Reduction of channel interference in OFDM systems through optimal phase space sampling

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ABSTRACT

Orthogonal frequency division multiplexing (OFDM) has gained considerable interest as an efficient technology for high-date-data transmission over wireless channels. Standard OFDM systems are associated with a rectangular grid in the time-frequency plane. However such a setup is in general not optimum for pulse shaping OFDM systems for doubly dispersive channels. We introduce lattice-OFDM systems (LOFDM), which are OFDM systems constructed with respect to general lattices in the time-frequency plane. We show how to design optimum pulse shaping LOFDM system. Our analysis is based on results from Heisenberg groups, Gabor frames, and sphere coverings. Numerical simulations confirm that LOFDM systems outperform ordinary OFDM systems with regard to robustness against intersymbol interference and interchannel interference.

Keywords: OFDM, sphere packings, time-frequency localization, intersymbol interference, interchannel interference.

1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is an efficient technology for wireless data transmission.[1] It is currently used in the European digital audio broadcasting standard,[2] in digital terrestrial TV broadcasting, and broadband indoor wireless systems.[3] The qualities of OFDM include robustness against multipath fading and high spectral efficiency.

Transmission over wireless channels can lead to intersymbol interference (ISI) as well as interchannel interference (ICI). ISI is caused by time dispersion due to multipath propagation and ICI results from frequency dispersion due to the Doppler effect.[4] We will refer to channels that are time dispersive and frequency dispersive as *time-frequency dispersive*. The distortion resulting from time-frequency dispersive channels depends crucially on the time-frequency localization of the pulse shaping filter. For instance OFDM systems with guard-time interval or cyclic prefix combat ISI, but do not prevent ICI, because they are based on rectangular-type pulses. The design of optimal pulse shaping OFDM systems is therefore an active area of research.[5–9]

It is natural — and has been proposed frequently in the literature — that the time duration and the effective bandwidth of the pulse as well as the symbol period and the carrier separation of the OFDM system should be matched to the maximal delay spread and the maximal Doppler spread of the channel in order to minimize the joint ISI/ICI. The symbol period and the carrier separation define the time-frequency lattice associated with an OFDM system. All work on optimal OFDM design so far has exclusively dealt with rectangular time-frequency lattices. However such a setup is suboptimal for time-frequency dispersive channels, as we will show in this paper. By using results from sphere covering theory we demonstrate that OFDM systems constructed for hexagonal-type lattices provide better protection against ISI/ICI. To this end we introduce Lattice OFDM systems, which are OFDM systems based on general (non-rectangular) time-frequency lattices. We show how to design LOFDM pulse shapes that are well localized in the time-frequency plane and at the same time optimally matched to the given channel conditions. for general time-frequency dispersive channels. Numerical simulations confirm our theoretical analysis, showing that LOFDM systems outperform OFDM systems. The improvement in the Signal-to-Noise-plus-Interference Ratio in our experiments was usually in the range of 1dB-2dB.

1.1. Notation

For a function $f \in \boldsymbol{L}_2(\mathbb{R})$ we define its Fourier transform $\hat{f}$ by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi i t\omega}dt. \tag{1}$$

We also write $\mathcal{F}f$ instead of $\hat{f}$.

The *ambiguity function* of $f \in \boldsymbol{L}_2(\mathbb{R})$ is defined to be

$$Af(t,\omega) = \int_{-\infty}^{+\infty} f\big(x+\frac{t}{2}\big)\overline{f\big(x-\frac{t}{2}\big)}e^{-2\pi i\omega x}dx. \tag{2}$$

We define the dilation operator $\mathcal{D}_\alpha$ by

$$\mathcal{D}_\alpha f(t) = \sqrt{\alpha}f(\alpha t), \qquad \alpha \in \mathbb{R}, \tag{3}$$

and the multiplication by a chirp is

$$\mathcal{C}_\beta f(t) = f(t)e^{-\pi i\beta t^2}, \qquad \beta \in \mathbb{R}. \tag{4}$$

Since $\mathcal{D}_\alpha$ and $\mathcal{C}_\beta$ are unitary operators we have

$$\mathcal{D}_\alpha^{-1} = \mathcal{D}_{\alpha^{-1}}, \qquad \mathcal{C}_\beta^{-1} = \mathcal{C}_{-\beta}. \tag{5}$$

Both $\mathcal{D}_\alpha$ and $\mathcal{C}_\beta$ are unitary operators.

