Grassmannian Beamforming for Multiple-Input Multiple-Output Wireless Systems*

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June 3, 2003

Abstract: Transmit and receive beamforming is an attractive, low-complexity technique for exploiting the significant diversity that is available in multiple-input and multiple-output (MIMO) wireless systems. Unfortunately, optimal performance requires either complete channel knowledge or knowledge of the optimal beamforming vector which is difficult to realize in practice. In this paper, we propose a quantized maximum signal-to-noise ratio beamforming technique where the receiver only sends the label of the best weight vector in a predetermined codebook to the transmitter. We develop optimal codebooks for i.i.d. Rayleigh fading matrix channels. We derive the distribution of the optimal transmit weight vector and use this result to bound the SNR degradation as a function of the quantization. A codebook design criterion is proposed by exploiting the connections between the quantization problem and Grassmannian line packing. The design criterion is flexible enough to allow for side constraints on the codebook vectors. Bounds on the maximum distortion with the resulting Grassmannian codebooks follow naturally from the nature of the code. A proof is given that any system using an overcomplete codebook for transmit diversity and maximum ratio combining obtains a diversity on the order of the product of the number of transmit and receive antennas. Bounds on the loss in capacity due to quantization are derived. Monte Carlo simulations are presented that compare the symbol error probability for different quantization strategies.

Index Terms-Diversity methods, MIMO systems, Rayleigh channels, Grassmannian line packing, limited feedback

*This work will appear in part in the *Proc. of the* 36^{th} *IEEE Asilomar Conf. on Sigs., Sys., and Comps.*, the *Proc. of the IEEE International Conf. on Comm.*, and the *Proc. of the* 40^{th} *Annual Allerton Conf. on Comm., Control, and Comp.*

†David J. Love was supported by a Microelectronics and Computer Development Fellowship and a Cockrell Doctoral Fellowship through The University of Texas at Austin. This material is based in part upon work supported by the Texas Advanced Research (Technology) Program under Grant No. 003658-0614-2001, the National Instruments Foundation, and the Samsung Institute of Applied Technology.

1 Introduction

Beamforming, or optimal combining, is a low complexity technique for exploiting the spatial diversity available from multiple antennas and can be employed at both the transmitter and the receiver [1]. Optimal combining at the receiver for single-input multiple-output (SIMO) wireless channels is well understood (see [2] and the references therein). Optimal combining for multiple-input single-output (MISO) channels is similar, with the important difference that full channel knowledge is required at the transmitter.

Multiple-input multiple-output (MIMO) wireless systems are of practical and theoretical interest primarily because of their ability to provide dramatic capacity gains over single antenna wireless systems [3], [4]. For this reason there is interest in MIMO systems in both third-generation wireless [5] and wireless LAN [6] systems. While the capacity gain of a MIMO system is most often thought of as manifesting in operation as a multiplexing system [7] where multiple symbol streams are transmitted in parallel, MIMO systems can provide resiliency against fading effects by operating as diversity only systems. This diversity only mode is understood to mean that only one substream is sent of the matrix channel via transmit beamforming and receive combining, both generalizations of the vector channel combining methods. Beamforming in MIMO communication channels, however, requires the selection of both transmit and receive antenna weights and naturally requires knowledge of the transmit beamforming weights at the transmitter ([8], [9], [10], [11]).

In this paper we consider beamforming in a narrowband MIMO communication system where the channel is well modeled by a Rayleigh fading matrix. The optimal beamforming solution for this system (in the sense of maximizing the received signal-to-noise ratio (SNR)) is to use the dominant right singular vector as the transmit beamforming vector and the dominant left singular vector as the optimum combiner ([11], [12]). This is the natural extension of maximum ratio combining [13] in SIMO channels and maximum ratio transmission [14] in MISO channels. Of course, this scheme requires full and complete knowledge of this beamforming vector (or equivalently full channel knowledge) at the transmitter. Further the feedback channel typically has low bandwidth thus requiring extremely coarse quantization of the transmit weight vectors. One approach to reducing complexity is to impose side constraints on the beamforming vectors to reduce the parameter space, e.g. selection diversity transmission (SDT) [15] or equal gain transmission (EGT) [8]. While quantization is easy in the case of selection diversity, optimal low-rate quantization strategies for more general MIMO transmission weighting schemes remain elusive.

Prior work has addressed the problem of quantizing the optimal beamforming vectors for MISO communication channels. Limited feedback equal gain transmit diversity weighting was explored in [16] and is also a part of the WCDMA closed-loop diversity mode [5]. Extension of the approach in [16] to the case of MIMO communication channels was recently proposed in ([8], [9]) and is called quantized equal gain transmission (QEGT). Unfortunately, ([8], [9]) use a codebook design method that is suboptimal since the designed systems were required to use codebooks containing M_t orthogonal vectors in order to satisfy the supposition for the proof of diversity order. Quantized maximum ratio transmission (QMRT) systems were first addressed in [17]. The transmit diversity codebooks proposed therein were obtained using the Lloyd Algorithm and a specific code design methodology was not developed. Due to the key role that multiple-input multiple-output (MIMO) technology will play in next generation wireless systems [18], finding efficient codebooks for quantizing the optimum transmit weight vector is paramount. Therefore in this paper a general codebook design criterion is developed for quantized beamforming systems.

In this paper we consider the problem of quantizing the optimum beamforming vector in independent identically distributed (i.i.d.) MIMO Rayleigh fading channels and conveying the vector from the receiver to transmitter used a fixed and limited number of bits. We derive the distribution of the optimal maximum ratio transmission weight vector for i.i.d. MIMO Rayleigh channels and show that it is uniformly distributed on the complex unit sphere. We then derive a codebook design criterion for a fixed amount of feedback by minimizing the average SNR degradation introduced by quantization. Using that fact that beamforming vectors that lie on the same line in a complex vector space provide the same performance and the isotropic distribution of the optimal beamforming vector, we show that the codebook should be designed by thinking of the codebook as a set, or packing, of lines. The SNR degradation bound then relates directly to the famous applied mathematics problem of Grassmannian line packing.

Recall that Grassmannian line packing is the problem of spacing N lines that pass through the origin in order to maximize the sine of the minimum angular separation between any two lines. Each of these lines can be represented by a unit vector whose entries are the coordinates of a point where the complex unit sphere and the line intersect. This paper addresses several new results in the area of Grassmannian line packing. We derive a closed form expression for the density of line packings based on probability of outage results in [19]. This result verifies the asymptotic subspace packing results presented in [20] and allows us to rederive the Hamming bound on the codebook minimum distance and the Gilbert-Varshamov bound on codebook size.

Grassmannian subspace packing techniques have previously been used in other areas of multi-antenna system design. The problem of designing multi-antenna constellations was shown to relate to Grassmannian subspace packing in [21]. In [22] it was shown that the capacity of a multi-antenna system fading system where the transmitter and receiver do not know the fading coefficients has a geometrical interpretation in the form of sphere packing on the Grassmannian manifold. We show that quantized beamforming for multiple transmit antenna systems relates directly to the problem of maximally spacing one-dimensional subspaces in the complex vector space. Interestingly, this result was proven for MISO systems using different derivation techniques concurrently and independently of ourselves in [19], [23], [24].

A common way to qualify the performance of a beamforming scheme is to inspect the diversity order, i.e. the asymptotic slope of the average probability of symbol error curve. This paper proves that all quantized transmit diversity systems that use an overcomplete codebook (i.e. a codebook whose codevectors span the M_t- dimensional complex vector space where M_t is the number of transmit antennas) with maximum ratio combining at the receiver achieve full diversity advantage. To the authors' knowledge this result has never been proven for an arbitrary codebook. It has, however, been proven for specific cases such as selection diversity transmission [15] or equal gain transmission [8]. We use various bounds on the minimum distance of Grassmannian line packings as a function of the codebook size to derive approximate lower bounds on the number of codevook vectors (and thus bits) necessary to minimize the quantization loss for two different criteria: capacity and SNR loss due to quantization.

This paper is organized as follows. Section 2 overviews MIMO weighting diversity systems. Grassmannian subspace packing and spherical codes are introduced in Section 3. Section 4 looks at the distribution of an optimal weight vector and ties the probabilistic results to Grassmannian line packing and proposes design criteria. A performance analysis of quantized transmit diversity systems is given in Section 5. Section 6 presents Monte Carlo simulation results. The paper concludes in Section 7.

