Inverse-Closedness of a Banach Algebra of Integral Operators on the Heisenberg Group

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Abstract

Let $\mathbb{H}$ be the general, reduced Heisenberg group. Our main result establishes the inverse-closedness of a class of integral operators acting on $L^p(\mathbb{H})$, given by the off-diagonal decay of the kernel. As a consequence of this result, we show that if $\alpha_1 I + S_f$, where S_f is the operator given by convolution with f, $f \in L^1_v(\mathbb{H})$, is invertible in $\mathcal{B}(L^p(\mathbb{H}))$, then $(\alpha_1 I + S_f)^{-1} = \alpha_2 I + S_g$, and $g \in L^1_v(\mathbb{H})$. We prove analogous results for twisted convolution on a locally compact abelian group and its dual group. We apply the latter results to a class of Weyl pseudodifferential operators, and briefly discuss relevance to mobile communications.

1 Introduction

We consider a class of integral operators defined for the general reduced Heisenberg group $\mathbb{H}$. We show that if the kernel of the integral operator N_1 has L^1_v-integrable off-diagonal decay (here v is a weight) and the operator $\alpha_1 I + N_1$ is invertible, then its inverse also has the form $\alpha_2 I + N_2$, and the off-diagonal decay is preserved in the kernel of N_2. As a consequence of this result, we establish the inverse-closedness of a Banach algebra of convolution operators on the general reduced Heisenberg group. Namely, we consider the operator S_f given by convolution with f and show that if $\alpha_1 I + S_f$, $f \in L^1_v(\mathbb{H})$, is invertible in $\mathcal{B}(L^p(\mathbb{H}))$, then $(\alpha_1 I + S_f)^{-1} = \alpha_2 I + S_g$, where $g \in L^1_v(\mathbb{H})$. While this result relies on other recent results, it has its roots in Wiener's lemma.

*B. F. was supported by NSF VIGRE grant DMS-0135345. T. S. was supported by NSF grant DMS-0511461.
2000 *Mathematics Subject Classification.* 43A20, 47G30

Wiener's lemma states that if a periodic function f has an absolutely summable Fourier series

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n t} \tag{1}$$

and is nowhere zero, then $1/f$ also has an absolutely convergent Fourier series [29]. We call this inverse-closed property the spectral algebra property. That is, let $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B} \subset \mathcal{A}$, be Banach algebras. $\mathcal{B}$ has the *spectral algebra property* if whenever $\mathbf{b} \in \mathcal{B}$ is invertible in $\mathcal{A}$, $\mathbf{b}^{-1} \in \mathcal{B}$. We equivalently say that $\mathcal{B}$ is *inverse-closed* in $\mathcal{A}$. In Wiener's lemma and in many of its descendants, the space $\mathcal{B}$ is given by some form of l^1 decay, for example summability of Fourier coefficients. Another perspective on inverse-closedness for a Banach algebra views the elements of the algebra as operators and considers to what degree the operator maps subspaces to other subspaces or to what degree it leaves subspaces invariant. This perspective becomes fruitful when we consider subspaces derived from the structure of the underlying group, rather than from a basis.

To view a periodic function with summable Fourier series $\mathbf{a} = \{a_n\}_{n\in\mathbb{Z}}$ as an operator, we consider the algebra of bi-infinite Toeplitz matrices with summable antidiagonals, and we let $T_{\mathbf{a}}$ be the Toeplitz matrix given by the sequence $\mathbf{a}$ as antidiagonal. To frame the Toeplitz case in the subspace perspective, we reformulate the summable antidiagonal property as

$$\sum_{n\in\mathbb{Z}} \sup_{i-j=n} |\langle T_{\mathbf{a}} e_i, e_j\rangle| = \sum_{n\in\mathbb{Z}} |\langle T_{\mathbf{a}} e_n, e_0\rangle| < \infty, \tag{2}$$

where $\{e_i\}_{i\in\mathbb{Z}}$ is the standard basis for $l^2(\mathbb{Z})$. In this case, the subspace perspective simply states that for the subspaces E_i and E_j given by e_i and e_j, $\|T_a : E_i \to E_j\| \to 0$ as $|i-j| \to \infty$ with decay given by (2).

Bochner and Phillips contributed the first essential step towards a general operator version of the spectral algebra property [6]. They showed that the a_n in (1) may belong to a–possibly noncommutative–Banach algebra. This key result enabled Gohberg, Kaashoek and Woerdeman [11] and Baskakov [3] to establish an operator version of the spectral algebra property. They considered subspaces X_i of the space X, indexed by a discrete abelian group $\mathbb{I}$, satisfying $X_i \cap X_j = \{0\}$ for $i \neq j$ and $X = \overline{\operatorname{span}}\{X_i\}_{i\in\mathbb{I}}$, and set P_i to be the projection onto X_i. For the linear operator $T:\ X \to X$ they set

$$a_n = \sum_{i-j=n} P_i T P_j,$$

and consider the operator-valued Fourier series

$$f(t) = \sum_{n\in\mathbb{I}} a_n e^{2\pi i n t}. \tag{3}$$

They then use Bochner and Phillips's work to establish that operators of the form (3) satisfying $\sum_{n\in\mathbb{Z}} \|a_n\| = \sum_{n\in\mathbb{Z}} \sup_{i-j=n} \|P_i T P_j\| < \infty$ form an inverse-closed Banach algebra in $\mathcal{B}(X)$ [11, 3, 4].

In the commutative setting, Gelfand, Raikov and Shilov [10] addressed the important question: what rates of decay of an element are preserved in its inverse? They answered this question by determining conditions on a weight function v such that series finite in the following weighted norm form an inverse-closed Banach algebra in $l_1^1(\mathbb{Z})$:

$$\|\mathbf{a}\|_{l_v^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \|a_n\| v(n) < \infty.$$

These three conditions on v are given later; the key condition is called the GRS condition, and a function satisfying all three is called *admissible.* Baskakov incorporated the GRS condition and proved the following operator version of the spectral algebra property[4, 5][1]: let v be an admissible weight; if the linear operator T satisfies

$$\sum_{n\in\mathbb{Z}} \sup_{i-j=n} \|P_i T P_j\| v(n) < \infty$$

and is invertible, then

$$\sum_{n\in\mathbb{Z}} \sup_{i-j=n} \|P_i T^{-1} P_j\| v(n) < \infty.$$

Kurbatov considered a class of operators satisfying

$$(Tf)(t) \leq \int \beta(t-s)|f(s)|ds \qquad (4)$$

for some $\beta \in L^1$. He showed, using results very similar to Baskakov's (but derived independently), that if $\alpha_1 I + T_1$ is invertible and T_1 satisfies (4) for $\beta_1 \in L^1$, then $(\alpha_1 I + T_1)^{-1} = \alpha_2 I + T_2$ and T_2 satisfies (4) for $\beta_2 \in L^1$ [20]. This theorem, as stated for integral operators in [19], is the point of departure for the research presented in this paper.

