QUANTITATIVE ESTIMATES FOR THE FINITE SECTION METHOD

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Abstract. The finite section method is a classical scheme to approximate the solution of an infinite system of linear equations. We present quantitative estimates for the rate of the convergence of the finite section method on weighted ℓ^p-spaces. Our approach uses recent results from the theory of Banach algebras of matrices with off-diagonal decay. Furthermore, we demonstrate that Banach algebra theory provides a natural framework for deriving a finite section method that is applicable to large classes of non-hermitian matrices. An example from digital communication illustrates the practical usefulness of the proposed theoretical framework.

1. Introduction

Many of the concrete applications of mathematics in science and engineering eventually result in a problem involving linear operator equations. This problem can be usually represented as a linear system of equations (for instance by discretizing an integral equation or because the operator equation is already given on some sequence space) of the form

$$Ax = b, \tag{1.1}$$

where A is an infinite matrix $A = (a_{kl})_{k,l\in\mathbb{Z}}$ and b belongs to some Banach space of sequences. Solving linear equations with infinitely many variables is a problem of functional analysis, while solving equations with finitely many variables is one of the main themes of linear algebra. Numerical analysis bridges the gap between these areas. A fundamental problem of numerical analysis is thus to find a finite-dimensional model for (1.1) whose solution approximates the solution of the original infinite-dimensional problem with any desired accuracy. This problem often leads to delicate questions of stability and convergence.

A simple and useful approach is the *finite-section method* [11, 17]. Let

$$P_n b = (\dots, 0, b_{-n}, b_{-n+1}, \dots, b_{n-1}, b_n, 0, \dots)$$

be the orthogonal projection onto a $2n+1$-dimensional subspace. We set

$$A_n = P_n A P_n \qquad \text{and} \qquad b_n = P_n b\,, \tag{1.2}$$

Key words and phrases. Finite section method, inverse-closed Banach algebras, spectral invariance.

K. G. and Z. R. were supported by the Marie-Curie Excellence Grant MEXT-CT 2004-517154, T. S. was partially supported by NSF DMS Grant 0511461.

and try to solve the finite system

$$A_n x_n = b_n \tag{1.3}$$

for properly chosen n. The crucial question is then: What is the relation between the numerical solution x_n and the actual solution x?

This problem has been analyzed in depth for the case of convolution operators and Toeplitz matrices in the pioneering work of Gohberg, e.g. see [11]. Important generalizations and extensions in the Toeplitz setting can be found in [4, 5]. Rabinovich et al. derive necessary and sufficient conditions for the convergence of the finite section method in terms of the so-called limit operator [20], which does not necessarily require any Toeplitz structure. These conditions, while intriguing, are not always easy to verify in practice.

A general theory for the approximation by finite-section is based on the powerful methods of C^*-algebras and has been developed by Böttcher, Silbermann, and coworkers, see for instance [4, 17]. Their framework leads to many attractive and deep results about the applicability of the finite section method as well as other approximation methods. William Arveson goes a step further and concludes that *"numerical problems involving infinite dimensional operators require a reformulation in terms of C^*-algebras"* [1]. However, C^*-algebras have some limitations. It was already pointed out in [17] that C^*-algebra techniques do not yield any information about the speed of convergence of the finite section method. An answer to this question is obviously not only of theoretical interest, but it is important for real applications. For instance, we want to choose n in (1.3) large enough to get a sufficiently accurate solution, but on the other hand, n should be small enough to bound the computational complexity which in general is of order $\mathcal{O}(n^3)$. Theorems about the speed of convergence will give a quantitative indication for how increasing n will impact the accuracy of the solution. Some results about the speed of convergence for the special case of Toeplitz matrices can be found in [12, 22, 25, 26]. In [12] the convergence in the ℓ^p-norm ($1 \le p < \infty$) is analyzed.

In this paper we present a thorough analysis of the convergence of the finite section method for positive definite matrices as well as for non-hermitian ones. Specifically, we solve the following problems.

(a) We study the finite section method on weighted ℓ^p-spaces. If the input vector b belongs to a weighted space ℓ^p_m, then, under suitable assumptions on the matrix A, the finite section method converges in the norm of ℓ^p_m.

(b) We obtain quantitative estimates for the rate of convergence of x_n to x in various weighted ℓ^p-norms.

(c) We define a modified version of finite sections, the *non-symmetric finite section method*, and show that this method converges also for non-symmetric matrices. The finite section method for non-symmetric matrices raises a number of rather difficult questions and has motivated a large part of [17]. Even for the classical case of Laurent operators (Toeplitz matrices) our approach enlarges considerably the class of matrices to which the finite section method can be applied.

As we work with Banach spaces of sequences, the methods will be taken from the theory of B^*-algebras (involutive Banach algebras) instead of C^*-algebras which suit only Hilbert spaces. The key property of the matrices A is their off-diagonal decay; we will rely heavily on recent results from the theory of Banach algebras of matrices. In fact, an important technical part of our analysis is to establish a finite section property of infinite-dimensional matrix algebras.

The paper is organized as follows. In Section 2 we recall the well known proof for the convergence of the finite section method for positive invertible matrices and take it as a model for more general statements. In Section 3 we introduce several Banach algebras of infinite matrices and collect their fundamental properties. Section 4 is devoted to the notion of inverse-closedness and spectral invariance in Banach algebras and their relation to the finite section method. In Section 5 we establish the convergence of the finite section method on weighted ℓ^p-spaces, in Section 6 we derive quantitative estimates. In Section 7 we investigate a version of the finite section method for non-symmetric matrices, and in the final Section 8 we briefly discuss an application to wireless communications.

2. Convergence of the finite section method

It is well known that for positive definite matrices the finite section method works in principle, see, e.g., [17]. The proof is instructive and exhibits what is necessary for an understanding of the finite section method.

Recall that if $\mathcal{A}$ is an algebra, then the spectrum of an element $A \in \mathcal{A}$ is defined to be the set $\sigma_{\mathcal{A}}(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \,\text{is not invertible}\}$. If the algebra is $\mathcal{B}(\mathcal{H})$, the bounded operators on some Hilbert space, we usually omit the reference to the algebra and write simply $\sigma(A)$ for the spectrum. For self-adjoint operators on $\mathcal{H}$ we denote the extremal spectral values of $\sigma(A)$ by $\lambda_- = \min \sigma(A)$ and $\lambda_+ = \max \sigma(A)$, so that $\sigma(A) \subseteq [\lambda_-, \lambda_+]$.

We will analyze the finite section method for multidimensional index sets of the form $\mathbb{Z}^d$. To that end we define the projection P_n in dimension $d > 1$. We set $C_n = [-n, n]^d \cap \mathbb{Z}^d$, the integer vectors in the cube of length $2n$ centered at the origin. Then the projection P_n is defined by $(P_n y)(k) = \chi_{[-n,n]^d}(k) y(k) = \chi_{C_n}(k) y(k)$ for $k \in \mathbb{Z}^d$. The range of P_n is a subspace of $\ell^2(\mathbb{Z}^d)$ of dimension $(2n+1)^d$ and will be identified with $\mathbb{C}^{(2n+1)^d}$. The finite section is then defined to be $A_n = P_n A P_n$. By definition, A_n is a (finite rank) operator acting on $\ell^2(\mathbb{Z}^d)$, but we often interpret A_n as a finite $(2n+1)^d \times (2n+1)^d$-matrix acting on $\mathbb{C}^{(2n+1)^d}$. In particular, by A_n^{-1} we understand the inverse of this finite matrix, but clearly A_n cannot be invertible on $\ell^2(\mathbb{Z}^d)$.

We mention that our results could also be formulated with respect to other index sets.

Theorem 1. *If A is a positive and (boundedly) invertible operator on $\ell^2(\mathbb{Z}^d)$, then x_n converges to x in $\ell^2(\mathbb{Z}^d)$.*

Proof. **Step 1.** Since by hypothesis, $\sigma(A) \subseteq [\lambda_-, \lambda_+] \subseteq (0, \infty)$, we have

$$\lambda_- \|P_n b\|_2^2 \le \langle A P_n b, P_n b\rangle = \langle A_n b, b\rangle \le \lambda_+ \|P_n b\|_2^2 \,.$$

Consequently on the invariant subspace $P_n \ell^2(\mathbb{Z}^d) \simeq \mathbb{C}^{(2n+1)^d}$

$$\sigma(A_n) \subseteq [\lambda_-, \lambda_+]$$

independent of n. In particular, each A_n is invertible on $\mathbb{C}^{(2n+1)^d}$ and

$$\sup_{n \in \mathbb{N}} \|A_n^{-1}\|_{op} \leq \lambda_-^{-1} = \|A^{-1}\|_{op} . \tag{2.1}$$

Step 2. Define an extension of A_n by

$$\widetilde{A_n} = A_n + \lambda_+(I - P_n) . \tag{2.2}$$

Then $\sigma(\widetilde{A_n}) \subseteq [\lambda_-, \lambda_+]$, and all matrices $\widetilde{A_n}$ are invertible on $\ell^2(\mathbb{Z}^d)$. Furthermore, $\widetilde{A_n}^{-1} = A_n^{-1} + \lambda_+^{-1}(I - P_n)$ and $\widetilde{A_n}$ converges to A in the strong operator topology.

Step 3. (Lemma of Kantorovich). Since

$$\begin{aligned} \|\widetilde{A_n}^{-1} b - A^{-1} b\|_2 &= \|\widetilde{A_n}^{-1}(A - \widetilde{A_n})A^{-1}b\|_2 \\ &\leq \sup_n \|\widetilde{A_n}^{-1}\|_{op} \, \|(A - \widetilde{A_n})A^{-1}b\|_2 \, , \end{aligned} \tag{2.3}$$

the strong convergence $\widetilde{A_n} \rightharpoonup A$ implies that $\widetilde{A_n}^{-1}$ converges strongly to A^{-1}.

