ON THE DESIGN OF OPTIMAL SPREADING SEQUENCES FOR CDMA SYSTEMS

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ABSTRACT

Signatures constructed from Welch bound equality sequences have been shown to be optimal in terms of sum capacity in synchronous CDMA systems. Unfortunately, these codebooks are a function of the number of active signatures and need to be reassigned as the number of active users changes to maintain optimality. We describe and analyze two subclasses of Welch bound equality sequences from a theoretical and practical viewpoint. We show that maximum Welch bound equality sequences are equiangular and thus each active interferer experiences the same amount of interference. We show that codebooks constructed by concatenating multiple orthonormal basis come closer to the Welch bound when not all signatures are active. Discussions and illustrations for both constructions are presented.

1. INTRODUCTION

Direct spread CDMA systems use signature sequences to distinguish between the signals of different users. In single-cell synchronous systems, orthogonal sequences have the advantage that it eliminates interference between users. It has been shown that, however, that non-orthogonal signatures are sum-capacity optimal in synchronous single-cell CDMA systems since CDMA is soft-interference limited (that is, each user can tolerate a certain amount of interference) [1, 2]. Optimal sequence sets are known as Welch bound equality (WBE) sets and have been of considerable interest (see [3] and the references therein). Unfortunately, as mentioned in [3] and proven herein, WBE sequences in general are a function of the number of active signatures. Thus whenever a user leaves or a new user arrives, all the codes will have to be reassigned. Alternatively, the subset of sequences will no longer be optimal. Therefore it is of interest to find WBE sequence sets that do not have to be reassigned as a function of the number of active users.

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In this paper we consider two different sequence sets that perform well even when subsets of the available codes are active. First we introduce a subclass of the WBE class known as maximum Welch bound equality (MWBE) sets [4, 3]. Second we consider a different class of WBE sequences that we define as multi-ONB (multiple orthonormal basis) sequences. These sets are constructed from unions of orthonormal basis. The advantage of this construction is that the first m signatures are orthogonal thus in a lightly loaded system the first m users will not be interference limited. We analyze both sets in terms of interference behavior. Simulations compare both arbitrary WBE sets, MWBE sets, and multi-ONB sets in terms of proximity to the Welch bound on sum squared total correlation and variability of the interference.

Much of the prior work on quasi-orthogonal signature construction for CDMA systems considers signatures that are constructed from sequences with elements from a finite-field mapped to a complex constellation symbol. This simplifies implementation and leads to many interesting constructions [5, 6, 7]. In this paper, however, we follow the approach of Welch [4] and consider the most general case when signatures have elements that are drawn from either the real or the complex field. Multi-ONB sets are discussed in [5, 6, 7]. A general discussion of such quasi-orthogonal constructions is available in [7]. Finding new signatures to work with the existing Walsh signatures has been performed in [5] for IS-2000 and in [6] for WCDMA. In this paper we derive the maximum number of basis that may be unionized to form a multi-ONB set with the minimum maximum correlation. The optimality of WBE sets when minimum mean squared error (MMSE) decoding is used at the receiver for the uplink and downlink cases and a variety of power control strategies is derived in [8].

2. MATHEMATICAL PRELIMINARIES

In this paper we consider the design of codes for short-code direct sequence CDMA systems. Let $\mathbf{s}_n$ be a $m \times 1$ vector that contains a spreading sequence of length m and suppose there are $n = 1, 2, \ldots, N$ active users. We assume that $\|\mathbf{s}_n\| = 1$ for all n.

The inner product between two (complex) sequences $\mathbf{s}_k$ and $\mathbf{s}_l$ is defined as $\langle \mathbf{s}_k, \mathbf{s}_l \rangle = \overline{\mathbf{s}}_k^T \mathbf{s}_l$. The norm $\|\mathbf{s}\|$ is the 2-norm of $\mathbf{s}$ unless otherwise stated. The unit sphere in $\mathbb{R}^m$ or $\mathbb{C}^m$ is denoted by Ω^m.