We have two motivations for the work presented here: on the one hand a question of abstract harmonic analysis and on the other hand research on the propagation channel of a mobile communication system. The abstract harmonic analysis question is: for what nonabelian groups does the spectral algebra property hold? One recent result in this direction is by Gröchenig and Leinert [13, 14]. They established the spectral algebra property for $(l_v^1(\mathbb{Z}^{2d} \times \mathbb{Z}^{2d}), \natural_\theta)$, where $\natural_\theta$ is the following form of twisted convolution:

$$(\mathbf{a}\natural_\theta \mathbf{b})_{(m,n)} = \sum_{k,l\in\mathbb{Z}^d} a_{kl} b_{m-k,n-l} e^{2\pi i\theta(m-k)\cdot l},$$

and they use this result to prove the spectral algebra property for convergent sums of time-frequency shifts $\sum_{\lambda\in\Lambda} c_\lambda T_{x_\lambda} M_{\omega_\lambda}$, $\sum_{\lambda\in\Lambda} |c_\lambda| v(\lambda) < \infty$. ($T_x$ and M_ω are

[1] The version presented here is slightly different from the theorems in [4, 5], but it can be easily extracted from the proof of Theorem 2 in [4].

defined below.) Balan recently generalized this result by relaxing the lattice requirement to solely a discrete *subset* of $\mathbb{R}^{2d}$. He showed that if $\sum_{\lambda\in\Lambda} c_\lambda T_{x_\lambda} M_{\omega_\lambda}$, $\sum_{\lambda\in\Lambda} |c_\lambda| v(\lambda) < \infty$, is invertible, then $(\sum_{\lambda\in\Lambda} c_\lambda T_{x_\lambda} M_{\omega_\lambda})^{-1} = \sum_{\sigma\in\Sigma} c_\sigma T_{x_\sigma} M_{\omega_\sigma}$ and $\sum_{\sigma\in\Sigma} |c_\sigma| v(\sigma) < \infty$, where Λ, $\Sigma \subset \mathbb{R}^{2d}$, and $|\Lambda|$, $|\Sigma| < \infty$, but $\Lambda \neq \Sigma$ (in general) [1].

Many of the other recent results of this nature are for integral operators, where the spectral algebra property is manifested in the kernel. This is true for the seminal paper by Sjöstrand in pseudodifferential operator theory [26]. In that paper he proves the matrix version of Baskakov's result, and uses this to prove the spectral algebra property for pseudodifferential operators with symbols in $M_v^{\infty,1}(\mathbb{R}^{2d})$. (Let $g \in \mathcal{S}(\mathbb{R}^{2d})$ be a compactly supported, C^∞ function satisfying $\sum_{k\in\mathbb{Z}} g(t-k) = 1$ for all $t \in \mathbb{R}^{2d}$. Then the symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ belongs to $M_v^{\infty,1}(\mathbb{R}^{2d})$ if $\int_{\mathbb{R}^{2d}} \sup_{k\in\mathbb{Z}^{2d}} |(\sigma \cdot g(. - k))\hat{}(\zeta)| d\zeta < \infty$.) Sjöstrand proved the matrix Baskakov result and his pseudodifferential operator result using techniques from "hard analysis". Gröchenig later also achieved this same result using techniques solely from harmonic analysis [12]. In fact, he proved more. He showed that a pseudodifferential operator L_σ has Weyl symbol $\sigma \in M_v^{\infty,1}(\mathbb{R}^{2d})$ if and only if the matrix given by $M_{m,n,m',n'} = \langle L_\sigma \psi_{m,n}, \psi_{m',n'}\rangle$, for a proper Gabor frame $\{\psi_{k,l}\}_{k,l\in\mathbb{Z}}$ [16], is in the Baskakov matrix algebra given by $\sum_{(k,l)\in\mathbb{Z}^2} \sup_{(m-m',n-n')=(k,l)} |A_{m,n,m',n'}| v(k,l) < \infty$ [12]. Gröchenig and one of us recently extended this result to pseudodifferential operators with symbols defined on $\mathbb{G} \times \hat{\mathbb{G}}$, where $\mathbb{G}$ is any locally compact abelian group and $\hat{\mathbb{G}}$ is its dual group [15]. Locally compact abelian groups, their dual groups and twisted convolution are standard features throughout the work just discussed; therefore, it is very natural to look directly at the general reduced Heisenberg group for a fundamental theorem.

In mobile communications a transmitted signal travels through a channel that is modeled by a pseudodifferential operator. When a single source transmits a signal it is reflected by objects in its environment, which results in different paths from transmitter to receiver, each with its own travel time. In the case of mobile communications, a moving transmitter and/or receiver gives rise to the Doppler effect [23], which results in a frequency shift. Thus, denoting time shift by $T_x f(t) = f(t - x)$ and modulation or frequency shift by $M_\omega f(t) = e^{2\pi i\omega t} f(t)$, the received signal can be represented as the following collection of weighted, delayed and modulated copies of the transmitted signal:

$$f_{rec}(t) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \hat{\sigma}(\omega, x) T_x M_\omega f_{trans}(t) dx d\omega.$$

We, therefore, consider Weyl pseudodifferential operators:

$$L_\sigma f(t) = \int_{\mathbb{G}} \int_{\hat{\mathbb{G}}} \hat{\sigma}(\omega, x) e^{-\pi i\omega\cdot x} T_{-x} M_\omega f(t) d\omega dx.$$

In particular, we posed the question, if $\alpha_1 I + L_\sigma$ is invertible and $\hat{\sigma} \in L_v^1(\hat{\mathbb{G}} \times \mathbb{G})$, does $(\alpha_1 I + L_\sigma)^{-1} = \alpha_2 I + L_\tau$, where $\hat{\tau} \in L_v^1(\hat{\mathbb{G}} \times \mathbb{G})$?

Throughout this paper, $\widetilde{\mathcal{A}}$ will denote the Banach algebra $\mathcal{A}$ with adjoined identity. In Section 2 we prove the spectral algebra property for $\widetilde{\mathcal{N}_v^1}(\mathbb{H})$, where $\mathcal{N}_v^1(\mathbb{H})$ is the space of integral operators with kernels having $L_v^1(\mathbb{H})$-integrable off-diagonal decay. The basis for our proof is establishing an operator class for which we can apply Baskakov's theorem, and using a dense, two-sided, proper ideal within that class. We use this result to prove the spectral algebra property for convolution operators on $L_v^1(\mathbb{H})$. In Section 3 we prove the spectral algebra property for $(\widetilde{L_v^1}(\mathbb{G}\times\hat{\mathbb{G}}), \natural)$, where $\natural$ is twisted convolution. This result is in the same spirit as work of Gröchenig and Leinert on $(L_v^1(\mathbb{Z}^d \times \mathbb{Z}^d), \natural_\theta)$ [13, 14], but more general, in that it holds for arbitrary locally compact abelian groups. We apply these results to the class of pseudodifferential operator with symbols σ satisfying $\hat{\sigma} \in L_v^1(\mathbb{G}\times\hat{\mathbb{G}})$. Lastly, we discuss the consequences of these theorems for mobile communication channels.

2 The Convolution Algebra on the General, Reduced Heisenberg Group

Our construction of the general, reduced Heisenberg group begins with the locally compact abelian group $\mathbb{G}$ and its dual group $\hat{\mathbb{G}}$, which is also locally compact and abelian [9]. We assume that $\mathbb{G}$ is second countable and metrizable. Throughout this paper, arbitrary groups will be denoted G, and locally compact abelian groups will be denoted $\mathbb{G}$. While $\mathbb{H}$ is not abelian, it will still be written in the same font as $\mathbb{G}$. Elements of $\mathbb{G}$ will be written in Latin letters and elements of $\hat{\mathbb{G}}$ in Greek letters. By Pontrjagin's duality theorem, $\hat{\hat{\mathbb{G}}} \cong \mathbb{G}$ [24]. For convenience, we will set $e^{2\pi i x\cdot\omega} = \langle x, \omega\rangle$, where $\langle x,\omega\rangle$ denotes the action of $\omega \in \hat{\mathbb{G}}$ on $x \in \mathbb{G}$.