Step 4. Recall $A_n x_n = b_n$ and $Ax = b$. Then

$$\begin{aligned} \|x - x_n\|_2 &= \|A^{-1}b - A_n^{-1}b_n\|_2 = \|A^{-1}b - A_n^{-1}P_n b\|_2 \\ &\leq \|(A^{-1} - \widetilde{A_n}^{-1})b\|_2 + \|\widetilde{A_n}^{-1}(b - P_n b)\|_2 = \mathrm{I} + \mathrm{II} \, . \end{aligned} \tag{2.4}$$

The first term goes to zero by Step 3, and the second term is estimated by

$$\mathrm{II} \leq \sup_n \|\widetilde{A_n}^{-1}\|_{op} \, \|b - P_n b\|_2 \leq \lambda_-^{-1} \|b - P_n b\|_2$$

and also goes to zero. □

The above theorem uses the $\ell^2(\mathbb{Z}^d)$-norm, so this is the realm of C^*-algebra techniques, cf. the work of Böttcher, Silbermann, et al. [4,17].

Several questions arise naturally in the context of the finite section method:
1. Does the finite section method also converge in other norms, e.g., in weighted ℓ^p-norms?
2. Can we derive quantitative estimates? If the finite section method works, how fast does x_n converge to x? What conditions on the matrix A and the input vector b are required to quantify the rate of convergence $x_n \to x$?
3. What conditions and modifications are required (if any) to make the finite section method work for matrices that are not hermitian?

For an answer of the first question, we make the following observation: The simple argument above extends almost word by word, provided we can show the following properties:

(1) Both A and A^{-1} are bounded on ℓ^p_m,
(2) $\sup_n \|\widetilde{A_n}^{-1}\|_{\ell^p_m \to \ell^p_m}$ is finite, and
(3) the finite sequences are dense in ℓ^p_m.

The answers to the other two questions also revolve around the above observation as well as on properties of certain involutive Banach algebras, which will be introduced in the next section.

3. A Class of Banach Algebras of Matrices

To understand the asymptotic behavior of the finite section method on Banach spaces, we need to resort to Banach algebra methods. We first consider some typical matrix norms that express various forms of off-diagonal decay. Our approach is partly motivated by some forms of off-diagonal decay that is observed in various applications, such as signal and image processing, digital communication, and quantum physics. A different way of describing off-diagonal decay of matrices (and operators) is given by the notion of band-dominated operators [22].

Weights. Off-diagonal decay is quantified by means of weight functions. A non-negative function v on $\mathbb{Z}^d$ is called an *admissible weight* if it satisfies the following properties:

(i) v is even and normalized such that $v(0) = 1$.

(ii) v is submultiplicative, i.e., $v(k+l) \le v(k)v(l)$ for all $k, l \in \mathbb{Z}^d$.

The assumption that v is even assures that the corresponding Banach algebra is closed under taking the adjoint A^*. The weight v is said to satisfy the *Gelfand-Raikov-Shilov (GRS) condition* [10], if

$$\lim_{n\to\infty} v(nk)^{\frac{1}{n}} = 1 \qquad \text{for all } k \in \mathbb{Z}^d. \tag{3.1}$$

This property is crucial for the inverse-closedness of Banach algebras, see Theorem 3 below. The standard weight functions on $\mathbb{Z}^d$ are of the form

$$v(x) = e^{ad(x)^b}(1 + d(x))^s\,,$$

where $d(x)$ is a norm on $\mathbb{R}^d$. Such a weight is submultiplicative, when $a, s \ge 0$ and $0 \le b \le 1$; v satisfies the GRS-condition, if and only if $0 \le b < 1$.

Consider the following conditions on matrices.

1. *The Jaffard class* is defined by polynomial decay off the diagonal. Let $\mathcal{A}_s$ be the class of matrices $A = (a_{kl}), k, l \in \mathbb{Z}^d$, such that

$$|a_{kl}| \le C(1 + |k-l|)^{-s} \qquad \forall k, l \in \mathbb{Z}^d \tag{3.2}$$

with norm $\|A\|_{\mathcal{A}_s} = \sup_{k,l\in\mathbb{Z}^d} |a_{kl}|(1 + |k-l|)^s$.

2. *More general off-diagonal decay.* Let v be an admissible weight on $\mathbb{Z}^d$ that satisfies the following additional conditions: $v^{-1} \in \ell^1(\mathbb{Z}^d)$ and $v^{-1} * v^{-1} \le Cv^{-1}$ (v is called *subconvolutive*). We define the Banach space $\mathcal{A}_v$ by the norm

$$\|A\|_{\mathcal{A}_v} = \sup_{k,l\in\mathbb{Z}^d} |a_{kl}|v(k-l)\,, \tag{3.3}$$

3. *Schur-type conditions.* Let v be an admissible weight. The class $\mathcal{A}_v^1$ consists of all matrices $A = (a_{kl})_{k,l\in\mathbb{Z}^d}$ such that

$$\sup_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} |a_{kl}|\, v(k-l) < \infty \quad \text{and} \quad \sup_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a_{kl}|\, v(k-l) < \infty \tag{3.4}$$

with norm

$$\|A\|_{\mathcal{A}_v^1} = \max\Big\{ \sup_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} |a_{kl}| v(k-l)\,,\, \sup_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |a_{kl}| v(k-l) \Big\}\,. \tag{3.5}$$

4. *The Gohberg-Baskakov-Sjöstrand class.* For any admissible weight v we define the class $\mathcal{C}_v$ as the space of all matrices $A = (a_{kl})_{k,l\in\mathbb{Z}^d}$ such that the norm

$$\|A\|_{\mathcal{C}_v} := \sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |a_{k,k-l}|\, v(l) \tag{3.6}$$

is finite. An alternative way to define the norm on $\mathcal{C}_v$ is

$$\|A\|_{\mathcal{C}_v} = \inf\{\|\alpha\|_{\ell^1_v} : |a_{kl}| \le \alpha(k-l)\}\,. \tag{3.7}$$

5. A further generalization is due to Sun [29]. Roughly speaking, Sun's class amounts to an interpolation between $\mathcal{C}_v$ and $\mathcal{A}_v$ or between $\mathcal{A}_v^1$ and $\mathcal{A}_v$. Our results also hold for Sun's class, but to avoid a jungle of indices, we stick to the simple classes defined above and leave the reformulation of our results in Sun's case to the reader.

These Banach spaces of matrices have the following elementary properties.

Lemma 2. *Let v be an admissible weight and $\mathcal{A}$ be one of the algebras $\mathcal{A}_s$ for $s > d$, $\mathcal{A}_v, \mathcal{A}_v^1, \mathcal{C}_v$. Then $\mathcal{A}$ has the following properties:*

(a) Both $\mathcal{A}_v^1$ and $\mathcal{C}_v$ are involutive Banach algebras (i.e., B^*-algebras) with the norms defined in (3.4) *and* (3.5)*. $\mathcal{A}_v$ and $\mathcal{A}_s, s > d$ can be equipped with an equivalent norm so that they become involutive Banach algebras.*

(b) If $A \in \mathcal{A}$, then A is bounded on $\ell^2(\mathbb{Z}^d)$.

(c) If $A \in \mathcal{A}$ and $|b_{kl}| \le |a_{kl}|$ for all $k,l \in \mathbb{Z}^d$, then $B \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \le \|\mathcal{A}\|_{\mathcal{A}}$. ($\mathcal{A}$ is a solid *algebra).*

Proof. Properties (a) and (c) are easy and follow directly from the definition of the matrix norms. The statements about $\mathcal{A}_s$ and $\mathcal{A}_v$ are proven in [16]. (b) is a consequence of Schur's test. □

Next we study the spectrum of matrices belonging to one of these Banach algebras.

Definition 1. We say that $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$, if for every $A \in \mathcal{A}$ that is invertible on $\ell^2(\mathbb{Z}^d)$ we have that $A^{-1} \in \mathcal{A}$.

Our next theorem states that the matrix algebras introduced above are inverse-closed as long as v satisfies the GRS-condition. The precise formulation is slightly more complicated, because we need to be a bit pedantic about the weights.

Theorem 3 (Inverse-closedness)**.** *Let v be an admissible weight that satisfies the GRS-condition, i.e., $\lim_{n\to\infty} v(nk)^{1/n} = 1$ for all $k \in \mathbb{Z}^d$.*

(a) Assume that $v^{-1} \in \ell^1(\mathbb{Z}^d)$ and $v^{-1} * v^{-1} \le Cv^{-1}$, then $\mathcal{A}_v$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. In particular $\mathcal{A}_s$ for $s > d$ possesses this property.

(b) If $v(k) \ge C(1+|k|)^\delta$ for some $\delta > 0$, then $\mathcal{A}_v^1$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$.

(c) $\mathcal{C}_v$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ for arbitrary admissible weights with the GRS-property.

Remark 3.1. The inverse-closedness is the key property and lies rather deep. While for C^*-(sub)algebras this property is inherent, for Banach algebras it is always hard to prove. Inverse-closedness for $\mathcal{A}_s$ is due to Jaffard [19] and Baskakov [2, 3], a simple proof is given in [29]. For $\mathcal{A}_v$ it was proved by Baskakov [3] and reproved in a different way in [16]. The result for $\mathcal{C}_v$ with $v \equiv 1$ is due to Gohberg-Kasshoek-Wordeman [13] and was rediscovered by Sjöstrand [24], the case of arbitrary weights is due to Baskakov [3], the algebra $\mathcal{A}_v^1$ was treated by one of us with Leinert [15]. More general conditions were announced by Sun [29].

The following properties are well-known consequences of inverse-closedness.