For $f, h \in \boldsymbol{L}_2(\mathbb{R})$ we have[10]

$$A(\mathcal{D}_\alpha f, \mathcal{D}_\alpha h)(t,\omega) = A(f,h)(\alpha t, \alpha^{-1}\omega) \tag{6}$$

$$A(\mathcal{C}_\beta f, \mathcal{C}_\beta h)(t,\omega) = A(f,h)(t, \omega + cx). \tag{7}$$

We will denote Gaussians by g^σ, i.e.,

$$g^\sigma(t) := (2\sigma)^{1/4}e^{-\pi\sigma t^2}. \tag{8}$$

If $\sigma = 1$ we skip the subscript and simply write g instead.

2. OFDM, WIRELESS CHANNELS, AND TIME-FREQUENCY LOCALIZATION

Let us briefly review the basic scheme of OFDM. The transmission functions of an orthogonal frequency division multiplexing (OFDM) system consist of translations and modulations of a single pulse shape φ, i.e.,

$$\varphi_{kl}(t) = \varphi(t-kT)e^{j2\pi lFt}, \tag{9}$$

for some $T, F > 0, k, \in \mathbb{Z}$ and $l = 0, \dots, N-1$ or $l \in \mathbb{Z}$. We denote such a function system generated via (9) by the triple (φ, T, F), whether or not the functions are orthogonal.

Here T is the symbol period T and F is the carrier separation. Let W be the given bandwidth, then $N = W/F$ is the number of subcarriers. Throughout our analysis it is convenient to assume an infinite number of subcarriers. Thus the frequency index l in (9) runs through the set of integers instead of from 0 to $N-1$. But many of the results presented in this paper also hold for finitely many subcarriers, either verbatim or with a negligible approximation error.

A necessary condition for perfect reconstruction of the data is that the functions $\varphi_{k,l}$ are linear independent (no matter if they are orthogonal or not). This implies that $TF \geq 1$, which follows easily from the connection between OFDM systems and Gabor frames, e.g., see [8] or the appendix. We note that orthogonality of the functions $\varphi_{k,l}$ is not a requirement for perfect reconstruction, but minimizes the error caused by AWGN.

As shown in[7] for a large number of subcarriers the *spectral efficiency* ρ of the OFDM system in terms of data symbols per second per Hertz is approximately given by $\rho = (TF)^{-1}$. Since $TF \geq 1$ we have of course that the maximal spectral efficiency is $\rho_{\max} = 1$.

Let $c_{k,l}, k \in \mathbb{Z}, l = 0, \dots N-1$ be the (already block-coded) data symbols to be transmitted. The OFDM signal is then given by

$$s(t) = \sum_{k,l} c_{k,l} \varphi_{kl}. \tag{10}$$

The received signal is

$$r(t) = (\mathbf{H}s)(t) + \varepsilon(t) = \int_{-\infty}^{+\infty} h(t,\tau) s(t-\tau) d\tau + \varepsilon(t), \tag{11}$$

where $h(t,\tau)$ is the impulse response of the time-varying channel $\mathbf{H}$[11] and $\varepsilon(t)$ is zero-mean stationary white noise. The data symbol $c_{k,l}$ is transmitted at the (k,l)-th lattice point of the rectangular time-frequency lattice $(kT, lF), k, l \in \mathbb{Z}$.

The transmitted data are recovered by computing the inner product of the received signal r with the functions φ_{kl}, i.e.,

$$\tilde{c}_{kl} = \langle r, \varphi_{kl} \rangle. \tag{12}$$

By (10) and (11) we have

$$\tilde{c}_{kl} = \langle \mathbf{H}s, \varphi_{kl} \rangle + \langle \varepsilon, \varphi_{k,l} \rangle = \sum_{k',l'} \langle \mathbf{H}\varphi_{k',l'}, \varphi_{k,l} \rangle + \langle \varepsilon, \varphi_{k,l} \rangle, \tag{13}$$

where the interchange of summation and integration is justified under mild smoothness and decay conditions on φ and $\mathbf{H}$. In the ideal (but unrealistic) case the wireless channel does not introduce any distortion. Thus the orthogonality between the functions $\psi_{k,l}$ is preserved, and if $\varepsilon = 0$ the data $c_{k,l}$ can be recovered exactly. However in practice wireless channels are time-dispersive and frequency-dispersive. Hence when $\varphi_{k,l}$ passes through the time-varying channel $\mathbf{H}$ it is spread out in time and frequency, causing ISI as well as ICI in the transmitted data. It is customary (cf. e.g. [5,4,7,8]) to describe the channel by the *maximal excess delay* τ_0 and the *maximal Doppler spread* ν_0 of the channel scattering function.