2 System Overview

A MIMO system with transmit beamforming and receive combining is described by Fig. 1 with M_t transmit antennas and M_r receive antennas. A symbol s ($s \in \mathbb{C}$, the field of complex numbers) to be transmitted is multiplied by w_l ($w_l \in \mathbb{C}$) at the l^{th} ($0 < l \leq M_t$) transmit antenna. The weighted data received by the k^{th} ($0 < k \leq M_r$) receive antenna, sent by the l^{th} transmit antenna, is multiplied by a gain $H_{[k,l]}$ where $H_{[k,l]}$ is circularly symmetric complex Gaussian distributed according to $\mathcal{CN}(0, \sigma^2)$. This paper assumes $H_{[k,l]}$ is independent of $H_{[m,n]}$ if $k \neq m$ or $l \neq n$. This channel model is reasonable when there is a significant amount of scattering [25]. The data received by the k^{th} receive antenna is added with white Gaussian noise n_k where n_k is distributed according to $\mathcal{CN}(0, N_0)$ and then multiplied by $\overline{z_k}$ ($z_k \in \mathbb{C}$ with $\bar{\cdot}$ denoting conjugation). The weighted output of each of the M_r receive antennas is summed yielding the combiner output x.

This formulation allows the MIMO system to be written in matrix form. Let $\mathbf{w} = [w_1\ w_2\ \ldots\ w_{M_t}]^T$ (with ^{T} denoting matrix transposition), $\mathbf{z} = [z_1\ z_2\ \ldots\ z_{M_r}]^T$, $\mathbf{n} = [n_1\ n_2\ \ldots\ n_{M_r}]^T$, and $\mathbf{H}$ be the $M_r \times M_t$ matrix with coordinate (k, l) equal to $H_{[k,l]}$. Then the combiner output is $x = (\mathbf{z}^H \mathbf{H} \mathbf{w})s + \mathbf{z}^H \mathbf{n}$ where ^{H} denotes the matrix conjugate transpose.

Given the system in Fig. 1, the key question is how to design $\mathbf{w}$ and $\mathbf{z}$ to maximize performance. It has been shown [26] that $\mathbf{w}$ and $\mathbf{z}$ should be chosen to maximize the SNR in order to minimize the average probability of error and maximize the capacity. For the proposed system, the SNR, γ_r,

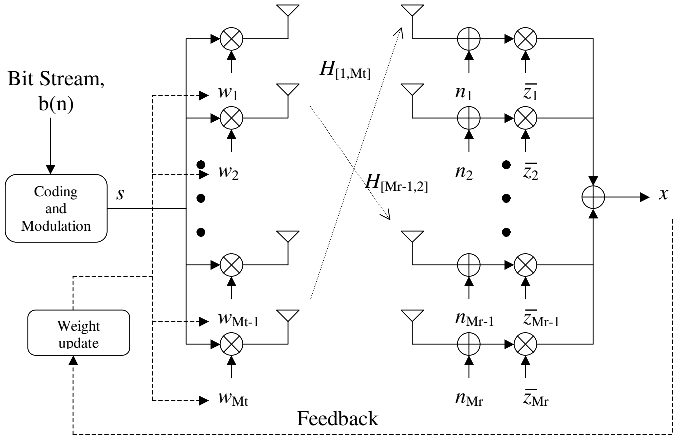

Figure 1: Block diagram of a MIMO system.

is

$$\gamma_r = \frac{\mathcal{E}_t|\mathbf{z}^H\mathbf{H}\mathbf{w}|^2}{\|\mathbf{z}\|_2^2 N_0} = \frac{\left(\mathcal{E}_t\|\mathbf{w}\|_2^2\right)\left|\frac{\mathbf{z}}{\|\mathbf{z}\|_2}^H\mathbf{H}\frac{\mathbf{w}}{\|\mathbf{w}\|_2}\right|^2}{N_0} \tag{1}$$

where $\|\cdot\|_2$ is the matrix 2-norm. Notice that in (1), $\|\mathbf{z}\|_2$ factors out therefore we can without loss of generality fix $\|\mathbf{z}\|_2 = 1$. We also can see that the transmitter transmits with total energy $\mathcal{E}_t\|\mathbf{w}\|_2^2$. Therefore, due to power constraints at the transmitter, we can take $\|\mathbf{w}\|_2 = 1$ and assume $\mathcal{E}_t$ is held constant. Now the receive SNR γ_r can be expressed as

$$\gamma_r = \frac{\mathcal{E}_r}{N_0} = \frac{\mathcal{E}_t|\mathbf{z}^H\mathbf{H}\mathbf{w}|^2}{N_0} = \frac{\mathcal{E}_t\mathbf{\Gamma}_r}{N_0} \tag{2}$$

where $\mathbf{\Gamma}_r = |\mathbf{z}^H\mathbf{H}\mathbf{w}|^2$ is the effective channel gain.

The beamforming vector $\mathbf{w}$ can be designed in many ways. Three common weighting methods are maximum ratio transmission, equal gain transmission, and generalized selection diversity transmission. A transmitter where $\mathbf{w}$ maximizes $|\mathbf{z}^H\mathbf{H}\mathbf{w}|$ for an arbitrary $\mathbf{z}$ is called maximum ratio transmission (MRT). A transmitter where w_l satisfies $|w_l| = \frac{1}{\sqrt{M_t}}$ for $0 < l \leq M_t$ is called equal gain transmission (EGT). This definition allows $\mathbf{w}$ to be expressed as $\mathbf{w} = \frac{1}{\sqrt{M_t}}e^{j\boldsymbol{\theta}}$ where $\boldsymbol{\theta} = [\theta_1\ \theta_2\ ...\ \theta_{M_t}]^T$ and $\theta_k \in [0, 2\pi)$. A transmitter where $\widetilde{\mathbf{w}}$ is the sum of columns of $\mathbf{I}_{M_t}$, the $M_t \times M_t$ identity matrix, and $\mathbf{w} = \widetilde{\mathbf{w}}/\|\widetilde{\mathbf{w}}\|_2$ is called generalized subset selection (GSS).

In MIMO channels, unlike in MISO channels, the receive vectors also need to be chosen. A receiver where $\mathbf{z}$ maximizes $|\mathbf{z}^H\mathbf{H}\mathbf{w}|$ for an arbitrary $\mathbf{w}$ is called a maximum ratio combiner (MRC). The form of this vector can be seen using the vector norm inequality

$$|\mathbf{z}^H\mathbf{H}\mathbf{w}|^2 \leq \|\mathbf{z}\|_2^2\|\mathbf{H}\mathbf{w}\|_2^2. \tag{3}$$

We already defined $\|\mathbf{z}\|_2^2 = 1$ thus the MRC vector must set

$$|\mathbf{z}^H\mathbf{H}\mathbf{w}|^2 = \|\mathbf{H}\mathbf{w}\|_2^2. \tag{4}$$

This is easily seen to be the unit vector $\mathbf{z} = \mathbf{H}\mathbf{w}/\|\mathbf{H}\mathbf{w}\|_2$. This paper will assume that the receiver always uses maximum ratio combining.

Given no design constraints on the form of the unit vectors $\mathbf{w}$ or $\mathbf{z}$ and a fixed N_0 the optimal solutions in an average probability of symbol error sense are the transmit and receive weights that maximize $\mathcal{E}_r$. For a combining scheme that solves for transmit weight vector $\mathbf{w}$ using the feasible set $\mathcal{W}$ ($\mathcal{W} \subseteq \Omega_{M_t}$ with Ω_{M_t} denoting the set of unit vectors in $\mathbb{C}^{M_t}$) with an MRC receiver, $\mathbf{w}$ is given by

$$\mathbf{w} = \arg\max_{\mathbf{x}\in\mathcal{W}} \|\mathbf{H}\mathbf{x}\|_2 \tag{5}$$

where arg max returns a global maximizer. Notice that if $\mathcal{W} = \Omega_{M_t}$, the case for an MRT system, then $\mathbf{w}$ is the dominant right singular vector of $\mathbf{H}$, the right singular vector of $\mathbf{H}$ corresponding to the largest singular value of $\mathbf{H}$ [27].