Corollary 4 (Spectral invariance). *Let $\mathcal{A}$ be one of the algebras $\mathcal{A}_s$, $\mathcal{A}_v$, $\mathcal{A}_v^1$, or $\mathcal{C}_v$ and assume that v satisfies the conditions of Theorem 3. Then*
(a) $\sigma_{\mathcal{A}}(A) = \sigma(A)$ (the spectrum in the algebra $\mathcal{A}$ coincides with the spectrum of A as an operator on $\ell^2(\mathbb{Z}^d)$)

(b) If A is bounded on ℓ^p_m for all $A \in \mathcal{A}$, then the operator norm satisfies

$$\|A\|_{\ell^p_m \to \ell^p_m} \le C\|A\|_{\mathcal{A}} \quad \text{for all } A \in \mathcal{A}, \tag{3.8}$$

and

$$\sigma_{\ell^p_m}(A) \subseteq \sigma(A)$$

(the spectrum is almost independent of the space A acts on).

Remark 3.2. Statement (a) is equivalent to inverse-closedness, the norm estimate in (b) follows from the closed graph theorem, the inclusion of the spectra is an immediate consequence of Theorem 3.

Let us emphasize that in our analysis of the finite section method we only need that the algebra $\mathcal{A}$ acts boundedly on ℓ^p_m. In order to understand how the weight m depends on the submultiplicative weight used to parametrize the off-diagonal decay, let us briefly discuss some sufficient conditions for the bounded action of $\mathcal{A}$ on ℓ^p_m. The weights m satisfy slightly different conditions. Let v be an admissible weight. The class of v-moderate weights is

$$\mathcal{M}_v = \Big\{ m \ge 0 : \sup_{k \in \mathbb{Z}^d} \frac{m(k+l)}{m(k)} \le Cv(l), \quad \forall\, l \in \mathbb{Z}^d \Big\}. \tag{3.9}$$

For example, if $a, s \in \mathbb{R}$ are arbitrary, then $m(x) = e^{ad(x)^b}(1 + d(x))^s$ is $e^{|a|d(x)^b}(1 + d(x))^{|s|}$-moderate.

The explicit examples of Banach algebras discussed above all act on the entire range of ℓ^p_m for $1 \le p \le \infty$ and a family of moderate weights associated to v. The following lemma provides some explicit sufficient conditions on m for $\mathcal{A}_v$, $\mathcal{A}_v^1$ or $\mathcal{C}_v$ to act boundedly on ℓ^p_m.

Lemma 5. *Let v be an admissible weight.*

(a) If $A \in \mathcal{A}_v^1$, then A is bounded simultaneously on all $\ell^p_m(\mathbb{Z}^d)$ for $1 \le p \le \infty$ and $m \in \mathcal{M}_v$.

(b) If $A \in \mathcal{A}_v$ and $v_0(k) := v(k)/(1 + |k|)^s$ is submultiplicative for some $s > d$, then A is bounded simultaneously on all $\ell^p_m(\mathbb{Z}^d)$ for $1 \le p \le \infty$ and $m \in \mathcal{M}_{v_0}$.

(c) If $A \in \mathcal{A}_v$, then A is bounded on $\ell^\infty_v(\mathbb{Z}^d)$.
(d) If $A \in \mathcal{C}_v$, then A is bounded on all $\ell^p_m(\mathbb{Z}^d)$ for $1 \le p \le \infty$ and $m \in \mathcal{M}_v$.

Proof. For completeness we sketch the easy proof.

(a) First, let $p = 1$, $c \in \ell^1_m(\mathbb{Z}^d)$ and $A \in \mathcal{A}^1_v$. Then, since $m(k) \le Cv(k-l)m(l)$, we obtain

$$\begin{aligned}\|Ac\|_{\ell^1_m} &= \sum_{k\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d} a_{kl}c_l\Big| m(k) \le C\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|a_{kl}|\,|c_l|v(k-l)m(l)\\ &\le C\sum_{l\in\mathbb{Z}^d}\Big(\sup_{l\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|a_{kl}|v(k-l)\Big)|c_l|m(l) = C\|A\|_{\mathcal{A}^1_v}\|c\|_{\ell^1_m}.\end{aligned}$$

Next, let $p = \infty$ and $c \in \ell^\infty_m$. Then, as before

$$\begin{aligned}\|Ac\|_{\ell^\infty_m} &= \sup_{k\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d} a_{kl}c_l\Big| m(k) \le C\sup_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|a_{kl}|\,|c_l|v(k-l)m(l)\\ &\le C\Big(\sup_{l\in\mathbb{Z}^d}|c_l|m(l)\Big)\sup_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|a_{kl}|v(k-l) = C\|A\|_{\mathcal{A}^1_v}\|c\|_{\ell^\infty_m}.\end{aligned}$$

The boundedness on $\ell^p_m(\mathbb{Z}^d)$ for $1 < p < \infty$ now follows by interpolation.

(b) and (d) follow from the easy embeddings $\mathcal{A}_v \hookrightarrow \mathcal{A}^1_{v_0}$, $C_v \subseteq \mathcal{A}^1_v$ and from (a).

(c) uses the subconvolutivity of v. Let $A \in \mathcal{A}_v$ and $c \in \ell^\infty_v(\mathbb{Z}^d)$. Then, $|a_{kl}| \le \|A\|_{\mathcal{A}_v} v(k-l)^{-1}$ and $|c_l| \le \|c\|_{\ell^\infty_v} v(l)^{-1}$. Consequently,

$$\begin{aligned}\|Ac\|_{\ell^\infty_v} &= \sup_{k\in\mathbb{Z}^d}\Big|\sum_{l\in\mathbb{Z}^d} a_{kl}c_l\Big| v(k)\\ &\le \|A\|_{\mathcal{A}_v}\|c\|_{\ell^\infty_v}\sup_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\frac{1}{v(k-l)}\frac{1}{v(l)}v(k) \le C\|A\|_{\mathcal{A}_v}\|c\|_{\ell^\infty_v},\end{aligned}$$

because $(v^{-1} * v^{-1})(k) \le Cv(k)^{-1}$. □

The matrices of the Banach algebras introduced above can be considered as approximate banded matrices. This might suggest that it would be sufficient to set those entries smaller than some threshold to zero and simply work with banded matrices, which are a special case of sparse matrices. At first sight this may seem appealing, since invertible banded matrices have inverses with exponentially fast off-diagonal decay [9]. However there is an important difference. In many applications, cf. [19,25,26,28] the matrix entries do decay off the diagonal, but thresholding would still leave us with banded matrices with the number of non-zero diagonals easily in the order of several dozens. The theoretical prediction for the decay of the inverse of such banded matrices is so slow that it is meaningless for practical purposes. The reason is that by resorting to banded matrices we have neglected the decay of the entries above the chosen threshold. Thus banded matrices are simply not the most suitable model to capture the decay behavior of those matrices and their inverses.

4. Finite Sections in Matrix Algebras

We first study the finite sections of matrices belonging to an inverse-closed matrix algebra and give a new characterization of inverse-closedness by means of finite sections. This is a necessary step in the qualitative analysis of the convergence properties of the finite section method on weighted ℓ^p-spaces, but should be of independent interest in the study of Banach algebras.

Let $\mathcal{A}^{FS}$ be the set of all finite sections of matrices in $\mathcal{A}$, formally

$$\mathcal{A}^{FS} = \{\vec{A} = (A_n)_{n\in\mathbb{N}} : A_n = P_nAP_n \text{ for some } A \in \mathcal{A}\}\,. \tag{4.1}$$

Although $\mathcal{A}^{FS}$ is not an algebra anymore, we may define a notion of inverse-closedness.

Definition 2. We say that $\mathcal{A}^{FS}$ is inverse-closed if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of invertible finite sections such that $\|A_n\|_{\mathcal{A}} \le C$ and $\|A_n^{-1}\|_{op} \le C$, we have that $\|A_n^{-1}\|_{\mathcal{A}} \le C'$, for some constants C and C' that do not depend on $n \in \mathbb{N}$.

The comparison of inverse-closedness of $\mathcal{A}$ and of $\mathcal{A}^{FS}$ indicates that the transition from the infinite-dimensional setting $\mathcal{A}$ to the finite-dimensional case $\mathcal{A}^{FS}$ is done by replacing the hidden word "bounded" by "bounded uniformly in dimension".

The definition of $\mathcal{A}^{FS}$ suggests as a next step to consider sequences of arbitrary finite square matrices instead of finite sections. To define a norm that is related to the $\mathcal{A}$-norm, we must assume that the norm $\|\cdot\|_{\mathcal{A}}$ can be applied to arbitrary finite matrices by defining an appropriate embedding of $(\mathbb{C}^{2n+1})^d$ into $\ell^2(\mathbb{Z}^d)$. For $j \in \mathbb{Z}^d$ and $C_n = \{-n,\dots,n\}^d \subseteq \mathbb{Z}^d$ we define an extension of the $(2n+1)^d \times (2n+1)^d$-matrix B to in infinite matrix B^J of B, say $B^J = (B^J)_{k,l\in\mathbb{Z}^d}$ such that

$$(B^J)_{j+k,j+l} = \begin{cases} (B)_{kl} & \text{for } k,l \in C_n \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Then we define the norm of B by

$$\|B\|_{\mathcal{A}} = \|B^J\|_{\mathcal{A}}\,. \tag{4.3}$$

In the big picture, this definition makes sense only when the norm does not depend on the embedding cube $J = j + C_n$. This requires an additional property of $\mathcal{A}$.

Let $T_l, l \in \mathbb{Z}^d$, denote the translation operator $T_lf(k) = f(k-l)$ acting on $f \in \ell^2(\mathbb{Z}^d)$. We say that the norm of $\mathcal{A}$ is *translation-invariant* if

$$\|T_{-l}AT_l\|_{\mathcal{A}} = \|A\|_{\mathcal{A}} \qquad \forall A \in \mathcal{A}, l \in \mathbb{Z}^d\,. \tag{4.4}$$

Clearly, if the norm of $\mathcal{A}$ is translation-invariant, then $\|B^J\|_{\mathcal{A}}$ does not depend on the cube $J = j + C_n$, and we can apply $\|\cdot\|_{\mathcal{A}}$ to finite matrices. From now on, let us assume that $\|\cdot\|_{\mathcal{A}}$ is translation-invariant.