In order to reduce ISI and ICI it often has been suggested to choose the time duration and the frequency duration of the transmission pulse as well as the parameters F and T such that they are adapted to the channel properties, cf. e.g. [12,7,8] and for a more theoretical approach see [13]. This means that for given maximal excess delay τ_0 and maximal Doppler spread ν_0 we choose $\Delta\tau_f$, $\Delta\nu_f$, T, and F such that

$$\frac{\tau_0}{\nu_0} \approx \frac{\Delta\tau_f}{\Delta\nu_f} \approx \frac{T}{F}, \tag{14}$$

where $\Delta\tau_f$ and $\Delta\nu_f$ are second order moments of f and its Fourier transform $\hat{f}$, respectively.

It is clear from (13) that the interference of the data symbol c_{kl} with neighbor symbols will be the smaller the better the time-frequency localization of $\varphi_{k,l}$ is and the larger the distance between $c_{k,l}$ and adjacent data symbols in the time-frequency domain, i.e, the larger T and F. The latter condition however comes at the cost of reduced spectral efficiency.

Ideally we would like to construct an OFDM system (φ, T, F) that satisfies the following three requirements simultaneously:

(i) the functions $\psi_{k,l}$ are mutually orthogonal,

(ii) φ is well-localized in time and frequency domain,

(iii) $TF = 1$, i.e., maximal spectral efficiency.

As is well-known from Gabor theory these three conditions cannot be satisfied simultaneously due to the Balian-Low theorem.[14] This holds true, even if we relax condition (i) and consider biorthogonality instead (which however would increase the sensibility of the system to AWGN).

Hence we relax condition (iii) and allow $TF > 1$. This loss in spectral efficiency is usually an acceptable price to pay to mitigate ISI/ICI (as long as TF is not too large). Alternatively we can use a slightly different modulation scheme, known as OQAM OFDM, see [6]. A comparison between these two approaches is beyond the scope of this paper.

3. LATTICE-OFDM, WIRELESS CHANNELS, AND INTERFERENCE

The robustness of an OFDM system against ISI and ICI depends essentially on two factors:

(i) The time-frequency localization of ψ_{kl};

(ii) the distance between adjacent time-frequency lattice points (kT, lF), $((k \pm 1)T, ((l \pm 1)F)$.

However increasing T and/or F results in an undesirable loss of spectral efficiency. Hence we are concerned with the following question:

Can we increase the distance between adjacent grid points without reducing the spectral efficiency?

We will show in this section that this is indeed possible. The solution lies in the introduction of a more general and flexible OFDM setup which we call *Lattice-OFDM* (LOFDM).

3.1. Lattice-OFDM

We briefly review some basic facts about lattices, see [15] for details. Any lattice Λ can be described by its (non-unique) generator matrix L. The generator matrix contains the coordinates of the (column-)vectors that describe the fundamental domain of a lattice. The density $\delta(\Lambda)$ of the lattice Λ is given by

$$\delta(\Lambda) = \frac{1}{\det(L)}. \tag{15}$$

For instance the generator matrix L_R for the rectangular lattice Λ_R displayed in Figure 3.2 is

$$L_R = \begin{bmatrix} T & 0 \\ 0 & F \end{bmatrix}, \tag{16}$$

the density of Λ_R is $\delta(\Lambda_R) = 1/\det(L_R) = 1/(TF)$. The hexagonal lattice Λ_H (shown in Figure 3.2(b) with $T = F$) with generator matrix

$$L_H = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt[4]{3}}T & \frac{\sqrt{2}}{2\sqrt[4]{3}}T \\ 0 & \frac{\sqrt[4]{3}}{\sqrt{2}}F \end{bmatrix} \tag{17}$$

has the same density, viz. $\delta(\Lambda_H) = 1/\det(L_H) = 1/(TF)$.