In this paper we consider a frequency division duplexing link where the receiver must send $\mathbf{w}$ back to the transmitter. Since $\mathbf{w}$ can be any unit vector in a possibly large feasible set (Ω_{M_t} for MRT) it is necessary to introduce some method of quantization due to the limited reverse-link feedback channel. A reasonable solution is to let the receiver and transmitter both use a codebook of N vectors. This allows the receiver to replace the costly singular value decomposition (SVD) of $\mathbf{H}$ needed for an MRT system [8] with a simple brute force search of the N vectors. The main benefit of using a reduced feasible set for $\mathbf{w}$ is that the number of feedback bits can be kept to a manageable number given by $\lceil \log_2 N \rceil$. Unfortunately, it is not obvious which N vectors should be included in the codebook.

To compare the performance of different quantized and unquantized weighting schemes we will use the average probability of symbol error defined as $\overline{P_e} = E_{\mathbf{H}}[P_e]$ where the expected value of the probability of symbol error P_e is taken over the channel $\mathbf{H}$. Two measures that are relevant when comparing average probability of symbol error are array gain and diversity gain. A system is said to have *array gain* A and *diversity gain* D if for SNR $\gg 0$ the average probability of symbol error is inversely proportional to $A(E_t/N_0)^D$ [28].

3 Grassmannian Line Packing

To provide sufficient background for the duration of this paper, in this section we summarize the key results from Grassmannian line packing. These applied mathematics problems will form the basis for our quantized beamforming codebook design method.

By the restrictions imposed, $\mathbf{w}$ is restricted to lie in the set Ω_{M_t}. Note that we can define an equivalence relation between two unit vectors defined by $\mathbf{w}_1 \equiv \mathbf{w}_2$ for some $\theta \in [0, 2\pi)$ $\mathbf{w}_1 = e^{j\theta}\mathbf{w}_2$. This equivalence relation says that two vectors are equivalence if they lay on the same line in $\mathbb{C}^{M_t}$. The quotient space with respect to this equivalence relation is called the set of all one-dimensional subspaces in $\mathbb{C}^{M_t}$ [22].

Definition: The complex Grassmann manifold $\mathcal{G}(m, 1)$ is the set of all one-dimensional subspaces of the space $\mathbb{C}^m$.

We can define a distance function on $\mathcal{G}(m, 1)$ by defining the distance between two lines the lines generated from unit vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ to be the sine of the angle $\theta_{1,2}$ between the two lines. This distance can also be expressed in the following way [20]

$$d(\mathbf{w}_1, \mathbf{w}_2) = \sin(\theta_{1,2}) = \sqrt{1 - |\mathbf{w}_1^*\mathbf{w}_2|^2}.$$

The *Grassmannian line packing problem* is the problem of finding the set, or packing, of N lines in $\mathbb{C}^m$ that has maximum minimum distance between any pair of lines. Because of the relation to Ω_m, the problem simplifies down to arranging N unit vectors so that the magnitude correlation between any two vectors is as small as possible. We will represent a packing of N lines in $\mathcal{G}(m, 1)$

by an $m \times N$ matrix $\mathbf{W} = [\mathbf{w}_1 \ \mathbf{w}_2 \ \ldots \ \mathbf{w}_N]$ where $\mathbf{w}_i$ is the vector in Ω_m whose column space is the i^{th} line in the packing.

The minimum distance of a packing is the sine to smallest angle between any pair of lines. This is written as

$$\delta(\mathbf{W}) = \min_{1 \le k \le l \le N} \sqrt{1 - \left|\mathbf{w}_k^* \mathbf{w}_l\right|^2} = \sin(\theta_{\min}) \tag{6}$$

where $\theta_{\min}$ is the smallest angle between any pair of lines in the packing. The problem of finding algorithms to design packings for arbitrary m and N has been studied by many researchers in applied mathematics (see [21], [29], [30] etc.). The Rankin bound (see for example [20], [29], [30]) gives a upper bound on the minimum distance for line packings as a function of m and N. In particular, this bound is given by [20], [30]

$$\delta(\mathbf{W}) \le \sqrt{\frac{(m-1)N}{m(N-1)}}. \tag{7}$$

The minimum distance is not the only quality measure that can be obtained from a line packing. The *covering radius* of a spherical code is defined as the most distant line in $\mathcal{G}(m,1)$ from any of the lines in the packing. This can be written as

$$\textit{covering radius} = \max_{\mathbf{x} \in \Omega_m} \min_{1 \le i \le N} \sqrt{1 - |\mathbf{x}^H \mathbf{w}_i|^2}. \tag{8}$$

The covering radius of a line packing is a worst case measure of a packing. A more useful measure is the density, which shows the performance on average of a line packing.

To define the density of a line packing, we must first define the notion of a metric ball in $\mathcal{G}(m,1)$. The ball of radius γ in $\mathcal{G}(m,1)$ around the line generated by $\mathbf{w}_i$ is defined as

$$\mathcal{B}_{\mathbf{w}_i}(\gamma) = \{\mathcal{P}_{\mathbf{v}} \in \mathcal{G}(m,1) \ : \ d(\mathbf{v}, \mathbf{w}_i) < \gamma\} \tag{9}$$

where $\mathcal{P}_{\mathbf{v}}$ is the line generated by $\mathbf{v} \in \Omega_m$. Note

$$\mathcal{B}_{\mathbf{w}_k}(\gamma) \cap \mathcal{B}_{\mathbf{w}_l}(\gamma) = \phi \tag{10}$$

when $\gamma \le \delta(\mathbf{W})/2$ where ϕ is the null set. Metric balls in $\mathcal{G}(m,1)$ can be geometrically visualized as spherical caps on Ω_m. Thus the ball $\mathcal{B}_{\mathbf{w}_k}(\gamma)$ is the set of lines generated by all vectors on the unit sphere that are within a chordal distance of γ from any point in $\Omega_m \cap \mathcal{P}_{\mathbf{w}_i}$.

The normalized Haar measure on Ω_m introduces a normalized invariant measure μ on $\mathcal{G}(m,1)$. This measure allows us to compute volumes in $\mathcal{G}(m,1)$, and thus will be of use in defining line packing density. The density of a line packing is defined as

$$\Delta(\mathbf{W}) = \mu\left(\bigcup_{i=1}^{N} \mathcal{B}_{\mathbf{w}_i}(\delta(\mathbf{W})/2)\right) = \sum_{i=1}^{N} \mu\left(\mathcal{B}_{\mathbf{w}_i}(\delta(\mathbf{W})/2)\right) = N\mu\left(\mathcal{B}(\delta(\mathbf{W})/2\right) \tag{11}$$

where $\mathcal{B}(\delta(\mathbf{W})/2)$ is an arbitrarily centered metric ball of radius $\delta(\mathbf{W})/2$.

Closed-form expressions for the density of Grassmannian subspace packings are often difficult to obtain [20]. However, we can give a closed-form expression for the density of line packings.

Theorem 1 *For any line packing in $\mathcal{G}(m,1)$,*

$$\Delta(\mathbf{W}) = N\left(\delta(\mathbf{W})/2\right)^{2(m-1)}. \tag{12}$$

Proof This reult will be proven using the outage probability results for MISO quantized beamforming in [19]. Let $\mathcal{C}_{\mathbf{w}_i}(\gamma) = \{\mathbf{v} \in \Omega_m \ : \ d(\mathbf{v}, \mathbf{w}_i) < \gamma\}$. Using our previous observation

$$\mu\left(\mathcal{B}_{\mathbf{w}_i}(\gamma)\right) = \frac{A(\mathcal{C}_{\mathbf{w}_i}(\gamma)}{A(\Omega_m)} \tag{13}$$

where $A(\cdot)$ is a function that computes area. It was shown in [19] that

$$\frac{A(\mathcal{C}_{\mathbf{w}_i}(\gamma)}{A(\Omega_m)} = (\gamma)^{2(m-1)}. \tag{14}$$

The result then follows. ∎

The result in Theorem 1 provides interesting insight into the rate at which the density grows as a function of the minimum distance. This result specifically verifies the asymptotic results in [20] for the one-dimensional subspace case.