Similarly to (4.1) and analogous to [17, Section 1.2.2] we introduce the set $\mathcal{A}^F$ by

$$\mathcal{A}^F = \{\vec{B} = (B_n)_{n\in\mathbb{N}} : B_n \text{ is a } (2n+1)^d \times (2n+1)^d \text{ matrix and } \sup_{n\in\mathbb{N}} \|B_n\|_{\mathcal{A}} < \infty\}\,,$$

and we endow $\mathcal{A}^F$ with the norm

$$\|\vec{B}\|_{\mathcal{A}^F} = \sup_{n\in\mathbb{N}} \|B_n\|_{\mathcal{A}}. \tag{4.5}$$

If the norm of $\mathcal{A}$ is translation-invariant, then $\mathcal{A}^F$ is a well-defined object. We note that $\mathcal{A}^F$ is a Banach algebra contained in $\mathcal{B} := \bigoplus_{n=1}^{\infty} \mathcal{B}((\mathbb{C}^{2n+1})^d)$.

Definition 3. We say that $\mathcal{A}^F$ is inverse-closed if for every sequence $\{B_m\}_{m\in\mathbb{Z}^d}$ of finite invertible matrices such that $\|B_m\|_{\mathcal{A}} \le C$ and $\|B_m^{-1}\|_{op} \le C$, we have that $\|B_m^{-1}\|_{\mathcal{A}} \le C'$, for some constants C and C' that do not depend on $m \in \mathbb{Z}^d$.

In view of Definition 1 this amounts to saying that the algebra $\mathcal{A}^F$ is inverse-closed in $\bigoplus_{n=1}^{\infty} \mathcal{B}((\mathbb{C}^{2n+1})^d)$.

The inverse-closedness of $\mathcal{A}$, $\mathcal{A}^{FS}$, and $\mathcal{A}^F$ depends on the original algebra $\mathcal{A}$. For the study of the relations between them we introduce some further natural conditions.

(C1) Weak solidity: For every $A \in \mathcal{A}$ there is a constant C such that $\|A_n\|_{\mathcal{A}} \le C\|A\|_{\mathcal{A}}$ for all $n \in \mathbb{N}$.

(C2) Weak inverse-closedness: For every $A \in \mathcal{A}$ that is invertible on $\ell^2(\mathbb{Z}^d)$ the condition $\sup_{n\in\mathbb{N}} \|A_n^{-1}\|_{\mathcal{A}} \le C$ implies that $A^{-1} \in \mathcal{A}$.

Our third condition concerns an infinite matrix B^{block} that is built from blocks of finite square matrices $\{B_m\}_{m\in\mathbb{N}}$ by stacking them "along the diagonal". For this we choose a sequence $j_m \in \mathbb{Z}^d$ such that sequence of cubes $J_m = j_m + C_m \subset \mathbb{Z}^d$ is disjoint. Now set

$$B^{block} = \sum_{m\in\mathbb{N}} B_m^{J_m}, \tag{4.6}$$

where $B_m^{J_m}$ is the extension of B_m given in (4.2).

(C3) Block norm equivalence: There exist constants $C, C' > 0$, such that for every $\vec{B} \in \mathcal{A}^F$

$$C\|B^{block}\|_{\mathcal{A}} \le \sup_{m\in\mathbb{N}} \|B_m\|_{\mathcal{A}} \le C'\|B^{block}\|_{\mathcal{A}} \tag{4.7}$$

Remark 4.1. We want to point out that in all settings $\mathcal{A}$, $\mathcal{A}^{FS}$, and $\mathcal{A}^F$ it suffices to show the inverse-closedness property for positive matrices. Therefore, if necessary, one could restrict conditions (C1)–(C3) to such matrices. We note that the upper bound in (4.7) follows already from condition (C1).

These three conditions are sufficient to show that the concepts of inverse-closedness in $\mathcal{A}$, $\mathcal{A}^{FS}$ and $\mathcal{A}^F$ are equivalent.

Theorem 6. *Let $\mathcal{A} \subset \mathcal{B}(\ell^2(\mathbb{Z}^d))$ be a Banach algebra such that the norm of $\mathcal{A}$ is translation-invariant and $\mathcal{A}$ satisfies conditions (C1)–(C3). Then the following are equivalent:*

a) *$\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$.*
b) *$\mathcal{A}^{FS}$ is inverse-closed.*
c) *$\mathcal{A}^F$ is inverse-closed in $\mathcal{B} := \bigoplus_{n=1}^{\infty} \mathcal{B}((\mathbb{C}^{2n+1})^d)$.*

Proof. c) $\Rightarrow$ b) This implication is clear, because each sequence of finite sections A_n belongs to $\mathcal{A}^F$ by (C1).

b) $\Rightarrow$ a) Since a matrix A is invertible, if and only if the matrices A^*A and AA^* are invertible, we may assume without loss of generality that A is positive and invertible on $\ell^2(\mathbb{Z}^d)$. So assume that $A \in \mathcal{A}$ is positive and invertible on $\ell^2(\mathbb{Z}^d)$. We want to show that, if $\mathcal{A}^{FS}$ is inverse-closed, then $A^{-1} \in \mathcal{A}$.

By condition (C1), we have that $\|A_n\|_\mathcal{A} \le C$ for all $n \in \mathbb{N}$ and some constant C independent of $n \in \mathbb{N}$. Moreover, (2.1) implies that $\|A_n^{-1}\|_{op} \le \|A^{-1}\|_{op}$. Since by assumption $\mathcal{A}^{FS}$ is inverse-closed, we obtain that $\|A_n^{-1}\|_\mathcal{A} \le C'$ for all $n \in \mathbb{N}$ and some $C' > 0$. By condition (C2) we obtain $A^{-1} \in \mathcal{A}$.

a) $\Rightarrow$ c) We argue by contradiction and show that if (c) fails, then so does (a). Assume that $\mathcal{A}^F$ is not inverse-closed. This means that there is a sequence of finite invertible matrices $\{B_m\}_{m\in\mathbb{N}}$ such that $\|B_m\|_\mathcal{A}$ and $\|B_m^{-1}\|_{op}$ are uniformly bounded in $m \in \mathbb{N}$, but $\sup_{m\in\mathbb{N}} \|B_m^{-1}\|_\mathcal{A} = \infty$.

Consider the corresponding matrix B^{block} as given in (4.6). Then its inverse $(B^{block})^{-1}$ is a block matrix that corresponds to the sequence $\{B_m^{-1}\}_{m\in\mathbb{Z}^d}$. Therefore, $\|(B^{block})^{-1}\|_{op} = \sup_{m\in\mathbb{N}} \|B_m^{-1}\|_{op} < \infty$, so B^{block} is invertible on $\ell^2(\mathbb{Z}^d)$. Condition (C3) implies that $\|B^{block}\|_\mathcal{A} \le C' \sup_{m\in\mathbb{Z}^d} \|B_m\|_\mathcal{A} < \infty$, so $B^{block} \in \mathcal{A}$. The same condition guarantees that $\|(B^{block})^{-1}\|_\mathcal{A} \ge C \sup_{m\in\mathbb{Z}^d} \|B_m^{-1}\|_\mathcal{A} = \infty$. Thus $(B^{block})^{-1} \notin \mathcal{A}$ and $\mathcal{A}$ cannot be inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. □

Next we apply Theorem 6 to the matrix algebras $\mathcal{A}_v$, $\mathcal{A}_v^1$, and C_v introduced in Section 2. As we mentioned in Lemma 2, all these algebras are contained in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. Moreover, the norms associated to these algebras are translation invariant. Indeed, to check that (4.4) holds, we denote the standard basis of $\ell^2(\mathbb{Z}^d)$ by $\{e_k\}_{k\in\mathbb{Z}^d}$ and observe that $\langle T_{-j}AT_je_l, e_k\rangle = \langle AT_je_l, T_je_k\rangle = \langle Ae_{l+j}, e_{k+j}\rangle$ for all $j \in \mathbb{Z}^d$. Therefore, $(T_{-j}AT_j)_{kl} = (A)_{k+j,l+j}$ and since the norms of $\mathcal{A}_v$, $\mathcal{A}_v^1$ and C_v use only the difference of k and l, they are translation invariant.

By Lemma 2(c) each of these algebras is solid, so condition (C1) holds for all of them. Condition (C3) is more problematic. Since the norm of a matrix in $\mathcal{A}_v$ and $\mathcal{A}_v^1$ is defined in terms of its rows and columns, it follows that $\|B^{block}\|_\mathcal{A} = \sup_{m\in\mathbb{Z}^d} \|B_m\|_\mathcal{A}$. So property (C3) holds for for $\mathcal{A}_v$ and $\mathcal{A}_v^1$. Condition (C3) fails, however, for C_v.

It remains to consider the weak inverse-closedness (C2).

Proposition 7. *Condition (C2) holds for each of the algebras $\mathcal{A}_v$ and $\mathcal{A}_v^1$.*

Proof. Assume that $A \in \mathcal{A}$ is invertible on $\ell^2(\mathbb{Z}^d)$ and that $\sup_{n\in\mathbb{N}} \|A_n^{-1}\|_\mathcal{A} = C < \infty$. Recall that $\widetilde{A_n} = P_nAP_n + \lambda_+(I - P_n)$ is the extension of A_n defined in (2.2). Clearly, our assumption that $\|A_n^{-1}\|_\mathcal{A}$ is uniformly bounded, implies immediately that $\|\widetilde{A_n}^{-1}\|_\mathcal{A}$ is uniformly bounded as well.

