# OPTIMAL OFDM SYSTEM DESIGN THROUGH OPTIMAL SPHERE COVERINGS

*Thomas Strohmer and Scott Beaver*

Department of Mathematics, University of California, Davis, CA-95616, USA

**ABSTRACT**

Standard OFDM systems are associated with a rectangular grid in the time-frequency plane. However such a setup is in general not optimal for pulse shaping OFDM systems for doubly dispersive channels. We introduce lattice-OFDM systems (LOFDM), which are OFDM systems constructed with respect to general lattices in the time-frequency plane. We show how to design optimal pulse shapes for LOFDM system. Furthermore we demonstrate by theoretical considerations and numerical simulations that LOFDM systems using hexagonal-type lattices outperform ordinary OFDM systems with regard to robustness against ISI/ICI.

## 1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is an efficient technology for wireless data transmission [1]. It is currently used in the European digital audio broadcasting standard [2], in digital terrestrial TV broadcasting, and broadband indoor wireless systems.

The basic idea of OFDM is to divide the available spectrum into several subchannels (subcarriers). A baseband OFDM system is schematically represented in Figure 1.

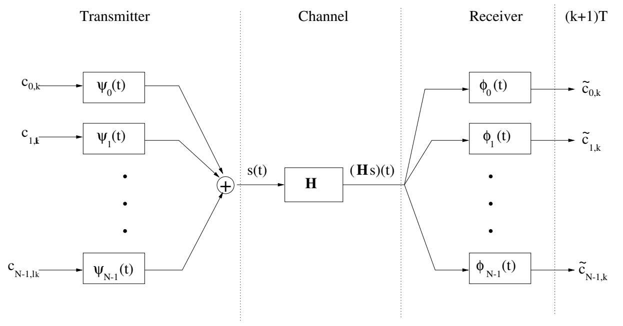


**Fig. 1**. A general baseband OFDM system configuration

Given $N$ subcarriers, a bandwidth of $W$ Hz, symbol length of $T$ seconds, and carrier separation $F := W/N$, the transmitter of a general OFDM system uses the following waveforms

$$\psi_{kl}(t) = \psi(t-kT)e^{j2\pi lF}, \quad k \in \mathbb{Z}, l = 0, \dots, N-1. \quad (1)$$

Assuming an infinite sequence of OFDM symbols to be transmitted, the OFDM signal is given by

$$s(t) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l}\psi_{kl}, \quad (2)$$

where $c_{k0}, c_{k1}, \dots, c_{k,N-1}$ are the complex-valued data symbols.

The OFDM system defined in (1) is associated with the rectangular time-frequency lattice $T\mathbb{Z} \times F\mathbb{Z}$. The data symbols $c_{kl}$ are transmitted at the lattice points $(kT, lF)$. As shown in [3] the spectral efficiency $\rho$ of the OFDM system in terms of symbols per second per Hertz can be represented by the density of the time-frequency lattice, thus we can set $\rho = TF$.

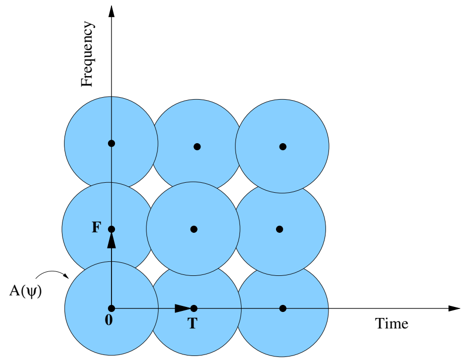


**Fig. 2**. Rectangular lattice in the time-frequency plane. The data symbols $c_{kl}$ are transmitted at the lattice points $(kT, lF)$. The spheres represent the effective support of the ambiguity function $A\psi_{k,l}$.

The OFDM receiver consists of a matched filter bank $\{\phi_{kl}\}$ of similar structure as the transmitter waveforms, i.e.,

$$\phi_{kl}(t) = \phi(t-kT)e^{j2\pi tlF}, \quad k \in \mathbb{Z}, l = 0, \dots, N-1. \quad (3)$$

The authors acknowledge partial support by NSF grant 9973373.

The received signal is

$$r(t) = (\mathbf{H}s)(t) + \varepsilon(t) = \int_{-\infty}^{+\infty} h(t,\tau)s(t-\tau)d\tau + \varepsilon(t), \quad (4)$$

where $h(t,\tau)$ is the impulse response of the time-varying channel $\mathbf{H}$ [4] and $\varepsilon(t)$ is noise, which we assume to be zero-mean stationary white noise.