This bound also allows us to derive a new upper bound on the minimum distance of Grassmannian line packings.

Theorem 2 *For any N line packing in $\mathcal{G}(m,1)$ we have*

$$\delta(\mathbf{W}) \leq 2\left(\frac{1}{N}\right)^{1/(2(m-1))}. \tag{15}$$

Proof This follows by using the Hamming bound on codesize,

$$N\mu(\mathcal{B}(\delta(\mathbf{W})/2)) \leq 1. \tag{16}$$

∎

We can also derive bounds on the existence of line backings of arbitrary radius.

Theorem 3 *There exists a packing of N lines in $\mathcal{G}(m,1)$ with minimum distance greater than δ if*

$$N < 1 + \delta^{-2(m-1)}. \tag{17}$$

Proof This is the Gilbert-Varshamov bound applied to line packings [20]. This is derived by observing that if $M\mu(\mathcal{B}(\delta)) < 1$, then there exists an $M+1$ line packing of minimum distance greater than δ. ∎

Finding the global maximizer of the minimum distance for arbitrary m and N is often difficult both analytically and numerically [30]. For this reason it is often most practical to resort to random computer searches (e.g., see the extensive tabulations on [31] that have been proposed for the real case). There are fortunately some cases however when exact solutions are available. Here we summarize a useful construction from [29] based on conference matrices.

4 Codebook Analysis and Design

In [8] it is shown that an optimal transmit weight vector for MRT systems is the dominant right singular vector of $\mathbf{H}$ with $\mathbf{H}$ defined as in Section 2. Therefore, $\mathbf{w}_{MRT}$ that satisfies (5) ($\mathcal{W} = \Omega_{M_t}$) is an optimal MRT solution. Another restatement of this is

$$\mathbf{w}_{MRT} = \arg \max_{\mathbf{x} \in \Omega_{M_t}} |\mathbf{x}^H \mathbf{H}^H \mathbf{H} \mathbf{x}|^2. \tag{18}$$

It is also important to note that if $\mathbf{w}_{MRT}$ satisfies (18), then $e^{j\phi}\mathbf{w}_{MRT}$ also satisfies (18) since $|\mathbf{w}_{MRT}^H \mathbf{H}^H \mathbf{H} \mathbf{w}_{MRT}|^2 = |e^{-j\phi}\mathbf{w}_{MRT}^H \mathbf{H}^H \mathbf{H} e^{j\phi}\mathbf{w}_{MRT}|^2$. Thus the beamforming solution is not unique.

Let us define this property in terms of points on a complex line instead of vectors. Let $\mathcal{P}_{\mathbf{w}}$ denote the column space of the vector $\mathbf{w}$. Because of the properties of the absolute value function, if $\mathbf{w} \equiv \tilde{\mathbf{w}}$ (using the equivalence relation defined in Section 3) then systems using $\mathbf{w}$ and $\tilde{\mathbf{w}}$ provide the same performance.

The distribution of the optimal weight vector must now be addressed. Let $\mathbf{H}$ be defined as in Section 2 with all entries independent. The distribution of $\mathbf{X} = \mathbf{H}^H\mathbf{H}$ is the complex Wishart distribution [32]. An important property of complex Wishart distributed random matrices is summarized in Theorem 1.

Lemma 1 *If $\mathbf{X}$ is complex Wishart distributed, then $\mathbf{X}$ is equivalent in distribution to $\mathbf{U}\mathbf{D}\mathbf{U}^H$ where $\mathbf{U}$ is Haar distributed on the group of $M_t \times M_t$ unitary matrices and $\mathbf{D}$ has distribution commonly found in [33].*

Thus a matrix of i.i.d. complex normal distributed entries is distribution invariant to multiplication by unitary matrices. From this it is easily proved that the complex Wishart distribution is invariant to transformation of the form $\mathbf{U}^H(\cdot)\mathbf{U}$ where $\mathbf{U} \in \mathcal{U}_{M_t}$ where $\mathcal{U}_{M_t}$ is the group of $M_t \times M_t$ unitary matrices. This is a trivial property in the case of the chi-squared distribution because of the commutativity of complex numbers, but this property has highly non-trivial implications for $M_t > 1$. A very important property of Haar distributed matrices that will be exploited later is given in the following theorem.

Lemma 2 *Let $\mathbf{U}$ be a Haar distributed $\mathrm{M}_t \times M_t$ random matrix. If $\mathbf{v} \in \Omega_{M_t}$ then $\mathbf{U}\mathbf{v}$ is uniformly distributed on Ω_{M_t}.*

The proof of this can be found in [34].

One solution to (18) has a distribution equivalent to $\mathbf{U}^H\mathbf{w}_{MRT} = [1\ 0\ 0\ \cdots\ 0]^T$ or rather $\mathbf{w}_{MRT} = \mathbf{U}[1\ 0\ 0\ \cdots\ 0]^T$ with $\mathbf{U}$ given in Lemma 1. Since $\mathbf{U}$ is Haar distributed on $\mathcal{U}_{M_t}$ and $[1\ 0\ 0\ \cdots\ 0]^T$ is a unit vector, Theorem 2 states that $\mathbf{w}_{MRT} = \mathbf{U}[1\ 0\ 0\ \cdots\ 0]^T$ is distributed uniformly on Ω_{M_t}. It similarly follows that each of the columns of $\mathbf{U}$ and any unit norm linear combination of columns of $\mathbf{U}$ is uniformly distributed on Ω_{M_t}.

This result taken along with Theorem 1 reveals a fundamental result about quantized transmit diversity systems that until this point, to the best of the authors' knowledge, has never been shown. The distribution of the optimal transmit weight vector is independent of the number of receive antennas. Thus *the problem of finding quantized transmit weights for MISO systems is the same problem as that of finding quantized transmit weights for MIMO systems.* Therefore the MISO quantized transmit diversity analysis contained in [17] and [16] is directly applicable to MIMO systems.

A corollary to Lemma 2 can be seen by understanding the weighting in terms of complex lines.

Corollary 1 *The line generated by the optimal beamforming vectors is a line in $\mathbb{C}^{M_t}$ passing through the origin that is isotropically oriented.*

Therefore, we wish to quantize an isotropically oriented line in $\mathbb{C}^{M_t}$.

We wish to obtain a codebook matrix $\mathbf{W}$ with unit code vector columns. Let us denote, as before, the set of $M_t \times N$ matrices with unit vector columns by $\mathcal{U}_{M_t}^{N}$. We will denote the i^{th} column of $\mathbf{W}$ by $\mathbf{w}_i$. Because we wish to maximize the receive SNR, we use an encoding function $\mathcal{Q}_{\mathbf{w}} : \mathbb{C}^{M_r \times \mathrm{M}_t} \to \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N\}$ given by

$$\mathcal{Q}_{\mathbf{w}}(\mathbf{H}) = \arg \max_{\mathbf{x} \in \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N\}} \|\mathbf{H}\mathbf{x}\|_2^2. \tag{19}$$

Notice that this encoding function is not solely a function of the maximum singular value direction in the matrix channel case. The reason for this is that situations arise where it is better to use the quantized vector that is close to some unit norm linear combination of the M_t singular vectors. For example, channels where several of the singular values are equal would fall into this case.

We will use the distortion function

$$G(\mathbf{W}) = E_{\mathbf{H}} \left[\lambda_1 - \|\mathbf{H}\mathcal{Q}_{\mathbf{w}}(\mathbf{H})\|_2^2 \right] \tag{20}$$

where λ_1 is the maximum eigenvalue of $\mathbf{H}^*\mathbf{H}$. This can be bounded as

$$G(\mathbf{W}) = E_{\mathbf{H}} \left[\lambda_1 - \sum_{i=1}^{M_t} \lambda_i \left| \mathbf{u}_i^* \mathcal{Q}_{\mathbf{w}}(\mathbf{H}) \right|^2 \right] \tag{21}$$

$$\leq E_{\mathbf{H}} \left[\lambda_1 - \lambda_1 \left| \mathbf{u}_1^* \mathcal{Q}_{\mathbf{w}}(\mathbf{H}) \right|^2 \right] \tag{22}$$

$$= E_{\mathbf{H}} \left[\lambda_1 \right] E_{\mathbf{H}} \left[1 - \left| \mathbf{u}_1^* \mathcal{Q}_{\mathbf{w}}(\mathbf{H}) \right|^2 \right] \tag{23}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{M_t}$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{M_t}$ are the eigenvalues and corresponding eigenvectors of $\mathbf{H}^*\mathbf{H}$. The result in (23) follows from the independence of the eigenvalues and eigenvectors of complex Wishart matrices [22], [32].