Since both A and A^{-1} are bounded on $\ell^2(\mathbb{Z}^d)$, $\widetilde{A_n}^{-1}$ converges strongly to A^{-1} in $\ell^2(\mathbb{Z}^d)$, as we have seen in the proof of Theorem 1, Step 3. Therefore, $\langle \widetilde{A_n}^{-1} e_l, e_k\rangle$ converges to $\langle A^{-1}e_l, e_k\rangle$ for all vectors of the standard basis of $\ell^2(\mathbb{Z}^d)$.

Case I: $\mathcal{A} = \mathcal{A}_v$**.** Since $\|\widetilde{A_n}^{-1}\|_{\mathcal{A}}$ is uniformly bounded, we have that

$$\|\widetilde{A_n}^{-1}\|_{\mathcal{A}} = \sup_{k,l\in\mathbb{Z}^d} |\langle \widetilde{A_n}^{-1} e_l, e_k\rangle| v(k-l) \le C.$$

Thus, we obtain that, for every $k, l \in \mathbb{Z}^d$,

$$|\langle A^{-1} e_l, e_k\rangle| v(k-l) = \lim_{n\to\infty} |\langle \widetilde{A_n}^{-1} e_l, e_k\rangle| v(k-l) \le C,$$

and so $A^{-1} \in \mathcal{A}$ with $\|A^{-1}\|_{\mathcal{A}} \le C = \sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{\mathcal{A}_v}$.

Case II: $\mathcal{A} = \mathcal{A}_v^1$**.** We use Fatou's Lemma. We have that

$$\|\widetilde{A_n}^{-1}\|_{\mathcal{A}} = \max\left\{ \sup_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} |\langle \widetilde{A_n}^{-1} e_l, e_k\rangle| v(k-l), \sup_{l\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} |\langle \widetilde{A_n}^{-1} e_l, e_k\rangle| v(k-l) \right\} \le C.$$

Therefore, we obtain, for every $k \in \mathbb{Z}^d$,

$$\sum_{l\in\mathbb{Z}^d} |\langle A^{-1} e_l, e_k\rangle| v(k-l) \le \liminf_{n\to\infty} \sum_{l\in\mathbb{Z}^d} |\langle \widetilde{A_n}^{-1} e_l, e_k\rangle| v(k-l) \le C$$

and for every $l \in \mathbb{Z}^d$

$$\sum_{k\in\mathbb{Z}^d} |\langle A^{-1} e_l, e_k\rangle| v(k-l) \le \liminf_{n\to\infty} \sum_{k\in\mathbb{Z}^d} |\langle \widetilde{A_n}^{-1} e_l, e_k\rangle| v(k-l) \le C.$$

Taking the supremum over $l \in \mathbb{Z}^d$ (or k respectively), we conclude that $A^{-1} \in \mathcal{A}_v^1$ and $\|A^{-1}\|_{\mathcal{A}_v^1} \le C = \sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{\mathcal{A}_v^1}$. □

Since we have verified that all assumptions of Theorem 6 are satisfied for $\mathcal{A}_v$ and $\mathcal{A}_v^1$, we obtain the following result.

Theorem 8. *If $\mathcal{A}$ is either $\mathcal{A}_v$ or $\mathcal{A}_v^1$, then $\mathcal{A}$ is inverse-closed, if and only if $\mathcal{A}^{FS}$ is inverse-closed if and only if $\mathcal{A}^F$ is inverse-closed.*

Remark 4.2. Theorem 8 holds for $\mathcal{A}_v^1$ even if $v \equiv 1$. In this case, $\mathcal{A}_v^1$ is the Schur class, i.e., the class of matrices that satisfy the Schur test [18] or, equivalently, the class of matrices that are bounded simultaneously on all $\ell^p, 1 \le p \le \infty$. It seems to be an open problem if this algebra is inverse-closed. Theorem 8 reduces this problem to an equivalent (and equally difficult) question about finite-dimensional matrices. However, if v satisfies a mild growth condition, see Theorem 3(b), then $\mathcal{A}_v^1$ is inverse-closed.

We need a few more facts before addressing questions of convergence of the finite section method.

Corollary 9. *Let v be an admissible weight satisfying the GRS-condition* (3.1)*.*

(a) If $A \in \mathcal{A}_v^1$ is positive and invertible on $\ell^2(\mathbb{Z}^d)$, then $\sup_{n\in\mathbb{N}} \|A_n^{-1}\|_{\mathcal{A}_v^1} < \infty$.

(b) Assume in addition that $v^{-1} \in \ell^1(\mathbb{Z}^d)$ and $v^{-1} * v^{-1} \le C v^{-1}$. If A is positive and invertible on $\ell^2(\mathbb{Z}^d)$, then $\sup_{n\in\mathbb{N}} \|A_n^{-1}\|_{\mathcal{A}_v} < \infty$.

Proof. Our assumptions on v imply that $\mathcal{A} \in \{\mathcal{A}_v, \mathcal{A}_v^1\}$ is inverse-closed. By Theorem 8, $\mathcal{A}^{FS}$ is inverse-closed as well. Therefore, to achieve that $\sup_{n\in\mathbb{N}} \|A_n^{-1}\|_{\mathcal{A}} < \infty$ it is enough to assure that $\|A_n\|_{\mathcal{A}}$ and $\|A_n^{-1}\|_{op}$ are bounded uniformly in $n \in \mathbb{N}$. This, however, follows from $\|A_n\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$ (solidity) and from Step 1 in the proof of Theorem 1, where we showed that $\|A_n^{-1}\|_{op} \leq \|A^{-1}\|_{op}$. □

As the block norm equivalence (C3) fails for the algebra $\mathcal{C}_v$, we do not know whether $(\mathcal{C}_v)^{FS}$ is inverse-closed. However, since $\mathcal{C}_v \subseteq \mathcal{A}_v^1$, we have the following result.

Corollary 10. *Let v be an admissible weight satisfying the GRS-condition* (3.1) *and $v(k) \geq C(1+|k|)^\epsilon$ for some $\epsilon > 0$. If $A \in \mathcal{C}_v$ is positive and invertible on $\ell^2(\mathbb{Z}^d)$, then $\sup_{n\in\mathbb{N}} \|A_n^{-1}\|_{\mathcal{A}_v^1} < \infty$.*

5. Convergence in ℓ^p_m

After the analysis of finite sections in matrix algebras, we are now in a position to show that the finite section method converges in weighted ℓ^p-spaces, whenever the matrix is in one of the algebras $\mathcal{A}_s$, $\mathcal{A}_v$, $\mathcal{A}_v^1$, and $\mathcal{C}_v$.

Theorem 11. *Let $\mathcal{A}$ be one of the inverse-closed algebras $\mathcal{A}_s, \mathcal{A}_v, \mathcal{C}_v$, or $\mathcal{A}_v^1$, where the weight satisfies the conditions stated in Theorem 3 for each case. Assume that $A \in \mathcal{A}$ is positive and invertible on $\ell^2(\mathbb{Z}^d)$ and acts boundedly on ℓ^p_m.*

If $b \in \ell^p_m$ and $p < \infty$, then the finite section method converges in the norm of ℓ^p_m.

If $b \in \ell^p_m$ and $p = \infty$, then the finite section method converges in the weak-topology. In particular, x_n goes to x entrywise.*

Proof. We expand the model proof of Theorem 1 and insert the results about Banach algebras obtained in Sections 3 and 4. Recall that $\widetilde{A_n} = P_nAP_n + \lambda_+(I - P_n)$ is the extension of A_n defined in (2.2). Throughout the proof C denotes a constant that may change from step to step.

Step 1 in the proof of Theorem 1 remains unchanged and yields that

$$\sigma(\widetilde{A_n}) \subseteq [\lambda_-, \lambda_+]$$

independent of n and that

$$\sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{op} \leq \lambda_-^{-1} = \|A^{-1}\|_{op}\,, \tag{5.1}$$

(where $\|\cdot\|_{op}$ is the operator norm on $\ell^2(\mathbb{Z}^d)$). Since A is positive and invertible on $\ell^2(\mathbb{Z}^d)$ and $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ by our hypotheses on the weights, Theorem 3 guarantees that $A^{-1} \in \mathcal{A}$ as well. By Corollary 4(b), the inverse A^{-1} is then bounded on ℓ^p_m. Furthermore, for $\mathcal{A} \in \{\mathcal{A}_s, \mathcal{A}_v, \mathcal{A}_v^1\}$ by Corollary 9 we know that

$$\sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{\ell^p_m \to \ell^p_m} \leq C \sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{\mathcal{A}} = C < \infty\,.$$

For $\mathcal{A} = \mathcal{C}_v$, Corollary 10 implies that

$$\sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{\ell^p_m \to \ell^p_m} \le C \sup_{n\in\mathbb{N}} \|\widetilde{A_n}^{-1}\|_{\mathcal{A}^1_v} = C < \infty\,.$$

Step 2. For $p < \infty$, $\widetilde{A_n}$ converges to A in the strong operator topology on ℓ^p_m. This follows from the inequality

$$\|A - P_n A P_n\|_{\ell^p_m \to \ell^p_m} \le \|(I - P_n)A\|_{\ell^p_m \to \ell^p_m} + \|P_n A (I - P_n)\|_{\ell^p_m \to \ell^p_m},$$

and the fact that $P_n f \to f \in \ell^p_m$ is equivalent to the density of the finite sequences in ℓ^p_m.