We define the signature matrix $\mathbf{S} := [\mathbf{s}_1, \ldots, \mathbf{s}_N]$. The sum-correlation of $\mathbf{S}$ is defined as

$$\mathcal{A}(\mathbf{S}) = \sum_{k=1}^{N} \sum_{l=1}^{N} |\langle \mathbf{s}_k, \mathbf{s}_l \rangle|^2$$

The max-correlation of $\mathbf{S}$ is defined as $\max_{k,l\neq k} |\langle \mathbf{s}_k, \mathbf{s}_l \rangle|$. We call a set of vectors equiangular if $|\langle \mathbf{s}_k, \mathbf{s}_l \rangle| = c$ for all k, l with $k \neq l$ for some constant $c \geq 0$.

3. WBE QUASI-ORTHOGONAL SEQUENCE SETS

In this section we review the synchronous CDMA system that be of frequent mention in the duration of the paper. We then present the quasi-orthogonal sequence design problem. Finally, we summarize WBE quasi-orthogonal sequence sets and uncover their limitations.

3.1. Review of Synchronous CDMA

Welch bound equality sequences typically often occur in the context of synchronous direct spread CDMA systems. To illustrate, consider the downlink of a single cell synchronous CDMA system with no multipath. Assuming perfect synchronization and chip-rate sampling, the receiver for user k observes the $m \times 1$ vector

$$\mathbf{y} \;=\; h_k \, \sigma_s \sum_{n=1}^{N} \mathbf{s}_n s_n \;+\; \mathbf{v} \tag{1}$$

where s_n is the complex symbol transmitted by the n-th user, σ_s^2 is the transmitted signal energy, h is the complex channel gain, and $\mathbf{v}$ is an $m \times 1$ vector sample of an independent identically distributed (i.i.d.) circular symmetric complex Gaussian random vector with distribution $\mathcal{CN}(0, \sigma_v^2 \mathbf{I})$. Using a conventional matched filter at the receiver, user k forms the statistic

$$\begin{aligned} x =& \langle h_k \sigma_s \mathbf{s}_k, \mathbf{y} \rangle = |h_k|^2 \sigma_s^2 \langle \mathbf{s}_k, \mathbf{s}_k \rangle \\ &+ \sigma_s^2 |h_k|^2 \sum_{n\neq k} \langle \mathbf{s}_k, \mathbf{s}_n \rangle s_n \;+\; \langle \sigma_s h \mathbf{s}_n, \mathbf{v} \rangle. \end{aligned}$$

The usual performance metric for this system is the signal-to-noise plus interference (SINR) ratio given by

$$\mathrm{SINR}_k \;=\; \frac{1}{\frac{\sigma_v^2}{|h|^2 \sigma_s^2} \;+\; \sum_{n\neq k} |\langle \mathbf{s}_n, \mathbf{s}_k \rangle|^2} \tag{2}$$

for $k = 1, 2, \ldots, N$. Performance is degraded by the total interference $\sum_{n\neq k} |\langle \mathbf{s}_n, \mathbf{s}_k \rangle|^2$ caused when the signatures of all the users are not orthogonal.

3.2. Quasi-Orthogonal Sequence Construction

The quasi-orthogonal sequence design problem is to find a set of suitable spreading sequences $\{\mathbf{s}_n\}_{n=1}^N$ where in general $N \geq m > 1$. Selection of "optimal" codeword vectors depends on the design constraints, which are motivated by improved performance or simplified implementation. Design constraints that improve the performance of matched filters are based on the autocorrelation or cross correlation of the sequence set. Implementation properties also influence the choice of spreading codes. More details along these lines are available in [7, 5, 6].

3.3. WBE Sequences

Recently many researchers have pointed out the importance of the WBE property for quasi-orthogonal CDMA sequences, see e.g. [1, 2, 3]. Recall that the Welch bound provides a lower bound on the sum of mutual correlations for a set of N signature sequences $\mathbf{s}_k$ in $\mathbb{C}^m$ [4], i.e., $\mathcal{A}(\mathbf{S}) \geq \frac{N^2}{m}$. Sequences that satisfy this bound with equality are called Welch bound equality sequences. A necessary and sufficient condition for a set of sequences to achieve equality is that [9]

$$\mathbf{S}\mathbf{S}^* \;=\; \frac{N}{m}\mathbf{I}_m \;\text{ and }\; \|\mathbf{s}_n\| = 1 \;\text{ for }\; n = 1, 2, \ldots, N.$$

From [9, 3] WBE sequences have a number of nice properties. For example, they have been shown to maximize the sum-capacity of the synchronous CDMA channel [1]. They are also optimal when minimum mean squared error (MMSE) decoding is used at the receiver for the uplink and downlink cases and a variety of power control strategies [8].