The intuition behind the bound is that the first factor is an indication of channel quality on average while the second factor is an indication of the beamforming codebook quality. Using the interpretation of $\mathbf{W}$ as a line packing and that $\mathbf{u}_1$ is uniformly distributed on Ω_{M_t} we can note that

$$Pr\left(1 - |\mathbf{u}_1^* \mathcal{Q}_{\mathbf{W}}(\mathbf{H})|^2 \leq \frac{\delta^2(\mathbf{W})}{4}\right) = \Delta(\mathbf{W}) = N\left(\frac{\delta(\mathbf{W})}{2}\right)^{2(M_t-1)}. \tag{24}$$

Thus,

$$\begin{aligned} G(\mathbf{W}) &\leq E_{\mathbf{H}}[\lambda_1]\left(\frac{\delta^2(\mathbf{W})}{4}\Delta(\mathbf{W}) + (1-\Delta(\mathbf{W}))\right) &(25)\\ &= E_{\mathbf{H}}[\lambda_1]\left(1 + N\left(\frac{\delta(\mathbf{W})}{2}\right)^{2(M_t-1)}\left(\frac{\delta^2(\mathbf{W})}{4} - 1\right)\right). &(26)\end{aligned}$$

Minimizing this bound corresponds to maximizing the minimum distance between any pair of lines spanned by the codebook vectors. Thus we propose the following criteria for designing Grassmannian beamforming codebooks.

Grassmannian Beamforming Criterion: Design the set of codebook vectors $\{\mathbf{w}_i\}_{i=1}^N$ such that the corresponding codebook matrix $\mathbf{W}$ maximizes $\delta(\mathbf{W}) = \min_{1\leq k<l\leq N}\sqrt{1 - \left|\mathbf{w}_k^*\mathbf{w}_l\right|^2}$.

As discussed in Section 3, this criterion is exactly the same as that of Grassmannian line packing. This problem is quite mathematically challenging from an analytical point of view [35]. The problem can also be approached from a brute force perspective where numerical simulations can yield good, if not optimal, results ([21], [36]). For a given M_t and N, the Rankin bound gives a firm upper bound on $\delta(\mathbf{W})$. Unfortunately in most cases the Rankin bound is not attainable and in effect quite loose [29]. Thus we have to use the notion of an achievable upper bound. In the real case this problem has been thoroughly studied and the best known packings are cataloged at [31]. While the real case is not what we are interested in directly here, these packings would still work well in most cases because they were designed using a restricted form of the *Grassmannian Beamforming Criterion.*

A quantized version of maximum ratio transmission, which poses no restrictions except unit energy on the transmit weight vector, can thus be designed using the criterion. Thus the optimal codebook matrix $\mathbf{W}$ would be given by

$$\mathbf{W} \in \arg \max_{\mathbf{X} \in \mathcal{U}_{M_T}^N} \delta(\mathbf{X}). \tag{27}$$

Codebook design becomes, for a given M_t and N, finding the matrix in $\mathcal{U}_{M_t}^N$ with the smallest maximum absolute correlation between any two columns. We can therefore use the design techniques discussed in [29]. Some example codebooks are given in Appendix A in Tables 1 to 6. Notice that when $N \leq M_t$ maximally spaced packings are easy to find: simply take N columns of any $M_t \times M_t$ unitary matrix. It follows that selection diversity represents a special form of Grassmannian beamforming.

Constrained weighting schemes are often of interest because they allow systems to reduce transmit hardware complexity by restricting the beamforming vectors to lie on a subset of Ω_{M_t}. We can also use the Grassmannian Beamforming Criterion to obtain codebooks with constrained unit vector weights. These weighting schemes will require us to restrict our line packing matrix $\mathbf{W}$ to be a matrix in the set $\mathcal{V}_{M_t}^N$ where $\mathcal{V}_{M_t}^N \subset \mathcal{U}_{M_t}^N$. One popular form of constrained weighting is equal gain transmission as was discussed in Section 2. In a quantized equal gain transmission system, $\mathcal{V}_{M_t}^N = \{\widetilde{\mathbf{W}} \in \mathcal{U}_{M_t}^N \;\mid\; \forall k, l \; |\widetilde{W}_{[k,l]}| = 1\}$. Thus we would design this codebook by attempting to find $\mathbf{W}$ given by

$$\mathbf{W} \in \arg \max_{\mathbf{X} \in \mathcal{V}_{M_T}^N} \delta(\mathbf{X}). \tag{28}$$

Numerical optimization techniques (such as those in [21]) are often ineffective in designing quantized equal gain transmission codebooks. For this reason random search based codebook design often yields codebooks with the best minimum distance. Suboptimal methods for designing quantized equal gain codebooks can be found in [8], [9]. Suboptimal codebooks can also be designed using the single antenna unitary space-time modulation codes presented in [36]. However, it must be noted that the methods in [8], [9] sometimes lack in performance when compared to random search codebooks (see *QEGT Experiment # 1* in Section 6 for an example). However, experience has shown (see also [19] that codebooks obtained from the technique in [36] are often optimal or near optimal. Therefore, *there is often no difference in performance between quantized maximum ratio transmission and quantized equal gain transmission.*

We can generalize selection diversity to systems that select any of the possible $2^{M_t} - 1$ non-empty subsets of the M_t antennas to transmit on. Thus we choose one of the non-empty members of the power set of $\{1, \ldots, M_t\}$ and transmit on this antenna subset. Generalized subset selection, as we call this transmission method, is a discrete system that can be represented via an M_t bit codebook. If M_t is large, we might wish to use fewer than M_t bits for our generalized selection diversity system. In this case we would pick the codebook matrix $\mathbf{W}$ that satisfies

$$\mathbf{W} \in \arg \max_{\mathbf{X} \in \mathcal{I}_{M_T}^N} \delta(\mathbf{X}). \tag{29}$$

where $\mathcal{I}_{M_t}^N$ is given by the set of matrices in $\mathcal{U}_{M_t}^N$ where each column can be represented as the normalized sum of $\mathbf{I}_{M_t}$ column vectors. For example, $\mathcal{I}_2^2$ is given by

$$\mathcal{I}_2^2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{pmatrix} \right\}. \tag{30}$$

The cardinality of $\mathcal{I}_{M_t}^N$ is given by

$$\left|\mathcal{I}_{M_t}^N\right| = \frac{\prod_{k=0}^{N-1} \left(2^{M_t} - 1 - k\right)}{N!}. \tag{31}$$

Since $\mathcal{I}_{M_t}^N$ has finite cardinality, the global maximum to (29) can be obtained by performing a brute force search over all matrices in $\mathcal{I}_{M_t}^N$.

5 Necessary Feedback Amount

It is important to understand the performance characteristics of quantized transmit diversity systems as a function of the number of bits fed back. In this section we look at the performance metrics of diversity order, capacity loss, and SNR loss and derive criterion for choosing N.

5.1 Diversity Order

Closed form results on the average probability of symbol error for quantized transmit diversity systems are difficult if not impossible to determine in closed-form. We will resort to the performance metrics defined in Section 2. It is important to determine if systems using the codebooks of quantized weight vectors designed in Section 4 combined with MRC at the receiver achieve a diversity advantage on the order of $M_r M_t$ since other quantized transmit diversity weighting schemes have been proven to achieve this performance ([15], [8]). The following analysis will hold for any transmit weighting technique that uses a codebook with at least M_t vectors. The only requirement is that the column vectors contained in the codebook span $\mathbb{C}^{M_t}$.

