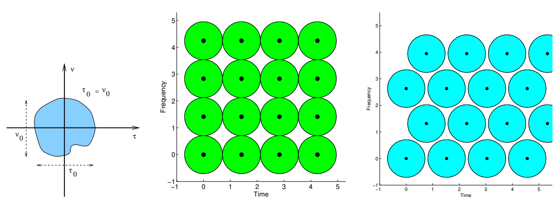
To prove (25) we recall that (g^σ, a, b) is a frame if $ab < 1$. Basic properties of the Fourier transform imply that $(\widehat{g^\sigma}, b, a)$ is also a frame. Since $\widehat{g^\sigma} = g^{\sigma^{-1}}$ we have that $\hat{\psi}^\sigma = \psi^{1/\sigma}$ converges to $\frac{1}{\sqrt{F}} \chi_{[-\frac{1}{2T}, \frac{1}{2T}]}$ for $\sigma \to \infty$ by (24) (with T and F swapped). Setting $\sigma' := 1/\sigma, g^{\sigma'} = g^{\sigma^{-1}}$, and $\psi^{\sigma'} = \psi^{1/\sigma}$, we see that $\lim_{\sigma'\downarrow 0} \psi^{\sigma'}(t) = \frac{1}{\sqrt{F}} \hat{\chi}_{[-\frac{1}{2T}, \frac{1}{2T}]} = \frac{1}{\sqrt{F}} \sin(\pi t/T)/\pi T$. This establishes (25).

Thus in the extreme case of a time-invariant channel ($\nu_0 = 0$ and hence $\sigma = 0$) we would get a sinc-type transmission pulse. If $\nu_0 = 0$ our main concern is ISI. A rectangular pulse with guard interval or cyclic prefix is known to provide excellent protection against ISI, whereas a sinc-type pulse is certainly not useful in this case. Conversely for a "spatially-invariant" channel ($\tau_0 = 0$ and hence $\sigma \to \infty$) we need be concerned only with ICI, but the theorem yields a rectangular pulse, which does not combat ICI. Thus the matching rule (22) makes sense only as long as the pulse and the lattice parameters are adapted simultaneously.

Of course any pulse shape method will suffer from the practical constraints discussed above.

3. LATTICE-OFDM

The robustness of an OFDM system against ISI and ICI is essentially determined by two factors: (i) The TFL of the pulses $\psi_{k,l}$, and (ii) the distance between adjacent lattice points. In the first part we have shown how to design ψ such that its TFL is optimized. Hence the only room for increasing the robustness of the OFDM system lies in increasing the distance between adjacent lattice points. It is known that OFDM systems constructed from hexagonal lattices provide better protection against ISI/ICI than standard OFDM systems while maintaining the same spectral efficiency as the corresponding OFDM system. Lattice OFDM systems, OFDM systems based on general time-frequency lattices, are the natural generalization of standard OFDM systems [20]. The pulse-shaping framework developed in Section 2 carries over to LOFDM systems, since Thm. 2.2 holds for general lattices as well. We present a simple method to adapt LOFDM design to given channel conditions in order to improve robustness against ISI/ICI without decreasing the spectral efficiency.



(a) (b) (c)

Fig. 2. Standard OFDM system (b), optimal LOFDM system (c), for the channel scattering shown in (a). The data symbols $c_{k,l}$ are transmitted at the lattice points. The green spheres represent $supp_\varepsilon A\psi_{k,l}$ for some ε. The use of the hexagonal lattice clearly increases the distance between adjacent spheres as compared to the rectangular lattice, thus leading to reduced ISI/ICI for time-frequency dispersive channels.

The following theorem extends the optimal LOFDM setup derived for the special case $\tau_0 = \nu_0$ to the general case. It shows how we can switch between different lattices by adapting the pulse shape accordingly while preserving the properties of the optimal hexagonal case. The theorem is essentially only a special case of a deep theorem by Gröchenig [13].

Theorem 3.1 Let (f_1, Λ_1) be a non-orthogonal, linearly independent system with $\|R(f_1, \Lambda_1)\| < \infty$ and denote $\varphi_1 = \text{orth}(f_1, \Lambda_1)$. Assume Λ_2 is a lattice with density $\delta(\Lambda_2) = \delta(\Lambda_1)$ and let

$$L_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix} \tag{30}$$

be the generator matrices of Λ_1 and Λ_2 respectively. If $f_2 = \mathcal{F}\mathcal{D}_{\alpha^{-1}}\mathcal{C}_{-\beta\alpha^2}\mathcal{F}^{-1} f_1$ with $\alpha = \frac{z_2}{z_1}$, $\beta = -\frac{y_2}{z_2} + \frac{y_1}{z_1}\frac{{z_1}^2}{{z_2}^2}$, then $R(f_2, \Lambda_2) = R(f_1, \Lambda_1)$. Furthermore define $\psi_2 = \text{orth}(\varphi_2, \Lambda_2)$, then $\varphi_2 = \mathcal{F}\mathcal{D}_{\alpha^{-1}}\mathcal{C}_{-\beta\alpha^2}\mathcal{F}^{-1}\varphi_1$.