The transmitted data are computed from $r$ via

$$\tilde{c}_{kl} = \langle r, \phi_{kl} \rangle = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{N-1} q_{k,l,k',l'} c_{k',l'} + n_{k,l}, \quad (5)$$

where $q_{k,l,k',l'} = \langle \mathbf{H}\psi_{k',l'}, \phi_{k,l} \rangle$ and $n_{k,l} = \langle \varepsilon, \phi_{k,l} \rangle$. Here the right-hand side of (5) follows from (1) and (4). The coefficients $q_{k,l,k',l'}$ describe the ISI and ICI introduced by the channel.

In the standard OFDM setup the functions $\psi_{kl}$ are designed to be mutually orthogonal. In this case $\phi \equiv \psi$. The situation where the sets $\{\psi_{kl}\}$ and $\{\phi_{kl}\}$ are biorthogonal is referred to as *biorthogonal frequency division multiplexing* (BFDM) [5, 6].

In practice wireless channels introduce time dispersion as well as frequency dispersion (in addition to the usual channel noise). The time dispersion is caused by multipath propagation and can lead to intersymbol interference (ISI). Frequency dispersion of the mobile radio channel is due to the Doppler effect and can cause interchannel interference (ICI). The distortion resulting from channel dispersion depends crucially on the time-frequency localization of the transmitter pulse shapes $\{\phi_{kl}\}$.

An optimal OFDM system in case of doubly dispersive channels would consist of orthogonal basis functions with $\rho = TF = 1$, such that the $\psi_{k,l}$ are well localized in time and frequency. The condition $TF = 1$ ensures maximal spectral efficiency of the transmission system. Unfortunately, such a system cannot exist due to the Balian-Low theorem [7].

Therefore other approaches (cyclic prefix, pulse-shaping, etc.) have been proposed to combat ISI and ICI, see e.g. [1, 3, 5, 6]. These approaches can be interpreted in the time-frequency plane as using an undersampled grid, i.e., $TF > 1$, which results in a set of basis functions that is incomplete in $\boldsymbol{L}_2(\mathbb{R})$ (or $\boldsymbol{\ell}_2(\mathbb{Z})$, respectively) [5]. Although the choice $TF > 1$ leads to a loss in the spectral efficiency of the transmission system, it is usually an acceptable price to pay to mitigate ISI/ICI.

## 2. OFDM, LATTICES, AND SPHERE COVERINGS

In what follows without loss of generality we normalize the time axis and the frequency axis, such that $T = F$. If we design the pulse shape $\psi$ such that $\psi$ and its Fourier transform $\hat{\psi}$ are well-localized around the origin, then the ambiguity function of $\psi_{kl}$, given by

$$(A\psi_{kl})(t,f) = \int_{-\infty}^{+\infty} \psi_{kl}(\tau)\psi_{kl}^*(\tau - t)e^{-j2\pi f\tau}d\tau \quad (6)$$

is essentially localized around the point $(kT, lF)$ in the time-frequency plane, cf. Figure 2.

As mentioned above, the stability of an OFDM system is essentially determined by two factors:
(i) The numerical support of $A(\psi_{kl})$: the smaller the numerical support of $A(\psi_{kl})$, the less interference between adjacent OFDM symbols.
(ii) The distance between adjacent lattice points: Larger distance between neighbor lattice points yields to better stability of the OFDM system against ISI and ICI.

The key problem is: how can we improve the ICI/ISI-stability of OFDM systems without giving up the orthogonality condition on the functions $\psi_{kl}$ – which would increase the sensibility to AWGN channels – and without decreasing the spectral efficiency of the system? Alternatively, we can ask: how can we increase the spectral efficiency of an OFDM system without decreasing its robustness against ISI/ICI?

In the following we show that for both problems the solution consists of so-called lattice-OFDM systems.

OFDM systems need not be associated with a rectangular time-frequency lattice.

**Definition 2.1** *We define a general* Lattice OFDM *system (LOFDM for short) via*

$$\psi_{kl}(t) = \psi(t-\lambda_k)e^{j2\pi t\nu_l}, \quad k \in \mathbb{Z}, l = 0,\ldots,N-1, \quad (7)$$

*with $(\lambda_k, \nu_l) \in \Lambda$, where $\Lambda \subseteq \mathbb{R}^2$ is a* general *lattice in $\mathbb{R}^2$, and not necessarily of the form $T\mathbb{Z} \times F\mathbb{Z}$.*

Every lattice $\Lambda \in \mathbb{R}^2$ can be described by means of its generator matrix $L$ [8] of the form

$$L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}.$$

For the rectangular lattice we have $a = T, b = 0, c = F$. Of course, this corresponds to the OFDM system described in (1). It is not difficult to see that the spectral efficiency $\rho$ of a general LOFDM system is given by $\rho = \det L$.