First consider the receive SNR, $\mathcal{E}_t \mathbf{\Gamma}_r / N_0$. Since $\mathcal{E}_t$ and N_0 are assumed fixed, we only need to consider $\mathbf{\Gamma}_r = |\mathbf{z}^H \mathbf{H} \mathbf{w}|^2$. It has been shown for a fixed realization of $\mathbf{H}$ [8] that the weight vectors $\mathbf{w}$ and $\mathbf{z}$ that maximize $|\mathbf{z}^H \mathbf{H} \mathbf{w}|$ are the left and right singular vectors of $\mathbf{H}$ corresponding to the largest singular value of $\mathbf{H}$. Antenna weighting systems of this form have been shown to achieve a diversity on the order of $M_r M_t$ [14]. Therefore the diversity order of our quantized transmit diversity and MRC system is always less than or equal to $M_r M_t$.

We will lower bound the system's diversity order by lower bounding the receive SNR of our quantized transmit diversity system. We assume an MRC receiver system and an overcomplete code (i.e. the columns of the codebook $\mathbf{W}$ span $\mathbb{C}^{M_t}$). For an N vector transmit diversity codebook system with MRC at the receiver, the effective channel gain is given by

$$\mathbf{\Gamma}_r = \max_{0<i\leq N} \|\mathbf{H}\mathbf{w}_i\|_2^2 \tag{32}$$

where $\mathbf{w}_i$ is the i^{th} column of the $M_t \times N$ codebook matrix $\mathbf{W}$. Because the columns of the codebook matrix $\mathbf{W}$ span $\mathbb{C}^{M_t}$, $\mathbf{W}$ can be factored via a singular value decomposition (SVD) into the form $\mathbf{W} = \mathbf{U}_1 [\mathbf{D}\ \mathbf{0}] \mathbf{U}_2$ where $\mathbf{U}_1$ is an $M_t \times M_t$ unitary matrix, $\mathbf{0}$ is an $M_t \times (N - M_t)$ matrix of zeros, $\mathbf{U}_2$ is an $N \times N$ unitary matrix, and $\mathbf{D}$ is a real diagonal matrix with $D_{[1,1]} \geq D_{[2,2]} \geq \ldots \geq D_{[M_t, M_t]} > 0$. By the invariance of complex Gaussian random matrices [32], $\mathbf{H}\mathbf{U}_1^H$ is equivalent in distribution to $\mathbf{H}$. Therefore,

$$\begin{aligned} \mathbf{\Gamma}_r &\overset{d}{=} \widetilde{\mathbf{\Gamma}}_r \\ &= \max_{0<i\leq N} \|\mathbf{H}\mathbf{U}_1^H \mathbf{w}_i\|_2^2. \end{aligned} \tag{33}$$

Now using matrix norm inequalities taken from [27], stated for the real case but easily seen to extend to the complex case, and letting $\|\cdot\|_1$ denote the matrix 1-norm, we find that

$$\begin{aligned}
\widetilde{\mathbf{\Gamma}}_r &= \max_{0<i\leq N} \|\mathbf{H}\mathbf{U}_1^H \mathbf{w}_i\|_2^2 && (34)\\
&\geq \max_{0<i\leq N} \frac{\|\mathbf{H}\mathbf{U}_1^H \mathbf{w}_i\|_1^2}{M_r} \\
&= \frac{1}{M_r}\|\mathbf{H}\mathbf{U}_1^H \mathbf{W}\|_1^2. && (35)
\end{aligned}$$

Using the SVD, (35) can be rewritten as

$$\begin{aligned}
\widetilde{\mathbf{\Gamma}}_r &\geq \frac{1}{M_r}\|\mathbf{H}\mathbf{U}_1^H \mathbf{U}_1[\mathbf{D}\ \mathbf{0}]\mathbf{U}_2\|_1^2. && (36)\\
&= \frac{1}{M_r}\left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\right\|_1^2
\end{aligned}$$

where $\widetilde{\mathbf{0}}$ is an $M_r \times (N - M_t)$ matrix of zeros.

By the matrix submultiplicative property [27],

$$\begin{aligned}
\left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\right\|_1 \|\mathbf{U}_2^H\|_1 &\geq \left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\mathbf{U}_2^H\right\|_1 \\
&= \left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\right\|_1 && (37)\\
&= \|\mathbf{H}\mathbf{D}\|_1
\end{aligned}$$

Then using an inequality property of the matrix 1- and 2-norm

$$\begin{aligned}
\|\mathbf{H}\mathbf{D}\|_1 &\leq \left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\right\|_1 \sqrt{N}\|\mathbf{U}_2^H\|_2 && (38)\\
&= \sqrt{N}\left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\right\|_1
\end{aligned}$$

or rather

$$\left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\right\|_1^2 \geq \frac{1}{N}\|\mathbf{H}\mathbf{D}\|_1^2. \qquad (39)$$

Applying this bound we find that

$$\begin{aligned}
\widetilde{\Gamma}_r &\geq \frac{1}{M_r}\left\|\left[\mathbf{H}\mathbf{D}\ \widetilde{\mathbf{0}}\right]\mathbf{U}_2\right\|_1^2 && (40)\\
&\geq \frac{1}{NM_r}\|\mathbf{H}\mathbf{D}\|_1^2 \\
&\geq \frac{D^2_{[M_t,M_t]}}{NM_r}\|\mathbf{H}\|_1^2 \\
&\geq \frac{D^2_{[M_t,M_t]}}{NM_r}\max_{i,j}|H_{[i,j]}|^2. && (41)
\end{aligned}$$

The lower bound on $\widetilde{\Gamma}_r$ is the effective channel gain of a system which selects the largest gain channel from among $M_r M_t$ i.i.d. complex Gaussian random variables when $D_{[M_t,M_r]} > 0$. Diversity systems of this form have been shown to achieve an $M_r M_t$ diversity order. The scale factor of $\frac{D^2_{[M_t,M_t]}}{NM_r}$ can simply be interpreted as a negative array gain, not affecting the asymptotic diversity slope.

Thus for high SNR, the average probability of symbol error for quantized diversity systems using maximum ratio combining receivers has been upper and lower bounded by the average probability of symbol error of $M_r M_t$ diversity order systems. Therefore, we can state the following criteria for choosing N.

Diversity Rule: Choose $N \geq M_t$ to guarantee full diversity order.

We now have a lower bound on choosing N. However choosing N to meet this lower bound would yield to square, unitary codebook matrices where $\mathbf{W}^*\mathbf{W} = \mathbf{I}_{M_t}$ and performance identical to SDT/MRC. Since we wish to obtain performance close optimal MRT/MRC we must obviously increase the value of N used.

5.2 Capacity

The capacity loss associated with using quantized beamforming is an important indicator of the quality of the quantization method. However, some amount of capacity loss must be expected because of the transmitter does not have complete channel knowledge. We will thus derive a criterion for choosing N based on the capacity loss.

The order to characterize the maximum achievable reliable data rate, we will follow the approach in [17], [37] and examine the mutual information between the input random variable and output random variable of our beamforming system over many transmissions. We can then define the mutual information as

$$I = \frac{1}{n} \lim_{n \to \infty} I_n \tag{42}$$

where $I_n = I(s_1, s_2, \ldots, s_n; y_1, y_2, \ldots, y_n)$, the mutual information between a block of n input and output random variables. For a fixed channel realization $\mathbf{H}$ and independent s_i, each distributed according to $\mathcal{CN}(0, \mathcal{E}_t)$, we can write the mutual information of our quantized system as

$$I_{quant} = \log_2 \left(1 + \frac{\|\mathbf{H}\mathcal{Q}_{\mathbf{w}}(\mathbf{H})\|_2^2 \mathcal{E}_t}{N_0}\right) \tag{43}$$

where $\mathcal{Q}_{\mathbf{w}}$ defined as before. If we take the expected value of I_{quant} with respect to $\mathbf{H}$ we can characterize the capacity of our effective channel $\mathbf{z}^H\mathbf{H}\mathbf{w}$ as

$$C_{quant} = E_{\mathbf{H}} \left[\log_2 \left(1 + \frac{\|\mathbf{H}\mathcal{Q}_{\mathbf{w}}(\mathbf{H})\|_2^2 \mathcal{E}_t}{N_0}\right)\right] \tag{44}$$

For high SNR we can approximate the capacity loss as

$$C_{quant} \approx E_{\mathbf{H}} \left[\log_2 \left(\frac{\|\mathbf{H}\mathcal{Q}_{\mathbf{w}}(\mathbf{H})\|_2^2 \mathcal{E}_t}{N_0}\right)\right]. \tag{45}$$