Step 3. (Lemma of Kantorovich). We know that $\widetilde{A_n}^{-1} = A_n^{-1} + \lambda_+^{-1}(I - P_n)$. Since

$$\begin{aligned} \|\widetilde{A_n}^{-1} b - A^{-1}b\|_{\ell^p_m} &= \|\widetilde{A_n}^{-1}(A - \widetilde{A_n})A^{-1}b\|_{\ell^p_m} \\ &\le \sup_n \|\widetilde{A_n}^{-1}\|_{\ell^p_m \to \ell^p_m} \|(A - \widetilde{A_n})A^{-1}b\|_{\ell^p_m} \qquad (5.2) \\ &\le \sup_n \|\widetilde{A_n}^{-1}\|_{\mathcal{A}} \|(A - \widetilde{A_n})A^{-1}b\|_{\ell^p_m} \qquad (5.3) \end{aligned}$$

the strong convergence $\widetilde{A_n} \rightharpoonup A$ on ℓ^p_m implies that $\widetilde{A_n}^{-1}$ converges strongly to A^{-1} on ℓ^p_m for $1 \le p < \infty$.

Step 4. Recall, that $A_n x_n = b_n$ and $Ax = b$. Then

$$\begin{aligned} \|x - x_n\|_{\ell^p_m} &= \|A^{-1}b - A_n^{-1}b_n\|_{\ell^p_m} = \|A^{-1}b - A_n^{-1}P_n b\|_{\ell^p_m} \\ &\le \|(A^{-1} - \widetilde{A_n}^{-1})b\|_{\ell^p_m} + \|\widetilde{A_n}^{-1}(b - P_n b)\|_{\ell^p_m} = \mathrm{I} + \mathrm{II}\,. \qquad (5.4) \end{aligned}$$

For $1 \le p < \infty$ the first term goes to zero by Step 3, and the second term is estimated by

$$\mathrm{II} \le \sup_n \|\widetilde{A_n}^{-1}\|_{\ell^p_m \to \ell^p_m} \|b - P_n b\|_{\ell^p_m} \le C^{-1}\|b - P_n b\|_{\ell^p_m}, \qquad (5.5)$$

and also goes to zero.

For $p = \infty$ we prove weak*-convergence. Assume $b \in \ell^\infty_m = \left(\ell^1_{1/m}\right)^*$ and $y \in \ell^1_{1/m}$. Then,

$$\langle x - x_n, y\rangle = \langle A^{-1}b - \widetilde{A}_n^{-1}P_n b, y\rangle = \langle b - P_n b, A^{-1}y\rangle + \langle P_n b, (A^{-1} - \widetilde{A}_n^{-1})y\rangle.$$

the first term tends to zero, because finite sequences are weak*-dense in ℓ^∞_m. The second term is majorized by $\|b\|_{\ell^\infty_m} \|(A^{-1} - \widetilde{A}_n^{-1})y\|_{\ell^1_{1/m}}$ and converges to zero by Step 3, (5.3). □

6. Quantitative Estimates

In Theorem 11 we have investigated the convergence of the finite section method in the norm of ℓ^p_m provided that the input vector b is in ℓ^p_m. For the quantitative analysis, we assume that the input is in ℓ^p_m and we study the convergence in a weaker norm.

We first work with the algebra $\mathcal{A} = \mathcal{A}_v$ defined by off-diagonal decay of the matrices and a subconvolutive weight satisfying the GRS-condition. Recall that $C_n = \{-n, \dots n\}^d$ is the cube of integer vectors, so that $\sum_{k \notin C_n} \dots$ becomes a tail estimate.

Theorem 12. *Assume that $A \in \mathcal{A}_v$ is invertible and $b \in \ell^\infty_v(\mathbb{Z}^d)$. Set*

$$\varphi(n) = \Big(\sum_{k \notin C_n} v(k)^{-2} \Big)^{1/2} .$$

Then the finite section method converges in the $\ell^2(\mathbb{Z}^d)$-norm with the asymptotic estimate for the error

$$\|x - x_n\|_2 \le C\varphi(n) . \tag{6.1}$$

Proof. We write

$$\begin{aligned} x - x_n &= A^{-1}b - A_n^{-1}b_n \\ &= A^{-1}(b - b_n) + A^{-1}(A_n - A)A_n^{-1}P_n b = \mathrm{I} + \mathrm{II} . \end{aligned}$$

Estimate of I:

$$\begin{aligned} \|\,\mathrm{I}\,\|_2 \le \|A^{-1}\|_{op}\,\|b - b_n\|_2 &= \|A^{-1}\|_{op} \Big(\sum_{|k|>n} |b_k|^2 \Big)^{1/2} \\ &\le \|A^{-1}\|_{op}\, \|b\|_{\ell^\infty_v} \Big(\sum_{k \notin C_n} v(k)^{-2} \Big)^{1/2} \\ &= \|A^{-1}\|_{op}\, \|b\|_{\ell^\infty_v}\, \varphi(n) . \end{aligned}$$

Estimate of II: Set $z_n = AP_nA_n^{-1}P_nb = AP_n\widetilde{A_n}^{-1}P_nb$, then $\mathrm{II} = A^{-1}(P_n - I)z_n$. Using Lemma 2, Corollary 9 and the obvious fact that $\|P_nb\|_{\ell^p_m} \le \|b\|_{\ell^p_m}$ (true for every solid sequence space) we obtain

$$\|z_n\|_{\ell^\infty_v} \le \|A\|_{\mathcal{A}_v} \|\widetilde{A_n}^{-1}\|_{\mathcal{A}_v} \|b\|_{\ell^\infty_v} ,$$

or the pointwise estimate

$$|(z_n)_k| \le Cv(k)^{-1}$$

which is independent of n. So

$$\begin{aligned} \|\,\mathrm{II}\,\|_2 &= \|A^{-1}(P_n - I)z_n\|_2 \\ &\le \|A^{-1}\|_{op}\,\|(P_n - I)z_n\|_2 \\ &= \|A^{-1}\|_{op} \Big(\sum_{k \notin C_n} |(z_n)_k|^2 \Big)^{1/2} \\ &\le \|A^{-1}\|_{op}\, C \Big(\sum_{k \notin C_n} v(k)^{-2} \Big)^{1/2} \\ &= \|A^{-1}\|_{op}\, C\varphi(n) . \end{aligned}$$

□

Remark 6.1. If $v(x) = (1+|x|)^s$, then $\varphi(n) = \Big(\sum_{k\notin C_n}(1+|k||)^{-2s}\Big)^{1/2} \sim n^{-s+d/2}$, and we recover the result of [26].

Theorem 12 can be generalized to other matrix algebras and sequence spaces. For this we note that the input b is in a "small" space, but that we measure the rate of convergence in a "large" space. The other item is that A and $\widetilde{A}_n$ have to be invertible on both the small and the large space with uniform bounds.

The rate of convergence will follow from the following tail estimates for the embedding of sequence spaces.

Lemma 13. *Assume that $\ell^p_m \subseteq \ell^q_w$ for $1 \le p,q \le \infty$ and m, w are moderate weights. Set $r^{-1} = \max\{q^{-1} - p^{-1}, 0\}$ and*

$$\varphi(n) = \left(\sum_{k\notin C_n} \frac{w(k)^r}{m(k)^r}\right)^{\frac{1}{r}}. \tag{6.2}$$

Then, $\|b - P_n b\|_{\ell^q_w} \le \varphi(n)\|b\|_{\ell^p_m}$.

Proof. We write $b - P_n b = (1-\chi_{C_n})b$, where χ_{C_n} is the characteristic function of C_n. Then, $\|b - P_n b\|_{\ell^q_w} = \|bm(1-\chi_{C_n})\frac{w}{m}\|_{\ell^q}$. If $p \le q$, then $\|c\|_{\ell^q} \le \|c\|_{\ell^p}$ and so

$$\|b - P_n b\|_{\ell^q_w} \le \Big\|bm(1-\chi_{C_n})\frac{w}{m}\Big\|_{\ell^p} \le \Big\|(1-\chi_{C_n})\frac{w}{m}\Big\|_{\infty}\|bm\|_{\ell^p} = \varphi(n)\|b\|_{\ell^p_m},$$

(since for $r=\infty$ formula (6.2) has to be interpreted with the supremum norm). If $p > q$, then $r = (q^{-1} - p^{-1})^{-1} > 1$. Thus, we use Hölder's inequality $\|bc\|_q \le \|b\|_p\|c\|_r$ and obtain

$$\|b - P_n b\|_{\ell^q_w} \le \Big\|(1-\chi_{C_n})\frac{w}{m}\Big\|_r \|bm\|_{\ell^p} = \varphi(n)\,\|b\|_{\ell^p_m},$$

□

Theorem 14. *Let $\mathcal{A}$ be one of the inverse-closed algebras $\mathcal{A}_s, \mathcal{A}_v, \mathcal{A}^1_v$ or $\mathcal{C}_v$. Assume that $\ell^p_m \subseteq \ell^q_w$ and that $\mathcal{A}$ acts boundedly on both ℓ^p_m and ℓ^q_w. If $A \in \mathcal{A}$ is invertible on ℓ^2 and $b \in \ell^p_m$ (the "smaller" space), then the finite section method converges in ℓ^q_w (the "larger" space) with the error estimate*

$$\|x - x_n\|_{\ell^q_w} \le C\|b\|_{\ell^p_m}\,\varphi(n), \tag{6.3}$$

where $C = \|A^{-1}\|_{\ell^q_w}(1 + \|A\|_{\ell^p_m}\|\widetilde{A}_n^{-1}\|_{\mathcal{A}})$ and $\varphi(n)$ is as in (6.2).

Proof. As in the proof of Theorem 12 we estimate the error by

$$\|x - x_n\|_{\ell^q_w} \le \|A^{-1}(b - b_n)\|_{\ell^q_w} + \|A^{-1}(P_n - I)z_n\|_{\ell^q_w}, \tag{6.4}$$

where $z_n = AP_n\widetilde{A}_n^{-1}P_n b$.

Since $A \in \mathcal{A}$ is invertible on ℓ^2, by inverse-closedness $A^{-1} \in \mathcal{A}$ and consequently A^{-1} is also bounded on ℓ^q_w. We note that, by Corollaries 9 and 10 we also have $\sup_{n\in\mathbb{N}}\|\widetilde{A}_n^{-1}\|_{\mathcal{A}} < \infty$.