DEFINITION 3.1. *Let Λ be a lattice with generator matrix*

$$L = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \qquad x, z \neq 0, \tag{18}$$

and let $\varphi \in \boldsymbol{L}_2(\mathbb{R})$ be given. We denote by (φ, Λ) the function system

$$\varphi_{kl}(t) = \varphi(t - \lambda_k)e^{j2\pi t\mu_l}, \qquad (\lambda_k, \mu_l) \in \Lambda, \quad k \in \mathbb{Z}, l = 0, \dots, N-1 \text{ or } l \in \mathbb{Z}. \tag{19}$$

If the functions $\varphi_{k,l}$ are mutually orthonormal we call (φ, Λ) a lattice-OFDM (LOFDM) system. Furthermore the Gram matrix $R(\varphi, \Lambda)$ associated with (φ, Λ) is defined via

$$R(\varphi, \Lambda) := \{R_{k,l,k',l'}\} \quad \text{with } R_{k,l,k',l'} = \langle \varphi_{k',l'}, \varphi_{k,l} \rangle. \tag{20}$$

Let (f, Λ) be a non-orthogonal set of functions with bounded invertible Gram matrix $R(f, \Lambda)$. There is a standard method to "orthogonalize" the set (f, Λ) (in fact any set of linear independent functions whose Gram matrix is bounded), see [14,9,16,17]. Define

$$\varphi = \sum_{k,l} R^{-\frac{1}{2}}_{k,l,0,0} f_{k,l}, \tag{21}$$

where

$$R = R(f, \Lambda). \tag{22}$$

We will write

$$\varphi = \text{orth}(f, T, F) \tag{23}$$

for this procedure.

Since the pulse shape obtained from the Gaussian via (23) will play a major role throughout the paper we introduce a separate notation for this pulse to distinguish it from general OFDM pulses. We denote

$$\psi^\sigma := \text{orth}(g^\sigma, \Lambda), \tag{24}$$

and if $\sigma = 1$ we skip the subscript and simply write ψ.

The following theorem shows that the orthogonalization procedure described above yields pulse shapes that are optimal in a certain sense.

THEOREM 3.2. *Let (f, Λ) be a Riesz basis for the separable Hilbert space $\mathcal{H}$.*
The function $\varphi_{opt} = orth(f, \Lambda)$ is optimally close to f in the sense that φ_{opt} solves the optimization problem

$$\arg\min_{\varphi} \|Af - A\varphi\|_2 \qquad \text{subject to } (\varphi, \Lambda) \text{ is an orthonormal basis for } \mathcal{H}. \tag{25}$$

See[18] for the proof.

This theorem implies that the OFDM pulse shapes $\psi^\sigma = \text{orth}(g^\sigma, \Lambda)$ are optimally time-frequency localized among all possible OFDM pulse shapes in the sense that they are closest (in the $\boldsymbol{L}^2$-sense) to the Gaussian g^σ. For a theoretical investigation of the decay properties of pulse shapes generated via (23) we refer to.[17]

This approach has many advantages over a Gram-Schmidt type procedure, since it often preserves important structural properties of the functions f_k, that are destroyed by the Gram-Schmidt method.

But we still need to match the pulse shape and the lattice to the given channel conditions in order to maximize the robustness of the LOFDM system against ISI/ICI.

3.2. Sphere packings and minimization of ISI/ICI

In this section we show how to design pulse shapes and lattices which provide optimal robustness for time-frequency dispersive channels. We start with a specific but important example. The analysis of the general case will be based on this example.

Assume that the parameters of the channel scattering function satisfy $\tau_0 = \nu_0$ (we will discuss more general choices for τ_0, ν_0 later). Based on theorem 3.2 the optimal pulse shape ψ in this case is given by $\psi = \text{orth}(g, T, F)$ with $g(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$ (since $\sigma = \frac{\nu_0}{\tau_0} = 1$) and $T = F$.