Here we approach the Heisenberg group from the perspective of pseudodifferential operators and time-frequency analysis, and thus motivate it from the operators translation and modulation. Translation is right addition by the inverse of an element in $\mathbb{G}$: $T_x f(y) = f(y-x)$; modulation is multiplication by the evaluation of a character in $\hat{\mathbb{G}}$: $M_\omega f(y) = \langle \omega, y\rangle f(y)$. However the set of operators $T_x M_\omega$ is not closed, as $(T_x M_\omega)(T_{x'} M_{\omega'}) = e^{2\pi i x'\cdot\omega} T_{x+x'} M_{\omega+\omega'}$, and therefore this set of operators is not parameterizable by $\mathbb{G}\times\hat{\mathbb{G}}$, but by $\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{T}$. The extension of $\mathbb{G}\times\hat{\mathbb{G}}$ to $\mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{T}$ is called the general, reduced Heisenberg group $\mathbb{H} = \mathbb{G}\times\hat{\mathbb{G}}\times\mathbb{T}$ [8, 16]. Elements of $\mathbb{H}$ will be written in bold, and elements of $\mathbb{G}$, $\hat{\mathbb{G}}$ and $\mathbb{T}$ will be written in the normal font. The group operation for $\mathbb{H}$ is written as multiplication, while the operations for $\mathbb{G}$ and $\hat{\mathbb{G}}$ are written additively and for $\mathbb{T}$ is written multiplicatively:

$$\mathbf{h}\mathbf{h}' = (x, \omega, e^{2\pi i\tau})(x', \omega', e^{2\pi i\tau'}) = (x+x', \omega+\omega', e^{2\pi i(\tau+\tau')} e^{\pi i(x'\cdot\omega - x\cdot\omega')}).$$

The identity on $\mathbb{H}$ is $\mathbf{e} = (0,0,1)$. The measure on $\mathbb{H}$ is $d\mathbf{h} = dx\,d\omega\,d\tau$, where $dx, d\omega$ and $d\tau$ correspond to the invariant measures on $\mathbb{G}, \hat{\mathbb{G}}$ and $\mathbb{T}$ respectively, normalized so that the measures of $U_{\mathbb{G}}, U_{\hat{\mathbb{G}}}$ and $U_{\mathbb{T}}$, to be defined shortly, are each

1. Since $\mathbb{G}, \hat{\mathbb{G}}$ and $\mathbb{T}$ are commutative, the invariant measure is both left and right invariant. The space $L^1_v(\mathbb{H})$ consists of those functions satisfying

$$\|f\|_{L^1_v(\mathbb{H})} = \int_{\mathbb{H}} |f(\mathbf{h})| v(\mathbf{h}) d\mathbf{h},$$

where v is an admissible weight, as defined below. We use $\star$ to denote the convolution of two functions defined on $\mathbb{H}$ as follows

$$(F_1 \star F_2)(\mathbf{h}_0) = \int_{\mathbb{H}} F_1(\mathbf{h}) F_2(\mathbf{h}^{-1}\mathbf{h}_0) d\mathbf{h}.$$

We now address three preliminaries: partitions, weight functions and the amalgam spaces.

Definition 2.1 *Let G be a group. $(\mathcal{I}, U)$ is a* partition *of G if $\mathcal{I}$ is a discrete set, U is subset of a locally compact group, $(iU) \bigcap (i'U) = \emptyset$ for $i \neq i'$, and $\bigcup_{i\in\mathcal{I}}(iU)$ covers G. For simplicity we assume that U contains the identity.*

Lemma 2.2 *The general reduced Heisenberg group $\mathbb{H}$ possesses a partition.*

Proof We call on the structure theorem, which states that for any locally, compact, abelian group $\mathbb{G}$, $\mathbb{G} \cong \mathbb{R}^d \times \mathbb{G}_0$, where the locally compact abelian group $\mathbb{G}_0$ contains a compact, open subgroup $\mathbb{K}$ [18]. $\mathbb{D} = \mathbb{G}_0/\mathbb{K}$ is a discrete group and $\mathbb{G}_0 = \bigcup_{d\in\mathbb{D}}(d\mathbb{K})$, $\mathbb{G} = \bigcup_{(i,d)\in\mathbb{Z}^d\times\mathbb{D}}((i,d)([0,1)\times\mathbb{K}))$, and different blocks are disjoint. If the group $\mathbb{G}_0$ contains the compact open subgroup $\mathbb{K}$, then $\hat{\mathbb{G}}_0$ contains the compact open subgroup $\mathbb{K}^\perp$ [24]; thus a partition of $\hat{\mathbb{G}}$ exists that is analogous to the partition used for $\mathbb{G}$. While the structure theorem does not apply in general to nonabelian groups, we can still partition $\mathbb{H}$ into blocks by the following construction. Set $\mathbb{D} = (\mathbb{G}_0 \times \hat{\mathbb{G}}_0)/(\mathbb{K}\times\mathbb{K}^\perp)$, $\mathcal{I} = \mathbb{Z}^{2d}\times\mathbb{D}\times\{0\}$, and $U = [0,1)^{2d}\times\mathbb{K}\times\mathbb{K}^\perp\times\mathbb{T}$. Then $\bigcup_{\mathbf{i}\in\mathcal{I}}(\mathbf{i}U)$ covers $\mathbb{H}$ and $(\mathcal{I}, U)$ is a partition for $\mathbb{H}$. Note that $\mathcal{I}$ is not closed, and hence is not a group. ❏

Definition 2.3 *Let G be a group. A weight function v defined on G is* admissible *if it satisfies the following three conditions:*

1. *v is continuous, symmetric, i.e. $v(x) = v(x^{-1})$, and normalized so that $v(0) = 1$.*
2. *v is submultiplicative, i.e. $v(xy) \leq v(x)v(y)$ for all $x, y \in G$.*
3. *v satisfies the Gelfand-Raikov-Shilov (GRS) [10] condition:*

$$\lim_{n\to\infty} v(nx)^{1/n} = 1 \quad \text{for all } x \in G.$$

Note: Throughout this paper the weight v will be assumed to be admissible.

Definition 2.4 *Let G be a locally compact group and v an admissible weight function. The* amalgam space $W(L^p(G), l^q_v)$ *is the space of functions finite in the local L^p norm and the global l^q_v norm as follows:*

$$\|f\|_{W(L^p(G),l^q_v)} = \left(\sum_{i \in \mathcal{I}} \|f\|^q_{L^p(iU)} v(i)^q \right)^{1/q},$$

for a partition $(\mathcal{I}, U)$ of G.

The definition of $W(L^p_v(G), l^q_v)$ is independent of the partition, as different partitions result in equivalent norms; see [27, 17]. Note that $L^1_v(G) = W(L^1(G), l^1_v)$. Kurbatov uses the amalgam spaces to prove the inverse-closedness of a class of integral operators given by the off-diagonal decay of the kernel. He considers integral operators N of the form

$$(Nf)(t) = \int_{s \in \mathbb{G}} n(t,s) f(s) ds,$$

where $\mathbb{G}$ is a locally compact abelian group. We introduce some notation and define three spaces: $\mathbf{N}^1_v(\mathbb{G})$ is the space of kernels n for which there exists $\beta \in L^1_v(\mathbb{G})$ s.t. $|n(t,s)| \leq \beta(ts^{-1})$ for all $t, s \in \mathbb{G}$. $\mathcal{N}^1_v(\mathbb{G})$ is the space of integral operators with kernel $n \in \mathbf{N}^1_v(\mathbb{G})$. An operator satisfying this property is said to be *majorized* by β. For $\mathbb{H}$ we define $Q_{i,d} = (i, d, 0)([0,1)^{2d} \times \mathbb{K} \times \mathbb{K}^\perp \times \mathbb{T})$, where $(i, d) \in \mathbb{Z}^{2d} \times \mathbb{D}$, as in Lemma 2.2. To make notation easier, μ, ν and γ will be elements of $\mathbb{Z}^{2d} \times \mathbb{D}$. Define P_μ to be the projection of $W(L^p(\mathbb{H}), l^q_v)$ onto $\{f | f \in W(L^p(\mathbb{H}), l^q_v), \mathrm{supp}(f) \subset Q_\mu\}$. For $N \in \mathcal{B}(W(L^p(\mathbb{H}), l^q_v))$, we set $N_{\mu,\nu} = P_\mu N P_\nu$, and $N_\gamma = \sum_{\mu\nu^{-1}=\gamma} N_{\mu,\nu}$. $\mathcal{N}^\infty_v(\mathbb{G})$ is the class of operators (not necessarily integral operators) satisfying

$$\sum_{\gamma \in \mathbb{Z}^{2d} \times \mathbb{D}} \sup_{\mu\nu^{-1}=\gamma} \|N_{\mu,\nu} : L^1(Q_\nu) \to L^\infty(Q_\mu)\| v(\gamma) < \infty.$$

The following is a slightly restricted version of Kurbatov's theorem as it applies to the work in this paper:

Theorem 2.5 Kurbatov, Theorem 5.4.7 [19]. *Let $\mathbb{G}$ be a non-discrete, locally compact abelian group. $\widetilde{\mathcal{N}^1_1}(\mathbb{G})$ is an inverse-closed subalgebra of $\mathcal{B}(W(L^p(\mathbb{G}), l^q_v))$ for $1 \leq p, q \leq \infty$.*

Remark : If $\mathbb{G}$ is discrete it is not necessary to adjoin the identity operator to $\mathcal{N}^1_1(\mathbb{G})$. The corresponding version of Theorem 2.5 for a discrete group $\mathbb{G}$ is a special case of Baskakov's more general and very significant Theorem 1 in [5].