But what is a good lattice for which pulse shape? Let us consider this question for the function that is optimally concentrated in time and frequency: the Gaussian $g(t) := 2^{1/4}e^{-\pi t^2}$. Note that $|Ag|$ is rotation invariant [9], thus the effective support of $Ag$ corresponds to a sphere. It is clear that the functions $g_{kl}$ obtained via (1) or (7) cannot form an orthogonal system. But they can be transformed into an orthogonal system by the following method [10, 6, 11]. For given lattice $\Lambda$ we compute

$$g_{kl}(t) = g(t-\lambda_k)e^{j2\pi t\nu_l}, \qquad (\lambda_k, \nu_l) \in \Lambda, \qquad (8)$$

and define the Gram matrix

$$R(\Lambda) := \{\langle g_{k'l'}, g_{kl}\rangle\}_{k,l,k',l' \in \mathbb{Z}}.$$

We set[1]

$$\overline{\psi}(\Lambda) := \sum_{k,l\in\mathbb{Z}} [R(\Lambda)^{-\frac{1}{2}}]_{k,l,0,0}\, g_{kl}. \qquad (9)$$

One can readily show that the LOFDM functions $\overline{\psi}_{kl}$ obtained from $\overline{\psi}$ via (7) are mutually orthogonal. For the standard rectangular lattice this approach coincides with existing pulse shape techniques [10, 6, 11].

The following theorem shows that the pulse shape $\overline{\psi}$ is in a certain sense the optimal pulse shape for (L)OFDM.

**Theorem 2.2** *For fixed lattice $\Lambda$ the pulse shape $\overline{\psi}$ obtained via* (9) *is optimal in the sense that it minimizes $\|g-\psi\|$ and $\|\hat{g}-\hat{\psi}\|$ among all pulse shapes $\psi$ for which the functions $\psi_{kl}$ are mutually orthogonal for given $\Lambda$*

The proof is based on the link between OFDM systems and Gabor functions and by extending the main theorem in [12] to general lattices.

We still have to answer the question of how to choose the best $\Lambda$ so that the distance between adjacent lattice points is maximized, in order to minimize the interference of the corresponding ambiguity functions $A(\mathbf{H}\overline{\psi})$. Note that the lattice $\Lambda$ has to be chosen in advance, before computing $\overline{\psi}$.

Although the ambiguity function of $\overline{\psi}$ is no longer exactly rotation invariant, its effective support is still approximately sphere-shaped. Thus the problem of finding the lattice $\Lambda$ such that we get minimal interference between adjacent data symbols is reminiscent of the sphere covering problem [8]. Recall that the sphere covering problem can be stated as "how can we arrange a fixed number of spheres of fixed diameter in $\mathbb{R}^2$ such that adjacent spheres have minimal overlap?" The well-known answer to this question is: Choose the midpoints of the sphere such that they form a hexagonal lattice as in Figure 3.

The generator matrix of the hexagonal lattice $\Lambda_H$ is

$$L_H = \alpha \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad (10)$$

where the scalar $\alpha$ depends on the desired spectral efficiency (lattice density) $\rho$. The two vectors $e_1, e_2$ in Figure 3 correspond to the rows of $L_H$.

The larger distance between adjacent lattice points and the smaller interference between the corresponding adjacent

[1] $R^{\frac{1}{2}}$ exists since $R$ is hermitian positive definite.

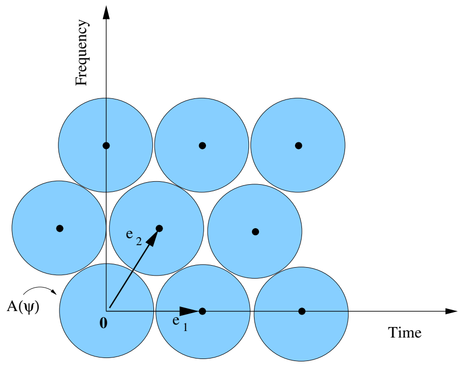


**Fig. 3**. Hexagonal lattice in the time-frequency plane. The data symbols $c_{kl}$ are transmitted at hexagonal lattice points.

ambiguity functions becomes obvious when comparing Figures 2 and 3, which show different lattices, but with the same density.