Thus the capacity loss due to quantization for high SNR (using the techniques that bounded distortion) is given by

$$\begin{aligned}
C_{loss} &\approx E_{\mathbf{H}} \left[\log_2 \left(\frac{\mathcal{E}_t}{N_0}\lambda_{\max}\right) - \log_2 \left(\frac{\|\mathbf{H}\mathcal{Q}_{\mathbf{w}}(\mathbf{H})\|_2^2 \mathcal{E}_t}{N_0}\right)\right] && (46)\\
&\leq E_{\mathbf{H}} \left[\log_2 \left(\frac{\mathcal{E}_t}{N_0}\lambda_{\max}\right) - \log_2 \left(\frac{\mathcal{E}_t}{N_0}\lambda_{\max} \left|\mathbf{u}_1 \mathcal{Q}_{\mathbf{w}}(\mathbf{H})\right|^2\right)\right] && (47)\\
&= C_{unquant} \left(1 - N\left(\frac{\delta(\mathbf{W})}{2}\right)^{2(M_t-1)}\right) - N\left(\frac{\delta(\mathbf{W})}{2}\right)^{2(M_t-1)} \log_2 \left(1 + \left(\frac{\delta(\mathbf{W})}{2}\right)^2\right) && (48)\\
&\leq C_{unquant} \left(1 - N\left(\frac{\delta(\mathbf{W})}{2}\right)^{2(M_t-1)}\right) && (49)
\end{aligned}$$

where $C_{unquant}$ is the capacity of an optimal beamforming and combining system [14][11]. Therefore a bound on the normalized capacity loss is given by $\left(1 - N\left(\frac{\delta(\mathbf{W})}{2}\right)^{2(M_t-1)}\right)$.

Note that for all cases of large N the Rankin bound is approximately $\frac{M_t-1}{M_t}$. Using the approximate Rankin bound and this result we obtain a selection criterion of N based on capacity loss.

Capacity Loss Criterion: Given an acceptable normalized capacity loss $\overline{C_{loss}}$, choose N such that

$$N \gtrapprox (1 - \overline{C_{loss}}) \left(\frac{2M_t}{M_t - 1}\right)^{2(M_t-1)}. \tag{50}$$

5.3 Signal-to-Noise Ratio

Just as we have derived an approximate bound for N given an acceptable capacity loss due to quantization, we can propose a criterion for choosing N based on SNR loss. In (26), a bound is given for the loss in receive SNR as a function of the codebook matrix. Once again, we will consider the normalized loss $0 \leq \overline{G(\mathbf{W})} \leq 1$ and the approximate Rankin upper bound $\frac{m-1}{m}$. Once again, this bound is only approximate but yields intuition into the choice of N.

SNR Criterion: Given an acceptable normalized SNR loss $\overline{G}$, choose N such that

$$N \gtrapprox \frac{1-\overline{G}}{\left(\frac{M_t-1}{2M_t}\right)^{2(M_t-1)}\left(1-\frac{(M_t-1)^2}{4M_t^2}\right)} \tag{51}$$

As an aside, we should point out that $E_{\mathbf{H}}\left[\lambda_1\right]$ can be expressed in a closed-form integral expression using techniques from [11], [38], [39]. In [38], [39] the probability density function of the largest singular value of a central, complex Wishart distribution is derived, while the cumulative distribution function is derived in [11]. These results can also be used to derive integral expressions for the outage probability as a generalization of results in [19].

6 Simulations

We simulate three different Grassmannian beamforming schemes: quantized maximum ratio transmission, equal gain transmission, and generalized selection diversity. All simulations used binary phase shift keying (BPSK) modulation and i.i.d. Rayleigh fading (where $H_{[k,l]}$ is distributed according to $\mathcal{CN}(0,1)$). The average probability of symbol error is estimated using at least 1.5 million iterations per SNR point. Codebooks for QEGT and QMRT systems were designed using optimal techniques when available [29], [36] and random searches when not available. GSS codebooks are globally optimal since searching over all possible codebooks is feasible. All of the simulations assume a maximum ratio combining receiver.

6.1 Quantized Maximum Ratio Transmission

QMRT Experiment # 1 In the first experiment, $M_r = M_t = 3$ is simulated. A quantized maximum ratio transmission system is shown for two different quantizations. The simulated error rate curve of an optimal unquantized weighting system and the actual error rate curve for an selection diversity system are shown for comparison. Notice that quantized maximum ratio transmission provides a 0.2dB gain over selection diversity for the same amount of feedback. Using 6 bits instead of 2 bits

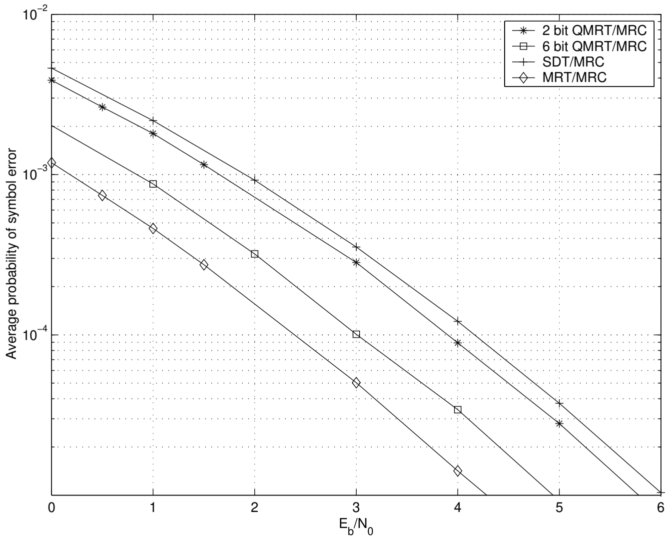

Figure 2: Bounds and symbol probability of error for 3 transmit and 3 receive antenna systems using QMRT/MRC.

of feedback provides around a 0.9dB gain. The system using 6 bits performs within 0.6dB of the optimal maximum ratio transmission weighting system. These error rate curves are shown in Fig. 2

QMRT Experiment # 2 In [17], vector quantization techniques were used to design quantized maximum ratio transmission codebooks. In the third experiment, we compare a system using Grassmannian beamforming with a system using a codebook designed using the Lloyd algorithm. Codebooks containing 8 vectors were designed for a $M_r = M_t = 2$ system. The results are shown in Fig. 3. This simulation verifies the validity of the proposed design criteria. Thus we are able to design codebooks that perform just as well as the codebooks performed using computationally complex vector quantization algorithms.

QEGT Experiment In this experiment, the two different methods of QEGT codebook design are compared on a 3 transmit and 3 receive antenna system. The results are shown in Fig. 4. The new method refers to equal gain Grassmannian beamforming. The old method refers to the codebook design method outlined in ([9], [8]).

A 3 bit new design method QEGT codebook performs approximately the same as a 5 bit old design method QEGT codebook. Thus we can use two fewer bits of feedback and actually improve

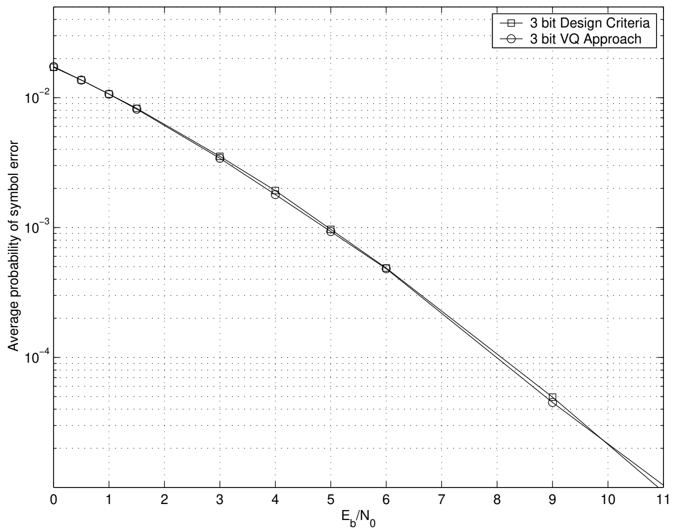

Figure 3: Symbol probability of error for 2 transmit and 2 receive antenna systems using QMRT codebooks designed with the proposed criteria and with vector quantization.