We show that Kurbatov's result, Theorem 2.5 above, holds with admissible weight functions for the nonabelian group $\mathbb{H}$.

Theorem 2.6 *Let $\mathcal{N}_v^1(\mathbb{H})$ denote those bounded integral operators N on $W(L_v^p(\mathbb{H}), l^q)$, $1 \leq p, q \leq \infty$, of the form*

$$(Nf)(\mathbf{h}_0) = \int_{\mathbb{H}} n(\mathbf{h}_0, \mathbf{h}) f(\mathbf{h}) d\mathbf{h},$$

for which there exists $\beta \in L_v^1(\mathbb{H})$ satisfying

$$|n(\mathbf{h}_0, \mathbf{h})| \leq \beta(\mathbf{h}_0^{-1}\mathbf{h})$$

for all $\mathbf{h}_0, \mathbf{h} \in \mathbb{H}$. Then $\widetilde{\mathcal{N}_v^1}(\mathbb{H})$ is an inverse-closed Banach algebra in $\mathcal{B}(W(L_v^p(\mathbb{H}), l^q))$.

Before we prove Theorem 2.6 we need some preparation. Kurbatov's proof of Theorem 2.5 can be adapted to the nonabelian group $\mathbb{H}$, Theorem 2.6, once two essential pieces are established. First, an appropriate partition must be developed for $L_v^1(\mathbb{H})$ that allows us to apply Baskakov's result. Second, one must establish that $\mathcal{N}_v^\infty(\mathbb{H})$ is a two-sided ideal in $\mathcal{N}_v^1(\mathbb{H})$. The proofs below of the intermediate results, Propositions 2.10 and 2.11 and Theorem 2.5 follow very closely Kurbatov's proofs of the analogous results for the abelian case, cf. Sections 5.3 and 5.4 of [19].

Lemma 2.7 *The identity operator I is not an element of $\mathcal{N}_v^\infty(\mathbb{H})$.*

Proof Let $(\mathcal{I}, U)$ be a partition for $\mathbb{H}$. Consider the indicator function χ_E, where $E \subset iU$ for some $i \in \mathcal{I}$, and the measure of E is $\mu(E) = \epsilon > 0$. Then $\|I : L^1(iU) \to L^\infty(iU)\| \geq 1/\epsilon$. ❑

Lemma 2.8 Kurbatov, Proposition 1.4.2 [19]. *Let $\mathcal{A}$ be a Banach algebra, and let $A, B \in \mathcal{A}$. If the element A is invertible and $\|B\|\|A^{-1}\| < 1$, then $A - B$ is invertible and $(A - B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \ldots$.*

Lemma 2.9 Kurbatov, Lemma 5.3.3 [19]. *Let Q and $\overline{Q}$ be locally compact topological spaces with measures λ and $\overline{\lambda}$, respectively. For any $N \in \mathcal{B}(L^1(Q), L^\infty(\overline{Q}))$ there exists a function $n \in L^\infty(Q \times \overline{Q})$ such that for all $f \in L^1(Q)$ one has*

$$(Nf)(t) = \int n(t, s) x(s) d\lambda(s).$$

Proposition 2.10 *The operator N is an integral operator majorized by $W(L^\infty(\mathbb{H}), l_v^1)$ if and only if $\sum_\gamma \sup_{\mu\nu^{-1}=\gamma} \|N_{\mu,\nu} : L^1(Q_\nu) \to L^\infty(Q_\nu)\| v(\gamma) < \infty$. That is, $\mathcal{N}_v^\infty(\mathbb{H})$ is the class of integral operators with kernels majorized by functions in $W(L^\infty(\mathbb{H}), l_v^1)$.*

Proof If N is majorized by $\beta \in W(L^\infty(\mathbb{H}), l_v^1)$, then $\|N_{\mu,\nu} : L^1(Q_\nu) \to L^\infty(Q_\mu)\| \leq \sup_{Q_{\mu\nu^{-1}}} |\beta|$, which proves the first claim. We prove the second claim. Let $f_\mu = P_\mu f$, where P_μ is the projection onto Q_μ, as defined following Definition 2.4. $N \in \mathcal{N}_v^\infty(\mathbb{H})$ implies $(Nf)_\mu = \sum_{\nu \in \mathbb{Z}^{2d} \times \mathbb{D}} N_{\mu\nu} f_\nu$. In the following, r, s and t will be elements of $\mathbb{R}^{2d} \times \mathbb{K} \times \mathbb{K}^{\perp} \times \mathbb{T}$. By Lemma 2.9, there exists $n_{\mu\nu} \in L^{\infty}(Q_\mu \times Q_\nu)$ such that

$$(N_{\mu\nu} f_\nu)(t) = \int_{Q_\nu} n_{\mu\nu}(t,s) f_\nu(s) d\lambda(s)$$

for $t \in Q_\mu$. Setting

$$\alpha_\gamma = \sup_{\gamma = \mu\nu^{-1}} \|N_{\mu\nu} : L^1(Q_\nu) \to L^\infty(Q_\mu)\|,$$

$\gamma \in \mathbb{Z}^{2d} \times \mathbb{D}$, we have $|n_{\mu\nu}(t,s)| \leq \alpha_{\mu\nu^{-1}}$ for all $t \in Q_\mu$ and $s \in Q_\nu$. Defining $n(t,s)$ to equal $n_{\mu\nu}(t,s)$ for $t \in Q_\mu$, $s \in Q_\nu$, we have

$$\begin{aligned}(Nf)(t) &= \sum_{\mu \in \mathbb{Z}^{2d} \times \mathbb{D}} \sum_{\nu \in \mathbb{Z}^{2d} \times \mathbb{D}} \int_{Q_\nu} n_{\mu\nu}(t,s) f(s) d\lambda(s) \\ &= \int n(t,s) f(s) d\lambda(s).\end{aligned}$$

We now must show that n is majorized by a function $\beta \in W(L^\infty(\mathbb{H}), l^1_v)$.

$$Q_\mu Q_\nu^{-1} = (i_\mu i_\nu^{-1}, d_\mu d_\nu^{-1}, 0)((-1,1)^{2d} \times \mathbb{K} \times \mathbb{K}^\perp \times \mathbb{T}),$$

where $\mu = (i_\mu, d_\mu)$ and $\nu = (i_\nu, d_\nu)$.