For a function f and $\varepsilon > 0$ define the *effective support* of the ambiguity function Af by

$$\text{supp}_\varepsilon(Af) = \{(t,\omega) : |Af(t,\omega)| > \varepsilon\}. \tag{26}$$

The ambiguity function of g is rotation-invariant, see [19], therefore $|Ag_{k,l}|$ is rotation invariant. Hence for any $\varepsilon > 0$ $\text{supp}_\varepsilon(Ag_{k,l})$ is a sphere with center (kT, lF). One can show that since the matrix R is well-conditioned then $A\psi \approx Ag$. Hence $\text{supp}_\varepsilon(A\psi_{k,l})$ is approximately a sphere with center (kT, lF). The OFDM system (ψ, T, F) with $T = F$ and $\sigma = 1$ can thus be represented schematically by Figure 3.2. The data symbol $c_{k,l}$ is transmitted at the lattice point (kT, lF). The gray spheres represent $\text{supp}_\varepsilon\, A\psi_{k,l}$. We have chosen ε such that the difference between the setup in Figure 3.2 and Figure 3.2(b), see below, becomes obvious. The reader should not be confused by the fact that the spheres may overlap for smaller ε. The $A\psi_{k,l}$ are of course mutually orthogonal, since the $\psi_{k,l}$ are mutually orthogonal. However after passing through the channel $\mathbf{H}$ the orthogonality between the $\psi_{k,l}$ (and thus between the $A\psi_{k,l}$) is lost and the support of $A\psi_{k,l}$ is spread out in time and frequency. And now the overlap between adjacent $A\psi_{k,l}$'s (or rather between $A\mathbf{H}\psi_{k,l}$ and $A\psi_{k',l'}$) plays a crucial role. Roughly speaking less overlap between the spheres results in less ISI and ICI.

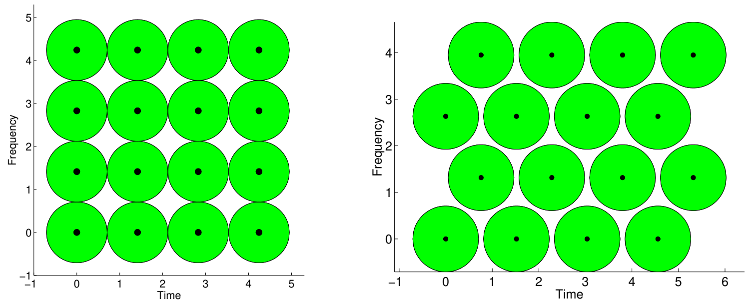
Note that since $\psi_{k,l}(t) = \mathcal{O}(e^{-|t|})$ we have

$$\langle \mathbf{H}\psi_{k,l}, \psi_{k\pm1,l\pm1}\rangle \gg \langle \mathbf{H}\psi_{k,l}, \psi_{k\pm n,l\pm n}\rangle, \quad \text{for } n > 1. \tag{27}$$

Therefore we can concentrate on the interference between $c_{k,l}$ and direct neighbor symbols $c_{k\pm1,l\pm1}$, since the interference of $c_{k,l}$ with symbols $c_{k\pm n,l\pm n}$ where $n > 1$ will be negligible small by comparison.

Hence the problem reduces to maximizing the distance between immediate neighbor lattice points (k,l) and $(k\pm1, l\pm1)$ without loosing spectral efficiency. The answer to this problem comes from the theory of sphere packings.[18] It is well-known that the sphere packing problem in 2-d is solved by the hexagonal lattice, cf. [15]. Thus for the case $\tau_0 = \nu_0$ an OFDM system constructed for a hexagonal lattice as shown in Figure 3.2(b) will provide optimal protection against ISI/ICI due to the increased distance between adjacent lattice points.

The effect of the increased distance becomes apparent when comparing Figure 3.2(a) with Figure 3.2(b). In both cases we have chosen the same ε as threshold for the effective support of the ambiguity function.



(a) An OFDM system with rectangular lattice

(b) Time-frequency representation of LOFDM system.

Of course the assumption $\tau_0 = \nu_0$ in our example is very restrictive. In the next subsection we consider more general channel conditions and derive a general LOFDM design rule in order to minimize the joint ISI/ICI.