the average symbol error rate performance by using Grassmannian beamforming. Performance improves by 0.5dB when changing from 3 bit new QEGT to 5 bit new QEGT. Thus we can either gain 0.5dB and use the same amount of feedback or we could keep the same performance and save 2 bits of feedback by using the *Grassmannian Beamforming Criterion.*

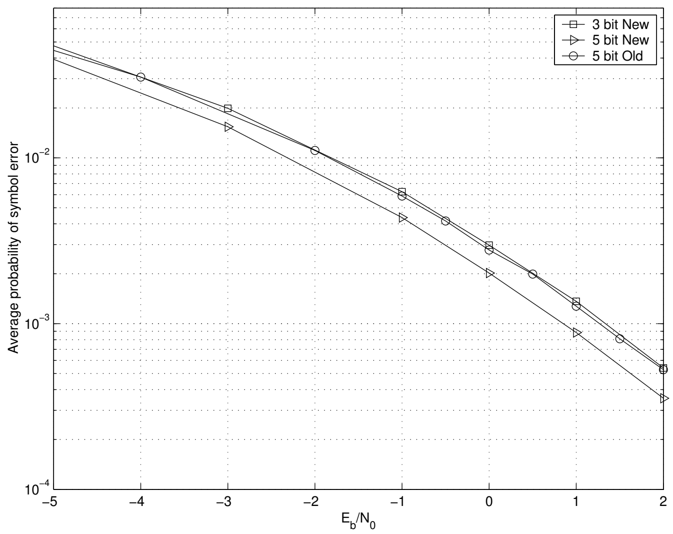

Figure 4: Comparison of symbol probability of error for 3 transmit and 3 receive antenna systems using QEGT/MRC with the old and new codebook designs.

Comparison Experiment The final experiment, shown in Fig. 5, compares all GSS and quantized maximum ratio transmission codebooks for a 4 transmit and 2 receive antenna system. A 4 bit QMRT system outperforms a 4 bit GSS system by approximately 0.5dB. A 6 bit QMRT system has a coding gain of approximately around 0.5dB compared to a 4 bit QMRT system. This shows the gains available by not requiring any constraints on the beamforming feasible set before quantization.

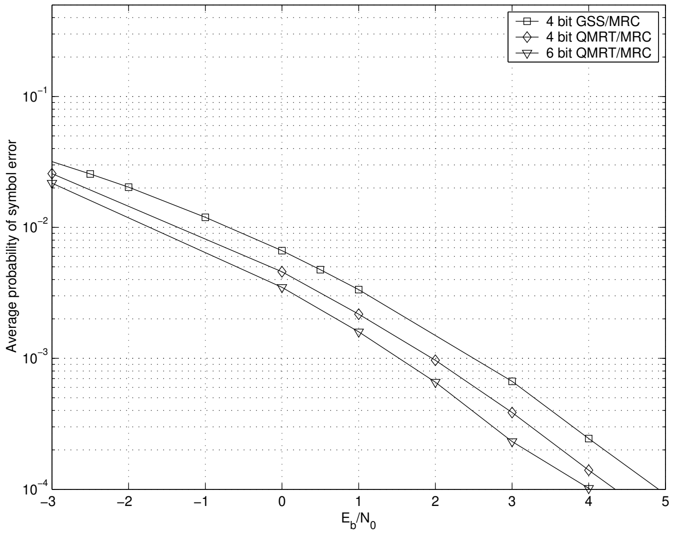

Figure 5: Comparison of symbol probability of error for 4 transmit and 2 receive antenna systems using QMRT/MRC and GSS/MRC.

7 Conclusion

We proposed Grassmannian beamforming as a new method for quantized transmit beamforming in MIMO wireless communication systems. The distribution of an optimal transmit weight vector in the sense of maximizing the receive SNR was found to be uniform on the complex unit sphere. By bounding the degradation in SNR for a fixed beamformer codebook size, it is found that the problem of designing beamformer codebooks relates directly to the problem of maximally spacing lines in Grassmann manifold. We derive several new results for line packings such as a closed-form density expression, the Hamming upper bound on the minimum distance, and the Gilbert-Varshamov bound on the codebook size. We use these new results along with the Rankin bound to provide approximate feedback requirements.

A point that we did not address in detail is implementation. Grassmannian beamforming schemes will likely be implemented in a look-up table format. While this would be feasible for systems with fixed bandwidth, it does not exploit the coherence time of the channel. When the channel is slowly varying, it may be possible to improve the estimate of the optimal beamforming vectors using some successive refinement techniques. One solution is to have a series of codebooks for different values of N that support successive refinement. Another solution would be to develop methods for optimally quantizing the update vectors.

Another important point is the effect of feedback error and delay in the feedback link. We did not address this issue in our work because we modeled the feedback link as error and delay free. An extensive simulation and/or analytical study of beamformer quantization such as that in [40] is needed. These effects will play an important role in performance in deployed MIMO systems using quantized beamforming.

Another limitation of the work proposed herein is that we considered only the transmission of a single data stream. It is well known, however, that MIMO channels can increase capacity by supporting a number of data streams ([3], [41], etc.). In the general case with full channel knowledge at the transmitter, it is possible to transmit on multiple right singular vectors with the number of vectors and the power on each vector being determined by the desired optimization criterion. A natural extension of our approach would be to derive codebooks for quantizing each of the singular vectors. While our Grassmannian codebooks could be used, a better approach may be to have codebooks that retain the orthogonality between the quantized singular vectors. We are currently investigating this topic.

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Appendix A

Examples of the packings found for various M_t are shown below. The codebooks were found using random searches or using results given in [29] when applicable.

1	0
0	1

Absolute Max Correlation = 0
Absolute Theoretical Max Correlation = 0

Table 1: Codebook generated for $M_t = 2$ and $N = 2$.

-0.1612 - 0.7348j	-0.0787 - 0.3192j	-0.2399 + 0.5985j	-0.9541
-0.5135 - 0.4128j	-0.2506 + 0.9106j	-0.7641 - 0.0212j	0.2996

Absolute Max Correlation = 0.57735
Absolute Theoretical Max Correlation = $\sqrt{1/3}$

Table 2: Codebook generated for $M_t = 2$ and $N = 4$.

0.8393 - 0.2939j	-0.3427 + 0.9161j	-0.2065 + 0.3371j
-0.1677 + 0.4256j	0.0498 + 0.2019j	0.9166 + 0.0600j

0.3478 - 0.3351j	0.1049 + 0.6820j	0.0347 - 0.2716j
0.2584 + 0.8366j	0.6537 +0.3106j	0.0935 - 0.9572j

-0.7457 + 0.1181j	-0.7983 + 0.3232j
-0.4553 - 0.4719j	0.5000 + 0.0906j

Absolute Max Correlation = 0.84152
Absolute Theoretical Max Correlation = $\sqrt{3/7}$

Table 3: Codebook generated for $M_t = 2$ and $N = 8$.

$1/\sqrt{3}$	$j/\sqrt{3}$	$-1/\sqrt{3}$	$-j/\sqrt{3}$
$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$
$1/\sqrt{3}$	$-j/\sqrt{3}$	$-1/\sqrt{3}$	$j/\sqrt{3}$

Absolute Max Correlation $= 1/3$
Absolute Theoretical Max Correlation $= 1/3$

Table 4: Codebook generated for $M_t = 3$ and $N = 4$.

$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}e^{2\pi j/3}$	$\frac{1}{\sqrt{2}}e^{2\pi j/3}$	$\frac{1}{\sqrt{2}}e^{4\pi j/3}$	$\frac{1}{\sqrt{2}}e^{4\pi j/3}$	0
$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}e^{4\pi j/3}$	0	0	$\frac{1}{\sqrt{2}}e^{2\pi j/3}$	$\frac{1}{\sqrt{2}}e^{4\pi j/3}$
0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}e^{4\pi j/3}$	$\frac{1}{\sqrt{2}}e^{2\pi j/3}$	0	$\frac{1}{\sqrt{2}}e^{2\pi j/3}$

Absolute Max Correlation $= 0.5$
Absolute Theoretical Max Correlation $= \sqrt{5/21}$

Table 5: Codebook generated for $M_t = 3$ and $N = 8$.