For $r \in \mathbb{R}^{2d} \times \mathbb{K} \times \mathbb{T}$, we define

$$\beta(r) = \sup\{\alpha_\mu : (r) \in (i_\mu, d_\mu, 0)((-1,1)^{2d} \times \mathbb{K} \times \mathbb{K}^\perp \times \mathbb{T})\}.$$

We then have $|n(t,s)| \leq \alpha_{\mu\nu^{-1}} \leq \beta(ts^{-1})$ for $t \in Q_\mu$ and $s \in Q_\nu$. We define $\Delta = \Delta(r)$ to be the set of all grid points $\mu \in \mathbb{Z}^{2d} \times \mathbb{D}$ such that $r \in (i_\mu, d_\mu, 0)((-1,1)^{2d} \times \mathbb{K} \times \mathbb{K}^\perp \times \mathbb{T})$. Now $\beta(r) \leq \max\{\alpha_\mu : \mu \in \Delta\}$, which implies that $\beta(r) \leq \sum_{\mu \in \Delta} \alpha_\mu$. By the definition of $\mathcal{N}_v^\infty(\mathbb{H})$, $\sum_\mu \alpha_\mu v(\mu) < \infty$. Δ has at most 2^{2d} elements for any r. Since β is constant on each block Q_μ, $\|\beta\|_{W(L^\infty(\mathbb{H}), l^1_v)} \leq 2^{2d} \sum_\mu \alpha_\mu v(\mu) < \infty$. ❑

Proposition 2.11 *$\mathcal{N}_v^\infty(\mathbb{H})$ is dense in $\mathcal{N}_v^1(\mathbb{H})$.*

Proof By Proposition 2.10 we may use that $W(L^\infty(\mathbb{H}), l^1_v)$ is dense in $W(L^1(\mathbb{H}), l^1_v)$, and choose $\overline{\beta} \in W(L^\infty(\mathbb{H}), l^1_v)$ such that $\|\beta - \overline{\beta}\|_{W(L^1(\mathbb{H}), l^1_v)} < \epsilon$. We may assume that $0 \leq \overline{\beta}(\mathbf{h}) \leq \beta(\mathbf{h})$ for all $\mathbf{h}$. Set

$$\overline{n}(\mathbf{h}_0, \mathbf{h}_1) = \begin{cases} \frac{\overline{\beta}(\mathbf{h}_0 \mathbf{h}_1^{-1})}{\beta(\mathbf{h}_0 \mathbf{h}_1^{-1})} & : \ \beta(\mathbf{h}_0 \mathbf{h}_1^{-1}) \neq 0 \\ 0 & : \ \beta(\mathbf{h}_0 \mathbf{h}_1^{-1}) = 0 \end{cases}$$

We then have $\overline{n}(\mathbf{h}_0, \mathbf{h}_1) \leq \overline{\beta}(\mathbf{h}_0 \mathbf{h}_1^{-1})$ and $0 \leq n(\mathbf{h}_0, \mathbf{h}_1) - \overline{n}(\mathbf{h}_0, \mathbf{h}_1) \leq \beta(\mathbf{h}_0 \mathbf{h}_1^{-1}) - \overline{\beta}(\mathbf{h}_0 \mathbf{h}_1^{-1})$. ❑

The following is one of the three cases covered by Baskakov's Theorem 1 in [4].

Theorem 2.12 *Let v be an admissible weight function, $\mathbb{I}$ a discrete abelian group, and $\{X_i\}_{i\in\mathbb{I}}$ subspaces of X satisfying $X_i \cap X_j = \{0\}$ for $i \neq j$ and $X = \overline{\text{span}}\{X_i\}_{i\in\mathbb{I}}$. Let P_i be the projection onto X_i. If T is invertible in $\mathcal{B}(X)$ and*

$$\sum_{i\in\mathbb{I}} \sup_{j-k=i} \|P_k T P_j\| v(n) < \infty$$

then

$$\sum_{i\in\mathbb{Z}} \sup_{j-k=i} \|P_j T^{-1} P_k\| v(n) < \infty.$$

Theorem 2.13 *If $N \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$ is invertible in $\mathcal{B}(W(L_v^p(\mathbb{H}), l^q))$, then $N^{-1} \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$.*

Proof We first define the space of operators

$$M_v = \{T | \sum_{\gamma\in\mathbb{Z}^{2d}\times\mathbb{D}} \sup_{\mu\nu^{-1}=\gamma} \|T : L^p(Q_\nu) \to L^p(Q_\mu)\| v(\gamma) < \infty \;\; \forall p \in [1,\infty]\},$$

where for each p, T is understood to be identically defined on the common part of different spaces. (Note that $L^\infty(Q_\mu)$ is dense in each $L^p(Q_\mu)$, $p \in [0,\infty]$.) By Theorem 2.12, if $T \in M_v$ and T is invertible in $\mathcal{B}(W(L^p(\mathbb{H}), l_v^q))$, then $T^{-1} \in M_v$. I is clearly an element of M_v, though it is not an element of $\mathcal{N}_v^\infty(\mathbb{H})$ by Lemma 2.7. For $N \in \mathcal{N}_v^\infty(\mathbb{H})$ and $T \in M_v$,

$$\begin{aligned}
&\sum_{\gamma\in\mathbb{Z}^{2d}\times\mathbb{D}} \sup_{\mu\nu^{-1}=\gamma} \|NT : L^1(Q_\nu) \to L^\infty(Q_\mu)\| v(\gamma) \\
&\leq \sup_{\mu,\nu} \|T : L^\infty(Q_\nu) \to L^\infty(Q_\mu)\| \sum_{\gamma\in\mathbb{Z}^{2d}\times\mathbb{D}} \sup_{\mu\nu^{-1}=\gamma} \|N : L^1(Q_\nu) \to L^\infty(Q_\mu)\| v(\gamma),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\gamma\in\mathbb{Z}^{2d}\times\mathbb{D}} \sup_{\mu\nu^{-1}=\gamma} \|TN : L^1(Q_\nu) \to L^\infty(Q_\mu)\| v(\gamma) \\
&\leq \sup_{\mu,\nu} \|T : L^1(Q_\nu) \to L^1(Q_\mu)\| \sum_{\gamma\in\mathbb{Z}^{2d}\times\mathbb{D}} \sup_{\mu\nu^{-1}=\gamma} \|N : L^1(Q_\nu) \to L^\infty(Q_\mu)\| v(\gamma);
\end{aligned}$$

therefore, $\mathcal{N}_v^\infty(\mathbb{H})$ is a proper two-sided ideal in M_v. If $\alpha_1 I + N_1 \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$ and $\alpha_1 I + N_1$ is invertible in $\mathcal{B}(L^p(\mathbb{H}))$, then Theorem 2.12 implies $(\alpha_1 I + N_1)^{-1} = T$ for some $T \in M_v$. Then $(\alpha_1 I + N_1)T = I$ implies $T = \frac{1}{\alpha_1} I - \frac{1}{\alpha_1} N_1 T$. By the ideal property of $\mathcal{N}_v^\infty(\mathbb{H})$, $T = \alpha_2 I + N_2$ for some $N_2 \in \mathcal{N}_v^\infty(\mathbb{H})$.

❑

Lemma 2.14 *$\mathcal{N}_v^\infty(\mathbb{H})$ is a two-sided ideal in $\mathcal{N}_v^1(\mathbb{H})$.*

Proof Assume $f \in L^1_v(\mathbb{H})$ and $g \in L^\infty_v(\mathbb{H})$. Then

$$\begin{aligned}
\|f \star g\|_{L^\infty_v(\mathbb{H})} &= \sup_{\mathbf{h}_0} \left| \int_{\mathbb{H}} f(\mathbf{h}) g(\mathbf{h}^{-1}\mathbf{h}_0) d\mathbf{h} \right| v(\mathbf{h}_0) \\
&\leq \int_{\mathbb{H}} |f(\mathbf{h})| \sup_{\mathbf{h}_0} |g(\mathbf{h}^{-1}\mathbf{h}_0) v(\mathbf{h}_0)| d\mathbf{h} \\
&\leq \int_{\mathbb{H}} |f(\mathbf{h})| \sup_{\mathbf{h}_0} |g(\mathbf{h}_0) v(\mathbf{h}_0)| v(\mathbf{h}) d\mathbf{h} \\
&= \|f\|_{L^1_v(\mathbb{H})} \|g\|_{L^\infty_v(\mathbb{H})}.
\end{aligned}$$

One similarly shows that $L^\infty_v(\mathbb{H}) \star L^1_v(\mathbb{H}) \subset L^\infty_v(\mathbb{H})$. By Theorem 11.8.3 in [17] and the discussion immediately following it concerning $\mathbb{H}$, $W(L^\infty(\mathbb{H}), l^1_v)$ is a two-sided ideal in $W(L^1(\mathbb{H}), l^1_v)$ with respect to convolution. The lemma then follows from the composition rule for majorized integral operators given at the start of the proof of Theorem 2.6.