For the first term in (6.4) we obtain, with Lemma 13, that

$$\|A^{-1}(b - b_n)\|_{\ell^q_w} \le \|A^{-1}\|_{\ell^q_w}\|b - b_n\|_{\ell^q_w} \le \|A^{-1}\|_{\ell^q_w}\|b\|_{\ell^p_m}\varphi(n).$$

The second term is estimated by

$$\|A^{-1}(P_n - I)z_n\|_{\ell^q_w} \le \|A^{-1}\|_{\ell^q_w \to \ell^q_w} \|(P_n - I)z_n\|_{\ell^q_w} \le \|A^{-1}\|_{\ell^q_w \to \ell^q_w} \|z_n\|_{\ell^p_m} \varphi(n).$$

Finally,

$$\|z_n\|_{\ell^p_m} = \|AP_n\widetilde{A}_n^{-1}P_n b\|_{\ell^p_m} \le \|A\|_{\ell^p_m \to \ell^p_m} \|\widetilde{A}_n^{-1}\|_{\ell^p_m \to \ell^p_m} \|b\|_{\ell^p_m},$$

and this expression is uniformly bounded by Corollary 9 and 10 and Corollary 4. Thus, we are done. □

If $\ell^\infty_v \subseteq \ell^2$, then we recover the simpler statement of Theorem 12.

7. Non-Symmetric Finite Section Method for Non-Symmetric Matrices

In the previous section we derived quantitative estimates for the convergence of the finite section method under the assumption that the matrices are positive definite. This assumption is crucial. For non-hermitian matrices it is already a difficult problem to derive merely qualitative statements about the convergence of the finite section method [6, 11, 12, 20]. Indeed, even for very simple non-hermitian matrices the finite section method may fail.

As an example, let us consider the Laurent operator given by the biinfinite Toeplitz matrix

$$A = \begin{bmatrix} \ddots & & & & & & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 1 & 0 & 0 & 0 & 0 & \\ & c & 1 & \boxed{0} & 0 & 0 & \\ & c^2 & c & 1 & 0 & 0 & \\ & c^3 & c^2 & c & 1 & 0 & \\ & & & & & & \ddots \end{bmatrix}.$$

Here, we assume $|c| < 1$ and, as usual in the finite section method literature, the box indicates the entry in the zero-zero position. An easy calculation shows that A is invertible on $\ell^2(\mathbb{Z})$ and its inverse is the Laurent operator with biinfinite Toeplitz matrix

$$A^{-1} = \begin{bmatrix} \ddots & & & & \\ & -c & 1 & 0 & \\ & 0 & \boxed{\text{-c}} & 1 & \\ & 0 & 0 & -c & \\ & & & & \ddots \end{bmatrix}.$$

Let us choose $b = e_0$, i.e, the right hand side is given by the zero-th unit vector. The solution to $Ax = b$ is obviously the zero-th column of A^{-1}, $x = e_{-1} + ce_0$. The finite section method as described in (1.2)–(1.3) applied to this system fails completely, because none of the matrices A_n is invertible. In theory the solution could be computed by solving the normal equations $A^*Ax = A^*b$. Thus, one might want to apply the finite section method to the positive definite system $A^*Ax = A^*b$ and invoke the results from the previous sections, since A (and thus A^*) belongs to $\mathcal{A}_v, \mathcal{A}_v^1$, or $\mathcal{C}_v$ with weight $v(k) = e^{|k|^\alpha}, 0 < \alpha < 1$. However, the computation of $P_n A^* A P_n$ involves the infinite matrices A, A^*, which makes this approach not feasible for numerical purposes.

It is easy to see how to alter the finite section method to make it work for this particular example. Our goal is more ambitious, and we want to derive a version of the finite section method that works for large classes of (algebras of) non-hermitian matrices, and not just for some individual cases. We will derive conditions for the convergence of the finite section method for non-hermitian matrices in some matrix algebras. For this, we consider a slightly generalized version of the finite section method.

Consider the system $Ax = b$ where A is an invertible, but not necessarily hermitian matrix. We set

$$A_{r,n} = P_r A P_n, \qquad \text{and} \qquad b_{r,n} = A^*_{r,n} b, \tag{7.1}$$

and try to solve the system

$$A^*_{r,n} A_{r,n} x_{r,n} = b_{r,n} \tag{7.2}$$

for properly chosen r and n. Observe that $A_{r,n}$ is a $(2r+1)^d \times (2n+1)^d$ matrix, and so $A^*_{r,n} A_{r,n}$ is a $(2n+1)^d \times (2n+1)^d$-matrix. In general we will need $r > n$, therefore we refer to (7.1)–(7.2) as *non-symmetric finite section method.*

Let us denote $B_n := P_n A^* A P_n$, $D_{r,n} := P_n A^* P_r A P_n = A^*_{r,n} A_{r,n}$. Analogously to (2.2) we define the extensions

$$\widetilde{B_n} = B_n + \lambda_+(I - P_n), \qquad \widetilde{D_{r,n}} = D_{r,n} + \lambda_+(I - P_n), \tag{7.3}$$

where $\sigma(A^*A) \subseteq [\lambda_-, \lambda_+]$.

Clearly, $B_n, n \in \mathbb{N}$, is the sequence of finite sections of A^*A and $D_{r,n}$ is an approximation of B_n. We study this approximation for matrices in $\mathcal{A}_v$.

Lemma 15. *Assume that $A \in \mathcal{A}_v$. Then there exists a sequence $R(n) \in \mathbb{N}$, such that for every $r(n) \geq R(n)$*

$$\lim_{n\to\infty} \|B_n - D_{r(n),n}\|_{\mathcal{A}_v} = 0\,. \tag{7.4}$$

If $v(k) = (1+|k|)^s$ and $\mathcal{A}_v = \mathcal{A}_s$, then we may choose $R(n) = n^\alpha$ for $\alpha > \frac{2s}{2s-d}$ and obtain the rate

$$\|B_n - D_{n^\alpha,n}\|_{\mathcal{A}_s} \leq C\|A\|_{\mathcal{A}_s}\, n^{\alpha(d-2s)+2s}\,,$$

Proof. We define $\widetilde{E_{r,n}} = \widetilde{B_n} - \widetilde{D_{r,n}} = B_n - D_{r,n}$. Clearly, $\widetilde{E_{r,n}}$ is hermitian and in $\mathcal{A}_v$, and $(\widetilde{E_{r,n}})_{kl} = 0$ for $k, l \notin C_n$. If $k, l \in C_n$, then

$$(E_{r,n})_{kl} = (B_n)_{kl} - (D_{r,n})_{kl} = \sum_{j\in\mathbb{Z}^d} (A^*)_{kj} a_{jl} - \sum_{j\in C_r} (A^*)_{kj} a_{jl} = \sum_{j\notin C_r} \overline{a_{jk}} a_{jl}\,,$$

and we obtain the estimate

$$|(\widetilde{E_{r,n}})_{kl}| \leq \sum_{j\notin C_r} |a_{jk}||a_{jl}|$$

for all entries. If $A \in \mathcal{A}_v$, then $|a_{jk}| \le \|A\|_{\mathcal{A}_v} v(j-k)^{-1}$. Consequently we estimate the norm of $\widetilde{E_{r,n}}$ by

$$\begin{aligned}\|\widetilde{E_{r,n}}\|_{\mathcal{A}_v} &= \sup_{k,l\in C_n} |(\widetilde{E_{r,n}})_{kl}| v(k-l) \\ &\le \sup_{k,l\in C_n} \|A\|^2_{\mathcal{A}_v} \sum_{j\notin C_r} v(j-k)^{-1} v(j-l)^{-1} v(k-l)\,.\end{aligned}$$

Since $v(j-l)^{-1} \le v(j-k)^{-1} v(k-l)$, we continue with

$$\|\widetilde{E_{r,n}}\|_{\mathcal{A}_v} \le \|A\|^2_{\mathcal{A}_v} \sup_{k,l\in C_n} \sum_{j\notin C_r} v(j-k)^{-2} v(k-l)^2\,.$$

Clearly, if $k,l \in C_n$ and $j \notin C_r$, then $k-l \in C_{2n}$ and $j-k \notin C_{r-n}$ and we arrive at the estimate

$$\|\widetilde{E_{r,n}}\|_{\mathcal{A}_v} \le \|A\|^2_{\mathcal{A}_v} \sup_{k\in C_{2n}} v(k)^2 \sum_{j\notin C_{r-n}} v(j)^{-2}\,. \tag{7.5}$$

As a consequence, we obtain that $\lim_{r\to\infty} \|\widetilde{E_{r,n}}\|_{\mathcal{A}_v} = 0$ and (7.4) is proved.

If $A \in \mathcal{A}_s$, i.e., $v(k) = (1+|k|)^s$, then $\sup_{k\in C_{2n}} v(k)^2 = \mathcal{O}(n^{2s})$ and $\sum_{j\notin C_{r-n}} v(j)^{-2} = \mathcal{O}((r-n)^{d-2s})$. For $r(n) = n^\alpha$, we obtain the explicit estimate

$$\|B_n - D_{n^\alpha,n}\|_{\mathcal{A}_s} \le C\|A\|_{\mathcal{A}_s}\, n^{\alpha(d-2s)+2s}\,,$$

which tends to 0 for $\alpha > \frac{2s}{2s-d}$. □

Theorem 16. *Let $\mathcal{A} \in \{\mathcal{A}_s, \mathcal{A}_v\}$ where the weight v satisfies the conditions stated in Theorem 3(a). Let $Ax = b$ be given. Assume that $b \in \ell^p_m$ and that $A \in \mathcal{A}$ is invertible on $\ell^2(\mathbb{Z}^d)$ and acts on ℓ^p_m.*

Consider the finite sections

$$A^*_{r,n} A_{r,n} x_{r,n} = A^*_{r,n} b. \tag{7.6}$$

Then, for every n there exists an $R(n)$ (depending on λ_- and v) such that $x_{r(n),n}$ converges to x in the norm of ℓ^p_m, for every choice $r(n) \ge R(n)$.