Perhaps the most interesting property is that, using WBE sequences, the interference is uniform across all users [9]. The sum total interference in the system is given by $\mathcal{A}(\mathbf{S}) - N^2$ which for WBE sequences is simply $\frac{N^2}{m} - N$. Using WBE sequences, the interference seen by user k is

$$I(k) \;=\; \sum_{l=1}^{N} |\langle \mathbf{s}_k, \mathbf{s}_l \rangle|^2 - 1 = \frac{N - m}{m} \tag{3}$$

and is *the same for every user*. Thus the SIR performance for any user k simplifies to

$$\text{SINR}_k = \left[\left(\frac{|h_k|^2 \sigma_s^2}{\sigma_v^2} \right)^{-1} + \left(\frac{m}{N-m} \right)^{-1} \right]^{-1} \tag{4}$$

and the performance only depends on the channel for the k^{th} user.

Despite the interesting mathematical implications in synchronous CDMA systems, as mentioned in [3] WBE sequences seem less useful in practical implementations. Even if one constructs WBE that satisfy additional properties they have a serious drawback. If we design a WBE set for $N > m$ users, then this set almost always ceases to be a WBE set if any $M < m$ sequences are removed from or added to the set as shown in Theorem 1 (see also discussion in [3])

Theorem 1 *Let* $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_N]$ *be a set of WBE sequences of length* m *and assume* $N > m$. *If we remove any* $M < m$ *sequences from or add any* $M < m$ *unit-norm sequences to this set, then the resulting set has no longer the WBE property.*

This means when the number of users changes in a cell, the WBE set is no longer optimal. Thus a system that fully exploits WBE sequences would need (i) a set of sequences for every possible N and (ii) would need to reassign *all* sequences every time a user arrived or departed from the system.

WBE sequences have the nice property that when the SIR requirements for each user are identical, the matched filter corresponds to the MMSE receiver thus simplifying the receiver complexity. Therefore when $\tilde{N} < N$ users are active an alternative to reassigning new WBE sequences is to instead use MMSE receivers in lieu of matched filters. This option maintains high performance but it alleviates the natural low complexity advantage offered by the matched filter solution. If the matched filter is used with a subset of the WBE sequences then the uniformity of the SIR would be lost. This leads to the undesirable property that users would see different amounts of interference as a function of their sequence assignment.

To mitigate the problems caused by the loss of the WBE property we present two different additional side constraints to improve performance when the number of active users is fluctuating. First we consider maximum Welch bound equality (MWBE) sequences. These sequences form a subset of WBE sequences, have a nice relation to Grassmannian packings, and are especially uniform. Second we construct WBE sequences through unions of certain orthonormal basis. While this approach has been taken by other authors [7, 5, 6], we present optimal constructions without restricting the entries to have identical constellations.

4. MWBE QUASI-ORTHOGONAL SEQUENCE SETS

4.1. Review of MWBE Sequences

The importance of general WBE sequence sets for S-CDMA has overshadowed the utility of a subclass of these sequences known as MWBE sequences. Recall that the Welch bound as originally stated is a lower bound on the maximum correlation for a set of N signature sequences $\mathbf{s}_k$ in $\mathbb{C}^m$ [4] defined as

$$\rho(N, m) := \max_{k,l,k \neq l} |\langle \mathbf{s}_k, \mathbf{s}_l \rangle|. \tag{5}$$

This problem is equivalent to finding a lower bound on the maximum angle between the lines generated by $\mathbf{s}_k$ and $\mathbf{s}_l$. In particular, this problem relates to the notion of packings of subspaces in Grassmannian spaces, see [10].

We summarize results from areas such as spherical design, algebraic geometry, graph theory, and sphere packings in the following theorem (e.g., see [10]).

Theorem 2 *Let* $\mathcal{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_N\}$ *be a set of unit vectors in* E^m *with* $N \geq m$. *Then*

$$\rho(N, m) \geq \sqrt{\frac{N-m}{m(N-1)}}. \tag{6}$$

If $E = \mathbb{R}$ *then equality in* (6) *can only hold if* $N \leq m(m+1)/2$ *and if* $E = \mathbb{C}$ *then equality in* (6) *can only hold if* $N \leq m^2$. *Furthermore if equality holds in* (6) *then the vectors in* $\mathcal{U}$ *are equiangular.*

Sequences for which the bound in (6) holds with equality are commonly known as MWBE sequences. It can be checked that they are a subclass of WBE sequences and thus have all the properties of WBE sequences in terms of capacity and uniformity of SIR mentioned in the previous section.