❑

Proof of Theorem 2.6 Let $N_1,\ N_2 \in \mathcal{N}^1_v(\mathbb{H})$ be majorized, respectively, by β_1 and β_2. Using Fubini's theorem, we have

$$\begin{aligned}
(N_1 N_2) f(\mathbf{h}_0) &= \int n_1(\mathbf{h}_0, \mathbf{h}_1) \int n_2(\mathbf{h}_1, \mathbf{h}_2) f(\mathbf{h}_2) d\mathbf{h}_2 d\mathbf{h}_1 \\
&= \iint n_1(\mathbf{h}_0, \mathbf{h}_1) n_2(\mathbf{h}_1, \mathbf{h}_2) f(\mathbf{h}_2) d\mathbf{h}_1 d\mathbf{h}_2 \\
&= \int n(\mathbf{h}_0, \mathbf{h}_2) f(\mathbf{h}_2) d\mathbf{h}_2.
\end{aligned}$$

Therefore, $N_1 N_2$ defines an integral operator of the same form.

$$\begin{aligned}
(N_1 N_2) f(\mathbf{h}_0) &= \int n_1(\mathbf{h}_0, \mathbf{h}_1) \int n_2(\mathbf{h}_1, \mathbf{h}_2) f(\mathbf{h}_2) d\mathbf{h}_2 d\mathbf{h}_1 \\
&\leq \int \beta_1(\mathbf{h}_0^{-1}\mathbf{h}_1) \int \beta_2(\mathbf{h}_1^{-1}\mathbf{h}_2) |f(\mathbf{h}_2)| d\mathbf{h}_2 d\mathbf{h}_1 \\
&= \iint \beta_1(\mathbf{h}_1) \beta_2(\mathbf{h}_1^{-1}\mathbf{h}_0^{-1}\mathbf{h}_2) d\mathbf{h}_1 |f(\mathbf{h}_2)| d\mathbf{h}_2 \\
&= \int \beta(\mathbf{h}_0^{-1}\mathbf{h}_2) |f(\mathbf{h}_2)| d\mathbf{h}_2
\end{aligned}$$

for $\beta = \beta_1 \star \beta_2 \in L^1_v(\mathbb{H})$. This establishes the Banach algebra property.

Assume that the operator $\alpha I + N$, $N \in \mathcal{N}^1_v(\mathbb{H})$, is invertible. We first show that $I \notin \mathcal{N}^1_v(\mathbb{H})$. If $I \in \mathcal{N}^1_v(\mathbb{H})$, then since $\mathcal{N}^\infty_v(\mathbb{H})$ is dense in $\mathcal{N}^1_v(\mathbb{H})$ (Proposition 2.11), $\mathcal{N}^\infty_v(\mathbb{H})$ would contain a sequence approaching I. Lemma 2.7 shows that such an operator would be unbounded in the $\mathcal{N}^\infty_v(\mathbb{H})$-norm. Therefore $\alpha \neq 0$.

By Proposition 2.11 we may choose $\overline{N} \in \mathcal{N}^\infty_v(\mathbb{H})$ such that $\|n - \overline{n}\|_{W(L^1(\mathbb{H}), l^1_v)} < \alpha/2$. Then by Lemma 2.8, $\alpha I + (N - \overline{N})$ is invertible in $\widetilde{\mathcal{N}^1_v}(\mathbb{H})$. As in the proof of Theorem 5.4.7 in [19] we consider the operator

$$\begin{aligned} K &= (\alpha I + (N - \overline{N}))^{-1}(\alpha I + N) \\ &= (\alpha I + (N - \overline{N}))^{-1}(\alpha I + (N - \overline{N}) + \overline{N}) \\ &= (\alpha I + (N - \overline{N})^{-1}(\alpha I + (N - \overline{N})) + (\alpha I + (N - \overline{N}))^{-1}\overline{N} \\ &= I + (\alpha I + (N - \overline{N}))^{-1}\overline{N}. \end{aligned}$$

K is invertible as the product of two invertible operators in $\widetilde{\mathcal{N}_v^1}(\mathbb{H})$. By the ideal property of $\mathcal{N}_v^\infty(\mathbb{H})$, Proposition 2.11, $(\alpha I + (N - \overline{N}))^{-1}\overline{N} \in \mathcal{N}_v^\infty(\mathbb{H})$; therefore Theorem 2.13 implies that $K^{-1} \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$. The composition of $K^{-1} \in \widetilde{\mathcal{N}_v^\infty}(\mathbb{H})$ and $(\alpha I + (N - \overline{N}))^{-1} \in \widetilde{\mathcal{N}_v^1}(\mathbb{H})$ is also in $\widetilde{\mathcal{N}_v^1}(\mathbb{H})$. Therefore, $(\alpha I + N)^{-1} = K^{-1}(\alpha I + (N - \overline{N}))^{-1} \in \widetilde{\mathcal{N}_v^1}(\mathbb{H})$. ❏

Theorem 2.6 allows us to prove the spectral algebra property for convolution operators on the Heisenberg group.

Corollary 2.15 *Let $\mathbb{H}$ be the general, reduced Heisenberg group, v an admissible weight function, and S_f the operator given by convolution with f. If $\alpha_1 I + S_f$, $f \in L_v^1(\mathbb{H})$, is invertible in $\mathcal{B}(L^p(\mathbb{H}))$, then $(\alpha_1 I + S_f)^{-1} = \alpha_2 I + S_g$, $g \in L_v^1(\mathbb{H})$.*

Remark : Barnes proves in [2] that the spectral algebra property for a convolution operator on $L^1(G)$ is equivalent to G being amenable and symmetric. Since $\mathbb{H}$ is nilpotent, it is symmetric [21]. Taking M as a mean on $L^\infty(\mathbb{G} \times \hat{\mathbb{G}})$, $M_{\mathbb{H}}(f) = \int_{\mathbb{T}} M(f(\cdot, \cdot, e^{2\pi i\tau}))d\tau$, is a shift-invariant mean on $L^\infty(\mathbb{H})$, and consequently $\mathbb{H}$ is amenable; see chapters 2 and 12 of [22]. Therefore, Corollary 2.15 also follows from Barnes's work in [2]. For the case when G is compactly generated, Corollary 2.15 is also a special case of Theorems 3.6 and 3.7 in [7].

Proof For $F_1, F_2 \in L_v^1(\mathbb{H})$, $\|F_1 \star F_2\|_{L_v^1(\mathbb{H})} \leq \int_{\mathbb{H}} \int_{\mathbb{H}} |F_1(\mathbf{h})||F_2(\mathbf{h}^{-1}\mathbf{h}_0)|v(\mathbf{h}_0)d\mathbf{h}_0 d\mathbf{h} = \int_{\mathbb{H}} |F_1|(\int_{\mathbb{H}} |F_2(\mathbf{h}_0)|v(\mathbf{h}_0)d\mathbf{h}_0)v(\mathbf{h})d\mathbf{h} = \|F_1\|_{L_v^1(\mathbb{H})}\|F_2\|_{L_v^1(\mathbb{H})}$. Consequently,

$$(\alpha_1\delta + F_1) \star (\alpha_2\delta + F_2) = \alpha_1\alpha_2\delta + \alpha_1 F_2 + \alpha_2 F_1 + F_1 \star F_2 = \alpha_3\delta + F, \quad F \in L_v^1(\mathbb{H}).$$