3.3. Optimal LOFDM design for the minimization of ISI/ICI

Recall that τ_0 and ν_0 describe the support of the channel scattering function. More precisely, assuming that the support of the scattering function is circumscribed by a rectangle (or ellipse), then τ_0 gives the length of the rectangle (in the time-direction) and ν_0 corresponds to the width (in the frequency direction). In the following we extend the result from the previous section to more general (and more realistic) choices for τ_0 and nu_0. In fact our approach will also include the case where the support of the scattering function may be concentrated along a rotated rectangle in the time-frequency domain (like a chirp). Such a setup has also been investigated in.[20]

The next theorem is the key to extend the optimal LOFDM setup derived for the special case $\tau_0 = \nu_0$ to the general case. It shows how we can switch between different lattices by adapting the pulse shape accordingly in order to preserve the properties of the optimal "hexagonal" case. The theorem is essentially only a special case of a deep theorem by Gröchenig.[10]

THEOREM 3.3. *Let (f_1, Λ_1) be a non-orthogonal, linear independent system with $\|R(f_1, \Lambda_1)\| < \infty$ and denote $\varphi_1 = \mathrm{orth}(f_1, \Lambda_1)$. Assume Λ_2 is a lattice with density $\delta(\Lambda_2) = \delta(\Lambda_1)$ and let*

$$L_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix} \tag{28}$$

be the generator matrices of Λ_1 and Λ_2 respectively. If

$$f_2 = \mathcal{F}\mathcal{D}_{\alpha^{-1}} C_{-\beta\alpha^2} \mathcal{F}^{-1} f_1 \tag{29}$$

with

$$\alpha = \frac{z_2}{z_1}, \quad \beta = -\frac{y_2}{z_2} + \frac{y_1}{z_1}\frac{{z_1}^2}{{z_2}^2}, \tag{30}$$

then

$$R(f_2, \Lambda_2) = R(f_1, \Lambda_1). \tag{31}$$

Furthermore define $\psi_2 = \operatorname{orth}(\varphi_2, \Lambda_2)$, *then*

$$\varphi_2 = \mathcal{F}\mathcal{D}_{\alpha^{-1}} C_{-\beta\alpha^2} \mathcal{F}^{-1} \varphi_1 \tag{32}$$

Proof: See[18] for the proof. ❑

The main feature of Theorem 3.3 is of course that it yields a simple rule for the design of optimal LOFDM systems for time-frequency dispersive channels. We will illustrate this fact by an example.

In the following example we consider LOFDM systems with spectral efficiency ρ. Λ_1 will denote the hexagonal lattice with generator matrix

$$L_1 = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt[4]{3}\sqrt{\rho}} & \frac{1}{\sqrt[4]{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt[4]{3}\sqrt{\rho}} \end{bmatrix} \tag{33}$$

and

$$\varphi_1 = \psi = \operatorname{orth}(g, \Lambda_H). \tag{34}$$

We have seen that the LOFDM system (φ_1, Λ_H) is optimally robust against ISI/ICI if $\tau_0 = \nu_0$. Based on Theorem 3.3 we are able now to consider more general situations.

Example: Assume $\tau_0 = \frac{1}{\sigma}\nu_0$ for some $\sigma > 0$ and fixed ρ. The LOFDM system that is optimally adapted to these channel conditions with respect to minimizing the joint ISI/ICI is given by $(\psi^\sigma, \Lambda_\sigma)$, where the generator matrix L_σ of Λ_σ can be computed by using Theorem 3.3 as

$$L_\sigma = \begin{bmatrix} T & \frac{T}{2} \\ 0 & F \end{bmatrix}, \quad \text{with } T = \frac{\sqrt{2\rho}}{\sqrt[4]{3}\sqrt{\sigma}}, \, , F = \frac{\sqrt[4]{3}\sqrt{\sigma\rho}}{\sqrt{2}}. \tag{35}$$

The optimal pulse shape is $\psi^\sigma = \operatorname{orth}(g^\sigma, \Lambda_\sigma)$ where $\sigma = \frac{\nu_0}{\tau_0}$ by assumption. This design rule is valid if σ is within the admissible range of F/T, i.e., if $\frac{F_{\min}}{T_{\max}} \le \sigma \le \frac{F_{\max}}{T_{\min}}$.