The relation between MWBE signatures and packings of lines is useful to provide insights not previously provided in other work on MWBE signature sets (see [3] and the references therein). For example, Theorem 2 provides bounds on the size of N over which such signatures exist. For example, sets that satisfy equality are guaranteed not to exist for certain choices of N and m, which also depends on whether the signatures are drawn from real or complex fields. Also, the fact that MWBE signatures are optimal packings reveals the difficulty in finding good candidate sequences.

4.2. Utility of MWBE Sequence Sets Versus Arbitrary WBE Sequence Sets

MWBE sequences have a number of properties that make them preferable over an arbitrarily chosen WBE sequence set. Perhaps the most interesting property is that with MWBE sequences, the interference level is only a function of the current number of active users and the total number of codes available. This is a byproduct of the equiangular property and is stated in the following Theorem.

Theorem 3 *The interference level for equiangular sequence sets is the same for all sequences and only depends on the total number of active sequences.*

Of course, there are many possible sets of equiangular sequences. A byproduct of Theorem 2, however, is that MWBE sequences are the best of all equiangular sequences.

The selection of MWBE sequence sets has a number of advantages of WBE sets. Unfortunately, in some applications the difficulty of code construction and the potential for non-constant modulus entries make it difficult to choose MWBE sequences in all applications. Therefore we investigate another subclass of WBE sequences in the following section.

5. MULTI-ONB SEQUENCE SETS

5.1. Review of Multi-ONB Sequences

Multi-ONB sequences are signatures that are constructed from unions of multiple orthonormal bases. They are useful when it is necessary to augment an existing orthogonal basis with additional codes. The main idea is to have a core set of m orthonormal sequences to which we add sequences from another orthonormal basis (ONB) if more than m codes have to be accommodated. To minimize the interference between the sets of bases (and for other reasons) it is useful to design these ONBs such that the correlation between members of different ONBs is minimal, i.e., equals $\frac{1}{m}$. These multi-ONB sequences have the nice property that for $\bar{N} \leq m$ users, there is no interference. For any $\bar{N}$ that is a multiple of m they are optimal from the point-of-view of sum capacity. They are also optimal for $N = \bar{N}$ and thus retain all the aforementioned properties of WBE sets. It is clear, however, that they do not have the constant interference property of MWBE sequences.

Multi-ONB codes have been of interest in practical systems since they are a convenient way to augment the existing set of Walsh codes. Typically a second basis is constructed by applying a cover code to the Walsh codes. The cover code is chosen so that the new set of orthogonal codes has optimal correlation with Walsh codes of the same length and further has low correlation to Walsh codes of smaller lengths. The latter property is a byproduct of the multi-spreading factor in CDMA 2000 and WCDMA. A thorough discussion of cover code construction for the binary case is treated in [7]. Extension to QPSK codes and application to IS-2000 is discussed in more detail in [5]. Analysis of certain previously proposed secondary spreading factors for WCDMA is available in [6].

5.2. Analysis of Multi-ONB Sequences

Construction of a multi-ONB sequence set begins with a set of m orthonormal signatures. First we consider the special case of adding $n \leq m$ signatures to this set. This is the approach taken in WCDMA [6] and IS-2000 [5]. In particular, we would like to know how to chose these n signatures and how far the resulting set is from a WBE set. This will determine the loss in optimality (if any). The following Theorem addresses this point.

Theorem 4 *Let $U = [u_1, \ldots, u_m]$ with $UU^* = U^*U = I_m$ and let $V_n = [v_1, \ldots, v_n]$ with $\|v_k\| = 1$ for $k = 1, \ldots, n$, $n \leq m$. Denote $W_n = [U, V_n]$. Then $\mathcal{A}(W_n)$ is minimized if $V_n^* V_n = I_n$. In this case*

$$\mathcal{A}(W_n) = \frac{(m+n)^2 + n(m-n)}{m}. \tag{7}$$

Thus if we want to add n sequences to an ONB, then the n sequences should form an orthonormal system. Note that the Welch bound for $m+n$ vectors is $\frac{(m+n)^2}{m}$. Thus if we add $n = m$ signatures, and the new signatures form an ONB, then the new set is also a WBE sequence set. If we add $n < m$ signatures then the term $\frac{n(m-n)}{m}$ gives the additional penalty factor.