To meet the conditions of Theorem 2.6, we define the function $F(\mathbf{h}_0, \mathbf{h}) = f(\mathbf{h}_0\mathbf{h}^{-1})$. Then αI plus the integral operator with kernel F is the same as $S_{\alpha\delta+f}$ and satisfies the conditions of Theorem 2.6. Assuming $S_{\alpha_1\delta+f}$ to be invertible in $\mathcal{B}(L^p(H))$, Theorem 2.6 states that $S_{\alpha_1\delta+f}^{-1} = \alpha_2 I + A$, where A is an integral operator majorized by a function $\beta \in L_v^1(\mathbb{H})$. We use an approximate identity $\{\psi_n\}_{n\geq 0} \subset L_v^1(\mathbb{H})$. We set $\theta_n = S_{\alpha_1\delta+f}^{-1}\psi_n$, and $\theta = \lim_{n\to\infty}(\alpha_2 I + A)\psi_n = \alpha_2\delta + \lim_{n\to\infty} A\psi_n$. Since A is majorized by $\beta \in L_v^1(\mathbb{H})$,

$$\begin{aligned} \lim_{n\to\infty} |A\psi_n|(\mathbf{h}_0) &\leq \lim_{n\to\infty} \int_{\mathbb{H}} \psi_n(\mathbf{h})\beta(\mathbf{h}^{-1}\mathbf{h}_0)d\mathbf{h} \\ &= \beta(\mathbf{h}_0) \end{aligned}$$

Set $g = \lim_{n\to\infty} A\psi_n \in L^1_v(\mathbb{H})$. Then $\theta = \alpha_2\delta + g$, and by the continuity of convolution, $(\alpha_1\delta + f) \star (\alpha_2\delta + g) = \delta$. For any $\phi \in C_0(\mathbb{H})$, the space of continuous, compactly supported functions on $\mathbb{H}$,

$$\begin{aligned} S_{\alpha_1\delta+f}(S_\theta - (\alpha_2 I + A))\phi &= S_{\alpha_2\delta+f} S_{\alpha_2\delta+g}\phi - S_{\alpha_1\delta+f} S^{-1}_{\alpha_1\delta+f}\phi \\ &= (\alpha_1\delta + f) \star (\alpha_2\delta + g) - \phi \\ &= \delta \star \phi - \phi \\ &= 0 \end{aligned}$$

$S_{\alpha_1\delta+f}$ is assumed invertible in $\mathcal{B}(L^p(\mathbb{H}))$, and both $S_{\alpha_2\delta+g}\phi$ and $(\alpha_2 I + A)\phi$ are in $L^1_v(\mathbb{H})$; therefore,$(S_{\alpha_2\delta+g} - (\alpha_2 I + A))\phi = 0$ for all $\phi \in C_0(\mathbb{H})$. Since the space of continuous compactly supported functions is dense in $L^1_v(\mathbb{H})$, $S_{\alpha_2\delta+g} = \alpha_2 I + A$, and $S^{-1}_{\alpha_1\delta+f} = S_{\alpha_2\delta+g}$, $g \in L^1_v(\mathbb{H})$. Equivalently, if $\alpha_1 I + S_f$, $f \in L^1_v(\mathbb{H})$, is invertible in $\mathcal{B}(L^p(\mathbb{H}))$, then its inverse is also of the form $\alpha_2 I + S_g$, and $g \in L^1_v(\mathbb{H})$. ❑

3 Spectral Algebra Property for Twisted Convolution and Pseudodifferential Operators

$\mathbb{T}$ is originally adjoined to $\mathbb{G} \times \hat{\mathbb{G}}$, thus creating the Heisenberg group, in order to obtain group structure for $\mathbb{G} \times \hat{\mathbb{G}}$. However, functions defined only $\mathbb{G} \times \hat{\mathbb{G}}$ are still of special interest, particularly for pseudodifferential operators. Here we discuss the Weyl pseudodifferential operator L_σ, given by a symbol $\sigma \in \mathcal{S}'(\mathbb{G} \times \hat{\mathbb{G}})$:

$$L_\sigma f(t) = \int_{\mathbb{G}} \int_{\hat{\mathbb{G}}} \hat{\sigma}(\omega, x) e^{-\pi i \omega \cdot x} T_{-x} M_\omega f(t) d\omega dx. \tag{5}$$

The map $\sigma \mapsto L_\sigma$ is called the Weyl transform, and σ and $\hat{\sigma}$ are called the symbol and spreading function of the operator L_σ. The composition rule for two Weyl pseudodifferential operators is $L_\sigma L_\tau = L_{\mathcal{F}^{-1}(\hat{\sigma}\natural\hat{\tau})}$, where $\natural$ denotes *twisted convolution*[8] and is defined by

$$F\natural G(x_0, \omega_0) = \int_{\mathbb{G}} \int_{\hat{\mathbb{G}}} F(x, \omega) G(x_0 - x, \omega_0 - \omega) e^{\pi i (x\omega_0 - \omega x_0)} d\omega dx.$$

Since $F\natural G(x, \omega) \leq |F| * |G|(x, \omega)$, twisted convolution is dominated by regular convolution. Therefore, $L^1_v(\mathbb{G} \times \hat{\mathbb{G}})$ is closed with respect to twisted convolution. In order to prove the spectral algebra property for twisted convolution on $L^1_v(\mathbb{G}\times\hat{\mathbb{G}})$ we need a weighted version of Kurbatov's Theorem 2.5.

Theorem 3.1 *Let $\mathbb{G}$ be a locally compact abelian group. Then $\widetilde{\mathcal{N}^1_v}(\mathbb{G})$ is an inverse-closed Banach algebra in $\mathcal{B}(L^p(\mathbb{G}), l^q)$, $1 \leq p, q \leq \infty$.*

Proof Kurbatov's proof of his Theorem 5.4.7 [19] holds here. The addition of weights is justified by Theorem 2.12. ❑

Corollary 3.2 *Let $\mathbb{G}$ be a locally compact abelian group and $\hat{\mathbb{G}}$ its dual group, and let $T_f \in \mathcal{B}(L^p(\mathbb{G} \times \hat{\mathbb{G}}))$ be the operator given by twisted convolution with f: $T_f\phi = f \natural \phi$, $\phi \in L^p(\mathbb{G} \times \hat{\mathbb{G}})$. If $\alpha_1 I + T_f$ is invertible in $\mathcal{B}(L^p(\mathbb{G} \times \hat{\mathbb{G}}))$ and $f \in L^1_v(\mathbb{G} \times \hat{\mathbb{G}})$, then $(\alpha_1 I + T_f)^{-1} = \alpha_2 I + T_g$ and $g \in L^1_v(\mathbb{G} \times \hat{\mathbb{G}})$.*

Proof The proof of Corollary 2.15 carries over exactly with the sole substitution of $\mathbb{G} \times \hat{\mathbb{G}}$ and $\natural$ for $\mathbb{H}$ and $\star$. ❑

Before applying these theorems to pseudodifferential operators, we briefly discuss the importance of pseudodifferential operators in the study of time-varying communication systems, such as wireless communications. We view $f(t)$ as a transmitted signal; then $T_x f(t)$, $x > 0$, corresponds to a time shift of the signal, and $M_\omega f(t)$ corresponds to a modulation or frequency shift. The received signal at time t_0 is a weighted collection of delayed, modulated copies of the transmitted signal. Therefore the received signal may be expressed as

$$f_{rec}(t_0) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\sigma}(x, \omega) T_{-x} M_\omega f_{trans}(t_0) dx d\omega, \tag{6}$$

where we have absorbed $e^{-\pi i \omega \cdot x}$ into $\hat{\sigma}$. The assumption that $\hat{\sigma} \in L^1_v(\mathbb{R}^2)$ is appropriate, as in practice the strength of the delayed copies of the signal decays quickly in time. The Doppler effect or frequency shift depends on the travel speed of the signal and the relative speeds and angles between the transmitter, any reflecting bodies, and the receiver. Since these quantities are all bounded in practice, the Doppler effect is also bounded. Hence if the Doppler effect is limited to $[-D, D]$, the support of $\hat{\sigma}(x, \cdot)$ is contained in $[-D, D]$ for all x [23, 28]. In practice one must "numerically invert" the operator in (6). Theorem 3.3 states that the inverse will have the same off-diagonal decay as the original operator. The resulting matrix may, therefore, be truncated to a small number of diagonals, which is essential for fast real-world computation.