Proof. We split the error $x - x_{r,n}$ into three terms as follows:

$$\begin{aligned}\|x - x_{r,n}\|_{\ell^p_m} &= \|(A^*A)^{-1}A^*b - D^{-1}_{r,n} A_{r,n} b\|_{\ell^p_m} \\ &\le \|(A^*A)^{-1}A^*b - B_n^{-1} P_n A^* b\|_{\ell^p_m} + \|B_n^{-1} P_n A^* b - B_n^{-1} A_{r,n} b\|_{\ell^p_m} + \\ &\quad + \|B_n^{-1} A_{r,n} b - D^{-1}_{r,n} A_{r,n} b\|_{\ell^p_m} = \|\,\mathrm{I}\,\|_{\ell^p_m} + \|\,\mathrm{II}\,\|_{\ell^p_m} + \|\,\mathrm{III}\,\|_{\ell^p_m}\,.\end{aligned} \tag{7.7}$$

We observe that the vector $B_n^{-1} P_n A^* b$ is exactly the result of the finite section method applied to the normal equation $A^*Ax = A^*b$. Since $A^*A \in \mathcal{A}_v$ and $A^*b \in \ell^p_m$, Theorem 11 is applicable and implies that $\|\,\mathrm{I}\,\|_{\ell^p_m} \to 0$ for $p < \infty$ and $\mathrm{I} \to 0$ weak* for $p = \infty$.

Since $A_{r,n} = P_r A P_n$ and $B_n^{-1} P_n = \widetilde{B_n}^{-1} P_n$ we can estimate the second term by

$$\begin{aligned}\|\,\mathrm{II}\,\|_{\ell^p_m} &= \|\widetilde{B_n}^{-1}(P_n A^* b - P_n A^* P_r b)\|_{\ell^p_m} \\ &\le \sup_{n\in\mathbb{N}} \|\widetilde{B_n}^{-1}\|_{\ell^p_m\to\ell^p_m} \|A^*\|_{\ell^p_m} \|b - P_r b\|_{\ell^p_m}\,.\end{aligned} \tag{7.8}$$

As in the proof of Theorem 11, Corollary 4 and 9 imply that $\sup_{n\in\mathbb{N}} \|\widetilde{B_n}^{-1}\|_{\ell^p_m\to\ell^p_m} \leq C\sup_{n\in\mathbb{N}} \|\widetilde{B_n}^{-1}\|_{\mathcal{A}_v} \leq C' < \infty$. Since the finite sequences are dense in ℓ^p_m for $p<\infty$, (7.8) yields $\|\,\mathrm{II}\,\|_{\ell^p_m} \to 0$, similarly $\mathrm{II}\to 0$ weak* for $p=\infty$.

For the third term, we start with the obvious estimate

$$\|\,\mathrm{III}\,\|_{\ell^p_m} = \|B_n^{-1}A_{r,n}b - D_{r,n}^{-1}A_{r,n}b\|_{\ell^p_m} \leq \|\widetilde{B_n}^{-1} - \widetilde{D_{r,n}^{-1}}\|_{\ell^p_m\to\ell^p_m}\,\|A^*_{r,n}b\|_{\ell^p_m}\,.$$

Here $\|A^*_{r,n}b\|_{\ell^p_m} = \|P_nA^*P_rb\|_{\ell^p_m} \leq \|A^*\|_{\ell^p_m\to\ell^p_m}\,\|b\|_{\ell^p_m}$ is uniformly bounded independent of n and r.

For the operator norm we use an estimate for inverses in Banach algebras, see e.g., [8], and obtain that

$$\begin{aligned}\|\widetilde{B_n}^{-1} - \widetilde{D_{r,n}}^{-1}\|_{\ell^p_m\to\ell^p_m} &\leq C\|\widetilde{B_n}^{-1} - \widetilde{D_{r,n}}^{-1}\|_{\mathcal{A}_v}\\ &\leq C\,\frac{\|\widetilde{B_n}^{-1}\|^2_{\mathcal{A}_v}\,\|\widetilde{B_n}-\widetilde{D_{r,n}}\|_{\mathcal{A}_v}}{1-\|\widetilde{B_n}^{-1}\|_{\mathcal{A}_v}\,\|\widetilde{B_n}-\widetilde{D_{r,n}}\|_{\mathcal{A}_v}}\,.\end{aligned}$$

Once again by Corollary 9 we have $\sup_{n\in\mathbb{N}}\|\widetilde{B_n}^{-1}\|_{\mathcal{A}_v} \leq C < \infty$, and by Lemma 15 $\lim_{r\to\infty}\|B_n - D_{r,n}\|_{\mathcal{A}_v} = 0$.

Consequently, for any positive sequence $\epsilon_n \to 0$, we may choose $R(n)$, such that $\|B_n - D_{r(n),n}\|_{\mathcal{A}_v} < \epsilon_n$ for $r(n)\geq R(n)$.

By combining the estimates for I, II, III, we have thus proved that $\|x - x_{r(n),n}\|_{\ell^p_m} \to 0$ for every sequence $r(n)\geq R(n)$ and we are done. □

Remark 7.1. If $v(k) = (1+|k|)^s$ for $s>d$, then $R(n)$ can be chosen to be n^α for $\alpha > \frac{2s}{2s-d}$ by Lemma 15.

Remark 7.2. It is well-known that, from a numerical viewpoint, the solution of the normal equations should be avoided whenever the condition number of the matrix is large. As an alternative to the normal equations one could use matrix factorization methods. Since $D_{r,n}$ is invertible, the matrix $A_{r,n}$ has full rank $(2n+1)^d$, and one could apply a QR-factorization of $A_{r,n}$ or some other factorization and compute an approximate solution to $Ax=b$ in that way. This idea raises a number of interesting questions: For instance, assume we can factorize a matrix $A\in\mathcal{A}$ into $A=QR$, where Q is unitary and R is upper triangular, do the individual components Q and R also belong to $\mathcal{A}$? How about other matrix factorizations such as LU- or polar-decomposition? We refer the reader to [27] for answers to these questions.

We return to the example in the beginning of this section. Clearly, A belongs to $\mathcal{A}_v$ for every weight $v(k) = e^{|k|^\alpha}, 0<\alpha<1$. Since the entries of A decay exponentially off the diagonal, it is not difficult to see that it is sufficient to choose $r(n) = sn$ for a sufficiently large $s>1$, independently of n. In this particular example it would even suffice to set $s=n+1$, but as pointed out, our goal was to derive a finite section technique that is applicable to large classes of matrices, not just to this particular one.

In light of Theorem 16 it is worthwhile to recall that a necessary and sufficient condition for the applicability of the finite section method (1.2)–(1.3) to Laurent operators is that the winding number of the invertible Laurent operator is zero, cf. [12,17]. For the non-symmetric finite section method the winding number is not relevant, the key property is the off-diagonal decay of the matrix. Thus Theorem 16 considerably enlarges the range of applicability of finite section type methods even for the classical and thoroughly analyzed cases of Laurent and Toeplitz operators.

8. An example from digital communication

In this section we demonstrate the practical relevance of the theoretical framework derived in this paper by analyzing a problem arising in digital communication. We highlight the details related to the finite section method and refer the reader to [21] for a more detailed description of the engineering aspects of the problem.

In a time-invariant digital communication system one is confronted with a linear system of equations $Ax = b$, where $x = \{x_l\}_{l\in\mathbb{Z}}$ is a sequence of information symbols to be transmitted and $b = \{b_k\}_{k\in\mathbb{Z}}$ is the received, discrete signal. We can assume that $x_k \in \{-1, 1\}$, thus $x \in \ell^\infty$. The entries of A are of the form

$$a_{kl} = \varphi(\cdot - kR) * h * \varphi(\cdot - lT), \tag{8.1}$$

where h is the *channel impulse response*, φ is a bandlimited function (the *transmission pulse*), T is the transmission period, and R is the receive sampling period. We do not go into detail about the particular choice of φ, T, and R. The only facts we need are: (i) for properly selected T and R we can choose φ to be a bandlimited function in $L^1_v(\mathbb{R})$, where v must satisfy the Beurling-Domar condition, i.e.,

$$\sum_{k=0}^{\infty} \frac{\log v(kx)}{k^2} < \infty, \qquad \text{for all } x \in \mathbb{R}; \tag{8.2}$$

(ii) under certain conditions on h, the matrix A has an inverse for $R = T$ and a left-inverse for $R < T$.

Furthermore, we note that h is a causal function that decays exponentially in time. This implies that A is non-hermitian and that $A \in \mathcal{A}_v$, the latter follows from well-known properties of Beurling convolution algebras [23] and the fact that a weight which satisfies (8.2) also satisfies the GRS condition (3.1), cf. [14].

There are two ways to approach the problem of recovering x from b. In the first case we try to recover the entries of x "on the fly", i.e., we solve the truncated system $A_{m,n}x_n = b_n$. In this case we only assume that $b \in \ell^\infty$ and the ℓ^p_m-convergence estimates of Theorem 16 apply. In the second case we precompute the inverse of A by solving $Az = e_0$ where e_0 is the zeroth unit vector. Due to the specific (block)-Toeplitz structure of A, the vector z contains all required information to fully determine the inverse of A, which is then used to recover x. In this case we apply the non-symmetric finite section method from Section 7 to $Az = e_0$. Since $A \in \mathcal{A}_v$ and $A^*e_0 \in \ell^1_v$, quantitative estimates as in Section 6 apply and we can approximate the true solution z with a rate of convergence depending on v. Since in this application v can be chosen to be $v(x) = e^{|x|^\alpha}$ with $\alpha < 1$, the (non-symmetric) finite section method achieves exponential rate of convergence.

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