Theorem 4 show how close multi-ONB sequence sets are to WBE sequence sets. However, they do not indicate how to construct the additional ONBs. Replicating the original ONB is not practical since this would lead to rather poor SIR performance. Several researchers considered enlarging an orthonormal set of CDMA codes $\mathbf{S}$ by sequences $\mathbf{u}_{m+1}, \ldots, \mathbf{u}_n$ such that the new sequences form an orthonormal set and $|\langle \mathbf{s}_k, \mathbf{u}_l \rangle| = 1/\sqrt{m}$ for $k = 1, \ldots, m$ and $l = 1, \ldots, n$, see [7, 5]. It is known that this is the minimum distance that can occur between pairs of ONBs. Prior work on quasi-orthogonal sequences for CDMA has emphasized adding a secondary ONB to a primary ONB. It is of interest though to know how many ONBs can be concatenated and yet retain the minimum inner product. The following theorem summarizes results from [11] by Calderbank, Cameron, Kantor, and Seidel.

Theorem 5 *Let $q = p^m$ where p is a prime number. Then there exist $q+1$ orthonormal bases $U_1, \ldots, U_{q+1}$ for $\mathbb{C}^q$, such that*

$$|\langle \mathbf{s}_k, \mathbf{s}_l \rangle| = \begin{cases} 0 & \text{if } \mathbf{s}_k \in U_r, \mathbf{s}_l \in U_s, r = s, k \neq l, \\ \frac{1}{\sqrt{q}} & \text{if } \mathbf{s}_k \in U_r, \mathbf{s}_l \in U_s, r \neq s. \end{cases}$$

If $p = 2$ then the entries of the vectors which do not belong to the standard basis take are entries from the set $\{\frac{1}{\sqrt{p^m}} e^{2\pi jk/4}\}_{k=0}^{3}$. If $p > 2$ then the entries of the vectors which do not belong to the standard basis are entries from the set $\{\frac{1}{\sqrt{p^m}} e^{2\pi jk/p}\}_{k=0}^{p-1}$.

6. COMPARISON OF WBE, MWBE, AND MULTI-ONB SIGNATURE SELECTION

For this comparison we consider $m = 64$ and $n = 2*m$. The purpose is to illuminate the variability in the interference in the different sequence sets as we increase the number of active sequences from m to n. We begin with m signatures. For the WBE nd MWBE sequence set, we choose the m signatures that have $\mathbf{S}$ with the smallest condition number. For the multi-ONB set, we pick any m signatures that form an orthonormal basis. Then we add we $0 < p \leq m$ vectors, randomly chosen, to that set and compute $I(k)$ for each signature $k = 1, 2, \ldots, m+p$. For each set p and randomly chosen set of vectors we compute $I_{min} := \min_k I(k)$, $I_{max} := \max_k I(k)$, $I_{avg} := \frac{1}{m+p}\sum_{l=1}^{m+p} I(l)$, and $\mathcal{A}(\mathbf{S})$. We averaged each of these metrics over 20 different randomly chosen sets of vectors.

In Fig. 1 we plot the results of the interference comparison. As expected, the signatures in the MWBE set experience the same amount of interference due to the equiangular property. The WBE set has a mean interference level that is almost the same as the MWBE set. However, the interference experienced by each code varies between the maximum and minimum average interference curves. This extra variability creates additional fluctuations in SIR that must be reduced in the power control algorithms. The multi-ONB set performs quite differently than the MWBE set. The average interference for the multi-ONB set is actually lower than the MWBE or the WBE sets. However the maximum and minimum are significantly more extreme.

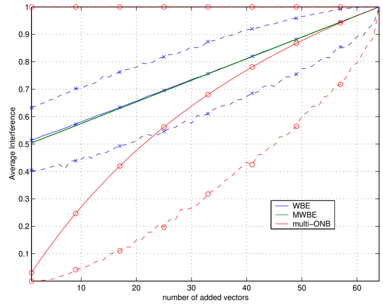

Fig. 1. Comparison of the average mean, maximum, and minimum interference levels as the number of active codes increases.

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