Above we showed that $\widetilde{L^1_v}(\mathbb{G} \times \hat{\mathbb{G}})$ is an inverse-closed Banach algebra with respect to twisted convolution. Due to the composition rule, the previous theorems easily establish the spectral algebra property for a class of Weyl pseudodifferential operators.

Theorem 3.3 *Let $\mathrm{OP}(\mathcal{F}^{-1} L^1_v(\hat{\mathbb{G}} \times \mathbb{G}))$ denote the space of pseudodifferential operators with Weyl symbol σ satisfying $\hat{\sigma} \in L^1_v(\hat{\mathbb{G}} \times \mathbb{G})$. Then $\widetilde{\mathrm{OP}}(\mathcal{F}^{-1} L^1_v(\hat{\mathbb{G}} \times \mathbb{G}))$ is an inverse-closed subalgebra of $\mathcal{B}(L^p(\mathbb{G}))$. That is*

(i) $\alpha I + L_\sigma$ *is bounded on all* $L^p(\mathbb{G})$.

(ii) *If* $\hat{\sigma}, \hat{\tau} \in L^1(\hat{\mathbb{G}} \times \mathbb{G})$, *then* $(\alpha_1 I + L_\sigma)(\alpha_2 I + L_\tau) = (\alpha_3 I + L_\gamma)$, *where* $\hat{\gamma} \in L^1(\hat{\mathbb{G}} \times \mathbb{G})$.

(iii) *If* $\alpha_1 I + L_\sigma$ *is invertible in* $\mathcal{B}(L^p(\mathbb{G}))$, *then* $(\alpha_1 I + L_\sigma)^{-1} = (\alpha_2 I + L_\tau)$ *where* $\hat{\tau} \in L^1(\hat{\mathbb{G}} \times \mathbb{G})$.

Proof
(i).

$$\begin{aligned}
\|L_\sigma f\|_{L^p}^p &\leq \int_{\mathbb{G}} \left| \int_{\hat{\mathbb{G}}} \int_{\mathbb{G}} \hat{\sigma}(\omega,x) e^{-\pi i \omega\cdot x} T_{-x} M_\omega f(t) dx d\omega \right|^p dt \\
&\leq \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \int_{\hat{\mathbb{G}}} |\hat{\sigma}(\omega,x)||f(t+x)| d\omega dx \right)^p dt \\
&= \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \|\hat{\sigma}(\cdot,-x)\|_{L^1} |f(t+x)| dx \right)^p dt \\
&= \| \, \|\hat{\sigma}(\cdot,u)\|_{L^1} * |f|(u) \|_{L^p}^p \\
&\leq \|\hat{\sigma}\|_{L^1}^p \|f\|_{L^p}^p \\
&\leq \|\hat{\sigma}\|_{L^1_v}^p \|f\|_{L^p}^p
\end{aligned}$$

Therefore, $\|(\alpha I + L_\sigma) f\|_{L^p(\mathbb{G})} \leq (|\alpha| + \|\hat{\sigma}\|_{L^1_v(\mathbb{G})}) \|f\|_{L^p(\mathbb{G})}$.
(ii). By Corollary 3.2, if $\hat{\sigma}, \hat{\tau} \in L^1(\hat{\mathbb{G}} \times \mathbb{G})$, $(\alpha_1 \delta + \hat{\sigma}) \natural (\alpha_2 \delta + \hat{\tau}) = (\alpha_3 \delta + \hat{\gamma})$, where $\hat{\gamma} \in L^1(\hat{\mathbb{G}} \times \mathbb{G})$. Then $\mathcal{F}^{-1}(\alpha_3 \delta + \hat{\gamma}) = \alpha_3 + \gamma$, and $(\alpha_1 I + L_\sigma)(\alpha_2 I + L_\tau) = (\alpha_3 I + L_\gamma)$.
(iii). Let $(\alpha_1 I + L_\sigma)^{-1} = A$, $A \in \mathcal{B}(L^p(\mathbb{G}))$. Using the Schwartz kernel theorem, Gröchenig shows in [16], that there exists a symbol $\gamma \in \mathcal{S}'(\mathbb{G})$, such that $A = L_\gamma$. In order to apply Corollary 3.2 we must show that the twisted convolution operator $T_{\hat{\gamma}}$ is bounded as an operator on $L^1(\hat{\mathbb{G}} \times \mathbb{G})$. By the closed graph theorem [25], $T_{\hat{\gamma}}$ is *not* bounded on $L^1(\hat{\mathbb{G}} \times \mathbb{G})$ if and only if there exists a sequence $\{\hat{\phi}_n\}_{n \in \mathbb{N}} \subset L^1(\hat{\mathbb{G}} \times \mathbb{G})$ such that $\{\hat{\phi}_n\} \to 0$, but $T_{\hat{\gamma}} \hat{\phi}_n \nrightarrow 0$. If $T_{\hat{\gamma}} \notin \mathcal{B}(L^1(\hat{\mathbb{G}} \times \mathbb{G}))$, then there exists an $\epsilon > 0$ and a subsequence $\{\hat{\phi}_{n_k}\}$ such that $\|T_{\hat{\gamma}} \hat{\phi}_{n_k}\|_{L^1(\hat{\mathbb{G}} \times \mathbb{G})} > \epsilon$ for all n_k. $L_\gamma \in \mathcal{B}(L^p(\mathbb{G}))$ by assumption, and $L_{\phi_{n_k}} \in \mathcal{B}(L^p(\mathbb{G}))$ by (i). Therefore $L_\gamma L_{\phi_{n_k}} \in \mathcal{B}(L^p(\mathbb{G}))$. By (i) we have

$$\|L_\gamma L_{\phi_{n_k}} - L_\gamma L_{\phi_{n_l}}\|_{\mathcal{B}(L^p(\mathbb{G}))} \leq \|L_\gamma\|_{\mathcal{B}(L^p(\mathbb{G}))} \|\hat{\phi}_{n_k} - \hat{\phi}_{n_k}\|_{L^1(\hat{\mathbb{G}} \times \mathbb{G})}.$$

Since $\{\hat{\phi}_{n_k}\}$ is a convergent sequence, $\{L_\gamma L_{\phi_{n_k}}\}$ is a Cauchy sequence in the Banach space $\mathcal{B}(L^p(\mathbb{G}))$. Since $\lim_{k\to\infty} \{\hat{\phi}_{n_k}\} \to 0$, $\lim_{k\to\infty} \{L_\gamma L_{\phi_{n_k}}\} \equiv 0$, where the latter convergence is in operator norm. By Theorem 14.6.1 in [16], $\lim_{k\to\infty} \|T_{\hat{\gamma}} \hat{\phi}_{n_k}\|_{L^2(\hat{\mathbb{G}} \times \mathbb{G})} = 0$, which implies $\lim_{k\to\infty} \|T_{\hat{\gamma}} \hat{\phi}_{n_k}\|_{L^1(\hat{\mathbb{G}} \times \mathbb{G})} = 0$ and contradicts the assumption. Thus $T_{\hat{\gamma}} \in \mathcal{B}(L^1(\hat{\mathbb{G}} \times \mathbb{G}))$. Therefore $(\alpha_1 I + L_\sigma) L_\gamma = I$ implies $(\alpha_1 \delta + \hat{\sigma}) \natural \hat{\gamma} = \delta$, and similarly $\hat{\gamma} \natural (\alpha_1 \delta + \hat{\sigma}) = \delta$. By Corollary 3.2., $\gamma = \alpha_2 \delta + \tau$, $\hat{\tau} \in L^1(\hat{\mathbb{G}} \times \mathbb{G})$.

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