Consider for instance $\tau_0 = \sqrt{3}\nu_0$ and $\rho = 2$. By using (35) we obtain

$$L_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \tag{36}$$

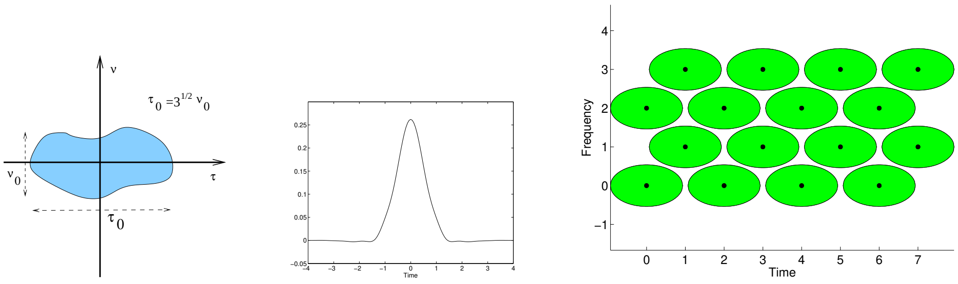
as generator matrix of the optimal lattice Λ_2. Using (31) and (29) with L_1 as in (33) and φ_1 as in (34) we compute $\alpha = \frac{1}{\sqrt[4]{3}}$, $\beta = 0$, hence

$$\varphi_2(t) = \mathcal{F}\mathcal{D}_{\alpha^{-1}} \mathcal{C}_{-\beta} \mathcal{F}^{-1} \varphi_1(t) = \mathcal{F}\mathcal{D}_{\alpha^{-1}} \varphi_1 = \mathcal{D}_\alpha \varphi_1(t), \tag{37}$$

where we have used the fact that $\mathcal{F}\mathcal{D}_{\alpha^{-1}} = \mathcal{D}_\alpha \mathcal{F}$. Thus the optimal LOFDM setup for the given conditions $\rho = 2, \tau_0 = \sqrt{3}\nu_0$ can be schematically represented as shown in Figure 1. Analog to Figures 3.2 and 3.2(b) the ellipses represent $\operatorname{supp}_\varepsilon(A\psi_2)_{k,l}$, where we have used the same ε as in those figures.

4. COMPARISON OF PERFORMANCE OF LOFDM VS OFDM

We compare the proposed LOFDM system to a standard pulse shaping OFDM system of (9) associated with the rectangular lattice. For both systems we use the pulse obtained from the Gaussian via $\operatorname{orth}(g^\sigma, \Lambda_R)$ and $\operatorname{orth}(g^\sigma, \Lambda_H)$ respectively. For the rectangular lattice the OFDM system obtained in



(c) Schematic plot of channel scattering function.

(d) Optimal LOFDM pulse shape.

(e) Time-frequency representation of LOFDM system.

Figure 1. Optimal LOFDM setup for a wireless channel with $\tau_0 = \sqrt{3}\nu_0$.

that way coincides with the setup suggested in.[12] In both cases we optimally adapt the pulse and the lattice parameters to the given delay spread and Doppler spread.

For the numerical simulations we assume the channel to be WSSUS.[11] We consider a Jakes-type Doppler behavior and exponential decay in temporal direction for the scattering function.[4] In addition we assume that $\tau_0\nu_0 < 1$, i.e., the channel is underspread.[11]

For both OFDM and OFDM We compute the Signal-to-Noise-Pulse-Interference-Ratio (SNIR) of the transmitted data and the received data (which have been equalized by a simple one-tap equalizer). The result is shown in Figure 2. As expected from the theoretical analysis LOFDM yields indeed a higher SIR than OFDM. On the average the improvement of LOFDM compared to OFDM is in the range of 1dB – 2dB.

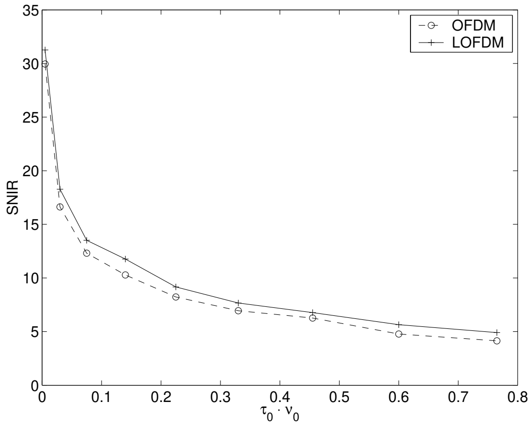


Figure 2. Comparison of Signal to Interference-plus-Noise Ratio (SINR) for OFDM and LOFDM. The proposed LOFDM scheme yields an improvement of 1dB–2dB.

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