For a proof and examples see [20].

A numerical comparison between OFDM and LOFDM shows that LOFDM can easily lead to an improvement of 1dB–2dB; see Figure 3.

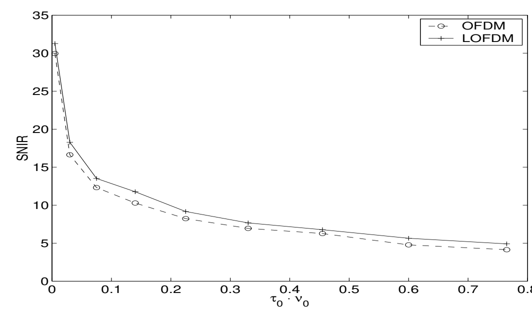


Fig. 3. Comparison of Signal to Noise-plus-Interference Ratio (SNIR) for OFDM and LOFDM. The proposed LOFDM scheme yields an improvement of 1dB–2dB.

4. REFERENCES

[1] W. Y. Zou and Y. Wu, "COFDM: An overview," IEEE Trans. Broadc., vol. 41, no. 1, pp. 1–8, March 1995.

[2] P.A. Bello, "Characterization of randomly time–variant linear channels," IEEE Trans. Comm. Syst., vol. 11, pp. 360–393, 1963.

[3] B. Le Floch, M. Alard, and C. Berrou, "Coded orthogonal frequency division multiplex," Proc. of IEEE, vol. 83, no. 6, pp. 982–996, June 1995.

[4] A. Vahlin and N. Holte, "Optimal finite duration pulses for ofdm," IEEE Trans. Comm., vol. 44, no. 1, pp. 10–14, 1995.

[5] R. Haas and J.-C. Belfiore, "A time–frequency well–localized pulse for multiple carrier transmission," Wireless Personal Communications, vol. 5, pp. 1–18, 1997.

[6] W. Kozek and A. Molisch, "Nonorthogonal pulseshapes for multicarrier communications in doubly dispersive channels," IEEE J. Sel. Areas Comm., vol. 16, no. 8, pp. 1579–1589, 1998.

[7] A.J.E.M. Janssen, "The Zak transform: A signal transform for sampled time-continuous signals." Philips J. Res., vol. 43, no. 1, pp. 23–69, 1988.

[8] A.J.E.M. Janssen, "Some Weyl-Heisenberg frame bound calculations," Indag. Mathem., vol. 7, no. 4, pp. 165-182, 1996.

[9] T. S. Rappaport, Wireless communications: Principles & Practice, Prentice Hall, New Jersey, 1996.

[10] H. Bölcskei, P. Duhamel, and R. Hleiss, "Efficient design of ofdm/oqam pulse shaping filters," 2000, submitted.

[11] T. Strohmer, "Approximation of dual Gabor frames, window decay, and wireless communications," Appl. Comp. Harm. Anal., vol. 11, no. 2, pp 243–262, 2001.

[12] A.J.E.M. Janssen, "On generating tight Gabor frames at critical density," 2001, Preprint.

[13] K.H. Gröchenig, Foundations of Time-Frequency Analysis, Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2001.

[14] A. Bonami, B. Demange, and P. Jaming, "Hermite functions and Uncertainty principles for the Fourier and the Windowed Fourier transforms," preprint, 2001.

[15] A.J.E.M. Janssen, "personal communication," 1998.

[16] B. Le Floch, M. Alard, and C. Berrou, "Coded orthogonal frequency division multiplex," IEEE Proc., vol. 83, no. 6, pp. 982–996, 1995.

[17] W. Kozek, "Adaptation of Weyl–Heisenberg frames to underspread environments," in Gabor Analysis and Algorithms: Theory and Applications, H.G. Feichtinger and T. Strohmer, Eds., chapter 10, pp. 323–352. Birkhäuser, Boston, 1998.

[18] J.H. Conway and N.J.A. Sloane, Eds., Sphere Packings, Lattices and Groups, Grundlehren der mathematischen Wissenschaften. Springer Verlag, New York, Berlin, Heidelberg, 1993.

[19] F. Riesz and B. Sz.-Nagy, Functional analysis, Frederick Ungar Publishing Co., New York, 1955.

[20] T. Strohmer and S. Beaver, Optimal OFDM design for time-frequency dispersive channels, IEEE Trans. Comm.,submitted.