The considerations above suggest that an LOFDM system with hexagonal lattice and associated pulse shape $\overline{\psi}(\Lambda_H)$ seems to be optimal with respect to maximizing the ISI/ICI robustness, and numerical experiments confirm this. However an LOFDM system associated with the hexagonal lattice is unfortunately not very attractive from the viewpoint of fast implementation. The reason is the incomensurability between the matrix entries 1 and $\sqrt{3}$ of the generator matrix in (10).

Let $(D_\beta g)(t)$ denote the dilation operator. It is a basic fact that the tripe $(g, a, b)$ establishes a Gabor frame if and only if $(D_\beta g, a/\beta, b\beta)$ is a Gabor frame and the frame bounds for both systems are the same. Moreover the corresponding canonical tight windows are related by dilation. A similar result holds for general lattices.[2] Thus we can equivalently use the an LOFDM system with the lattice $\Lambda_C$ with generator matrix

$$L_C = \alpha \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \qquad (11)$$

and corresponding pulse shape $\overline{\psi}(\Lambda_C)$ computed via (9) from a properly dilated Gaussian with dilation factor $\alpha = \sqrt{2}/\sqrt[4]{3}$. The lattice $\Lambda_C$ avoids the incomensurability of $\Lambda_H$, thus allows a simple FFT-based implementation of the transmitter and the receiver.

[2] This can be shown by exploiting the fact that $D$ is a unitary operator. A rather abstract way to prove this - and in fact a much more general result - would make use of the Stone-von Neumann theorem on the equivalence of the irreducible unitary representations of the Schrödinger representation of the Heisenberg group.

## 3. NUMERICAL SIMULATIONS

We compare the proposed LOFDM system using the lattice generated by $L_C$ of (11) to a standard OFDM system of (1) associated with the rectangular lattice. For both systems we use the pulse obtained from the Gaussian by the orthogonalization procedure outlined in (9). For the rectangular lattice the OFDM system obtained in that way coincides with the setup suggested in [10].

For the numerical simulations we assume the channel to be WSSUS [4]. We consider a Jakes-type Doppler behavior and exponential decay in temporal direction for the scattering function [13]. In addition, defining $\tau_0$ as the maximum excess delay and $\nu_0$ as the maximum Doppler shift, we assume that $\tau_0\nu_0 < 1$, i.e., the channel is underspread [4]. For underspread channels and pulse shapes that are well concentrated in time and frequency, one can show that the coefficients $q_{k,l,k',l'}$ in (5) are approximately 0 for $k \neq k', l \neq l'$, cf. [5].

Since both, OFDM and LOFDM are based on orthogonal systems, they are all equally well adapted to AWGN. Hence for our simulations we can assume the absence of thermic noise, so that any error in the received data is caused only by ISI and ICI. Thus in absence of thermic noise the received data can be equalized by computing

$$d_{kl} = \frac{\tilde{c}_{kl}}{q_{k,l,k,l}}, \qquad \text{for } q_{k,l,k,l} \neq 0, \tag{12}$$

where $q_{k,l,k,l} \approx S_{\mathbf{H}}(\lambda_k, \nu_l), (\lambda_k, \nu_l) \in \Lambda$, with the time-varying transfer function [4, 14]

$$S_H(t, f) = \int h(t, \tau) e^{-j2\pi f \tau} d\tau.$$

We compare the mean square error between the transmitted data $c_{k,l}$ and the received data $d_{k,l}$ (which have been equalized according to (12)). Figure 4 shows that LOFDM yields indeed a better SNR in case of ISI/ICI. On the average the improvement of LOFDM compared to OFDM is in the range of 1dB – 2dB.


## Acknowledgments

The authors thank F. Hlawatsch and his collaborators as well as W. Kozek for making their MATLAB files on the simulation of underspread channels available to us.


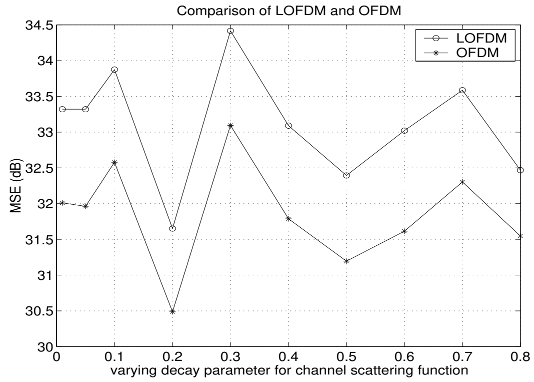


**Fig. 4**. Mean-square coefficient error for OFDM and LOFDM. The proposed LOFDM scheme yields an improvement of 1dB–2dB.

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