A note on equiangular tight frames

Thomas Strohmer [1]

Department of Mathematics, University of California, Davis, CA 95616, USA

Abstract

We settle a conjecture of Joseph Renes about the existence and construction of certain equiangular tight frames.

Key words: Equiangular tight frames, Sherman-Morrison-Woodbury formula, conference matrix

1 Introduction

An *equiangular tight frame* is a family of vectors $\{f_k\}_{k=1}^n$ in $\mathbb{E}^d$ (where $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$) that satisfies the conditions (10)

$$\|f_k\|_2 = 1 \qquad \text{for } k = 1, \dots, n, \tag{1}$$

$$|\langle f_k, f_l \rangle| = c, \qquad \text{for all } k \neq l \text{ and some constant } c, \tag{2}$$

$$\frac{d}{n}\sum_{k=1}^{n} \langle f, f_k \rangle f_k = f, \qquad \text{for all } f \in \mathbb{E}^m. \tag{3}$$

In fact, conditions (2) and (3) together imply that

$$|\langle f_k, f_l \rangle| = \sqrt{\frac{n-d}{d(n-1)}}, \qquad \text{for all } k \neq l, \tag{4}$$

which is the smallest possible value for c for a set of n equiangular unit-norm vectors in $\mathbb{E}^d$, cf. (4; 10).

Due to their rich theoretical properties and their numerous practical applications, equiangular tight frames are arguably the most important class of

Email address: strohmer@math.ucdavis.edu (Thomas Strohmer).

[1] T.S. acknowledges support from NSF DMS grant 0511461.

finite-dimensional frames, and they are the natural choice when one tries to combine the advantages of orthonormal bases with the concept of redundancy provided by frames (10).

Yet, despite their importance, we are far from having a complete understanding about the existence of equiangular tight frames. Some results can be found in (12; 8; 4; 3; 13; 10; 6; 1; 7; 9; 11).

A popular method to construct equiangular tight frames is based on *conference matrices* (8; 3; 10). Before we proceed to this construction and the statement of Renes' conjecture, we need to introduce some notation. For general background on frames we refer to (2). Given a frame $\{f_k\}_{k=1}^n$ for $\mathbb{E}^d$, the frame operator S is defined by

$$Sf = \sum_{k=1}^n \langle f, f_k \rangle f_k.$$

Note that the frame operator of a tight frame is a multiple of the identity operator. The *tight frame canonically associated to* $\{f_k\}_{k=1}^n$ is $\{S^{-\frac{1}{2}} f_k\}_{k=1}^n$. We will write (n, d)-ETF for an equiangular tight frame $\{f_k\}_{k=1}^n$ in $\mathbb{E}^d$.

Now, recall that an $n \times n$ conference matrix C has zeros along its main diagonal, ± 1 as its other entries. and satisfies $CC^* = (n-1)I_n$ (I_n denotes the $n \times n$ identity matrix). Given an $n \times n$ conference matrix with $n = 2d$, one can construct a $(2d, d)$-ETF $\{f_k\}_{k=1}^n$ via its Gram matrix $R = \{\langle f_l, f_k \rangle\}_{k,l=1}^n$. If C is symmetric, one computes $R = \sqrt{n-1}C + I_n$, if C is skew-symmetric (i.e., $C = -C^T$) one computes $R = i\sqrt{n-1}C + I_n$. One can then extract the $(2d, d)$-ETF $\{f_k\}_{k=1}^n$ from R via a singular value decomposition, see (10).

In (9), Joseph Renes conjectured that, given an $(2d, d)$-ETF $\{f_k\}_{k=1}^n$ associated with a skew-symmetric conference matrix, one can always construct a $(2d-1, d)$-ETF by removing an arbitrary frame element from $\{f_k\}_{k=1}^n$ and then computing the tight frame canonically associated with the remaining frame elements. This conjecture is supported by numerical simulations as well as by a proof by Renes for the special case when d satisfies the property $2d-1 = p^k = 3$ mod 4, where p is a prime number. Renes' proof relies on Zauner's construction of $(2d, d)$-ETFs and specific properties of finite fields and Gauss sums.

In the following we settle Renes's conjecture for general d. Our proof uses only elementary linear algebra.

Theorem 1.1 *Let $\{f_k\}_{k=1}^n$ be an equiangular tight frame for $\mathbb{C}^d$ with $n = 2d$ and assume that*

$$\langle f_k, f_l \rangle = \pm i \sqrt{\frac{1}{n-1}}, \qquad \text{for all } k \neq l. \tag{5}$$

Define the frame $\{\varphi_j^{(l)}\}_{j=1}^{n-1} := \{f_k\}_{k \neq l, k=1}^n$ and denote the frame operator asso-

ciated with $\{\varphi_j^{(l)}\}_{j=1}^{n-1}$ *by* S_l. *Set*

$$g_j = \sqrt{\frac{n-1}{d}} S_l^{-\frac{1}{2}} \varphi_j^{(l)}, \qquad \text{for } j = 1, \dots, n-1,$$

then $\{g_j\}_{j=1}^{n-1}$ *is an equiangular tight frame for* $\mathbb{C}^d$.

Proof: Without loss of generality we let $l = n$ and set $\varphi_k := \varphi_k^{(1)} = f_k$, for $k = 1, \dots, n-1$ (i.e., we remove the last element of the frame $\{f_k\}_{k=1}^n$). We denote by F the $d \times n$ matrix containing the frame vectors $f_k, k = 1, \dots, n$ as columns and similarly Φ and G are $d \times (n-1)$ matrices having the φ_k and g_k as their columns, respectively. There holds

$$G^* G = \frac{n-1}{d} \Phi^* (\Phi\Phi^*)^{-\frac{1}{2}} (\Phi\Phi^*)^{-\frac{1}{2}} \Phi = \frac{n-1}{d} \Phi^* (\Phi\Phi^*)^{-1} \Phi. \tag{6}$$

We apply the *Sherman–Morrison–Woodbury* formula (5) and compute

$$(\Phi\Phi^*)^{-1} = (FF^* - f_n f_n^*)^{-1} = \frac{d}{n}\Big(I_d - \frac{f_n f_n^*}{1 - \frac{n}{d}}\Big) = \frac{1}{2}\Big(I_d + f_n f_n^*\Big), \tag{7}$$

where we have used that $\{f_k\}_{k=1}^n$ is a tight frame. We insert (7) into (6) and obtain

$$G^* G = \frac{n-1}{n}\Big(\Phi^*\Phi + \Phi^* f_n f_n^* \Phi\Big).$$

Now consider $(G^*G)_{k,l}$ for $k, l = 1, \dots, n-1; k \neq l$:

$$(G^*G)_{k,l} = \frac{n-1}{n}\Big(\langle f_l, f_k\rangle + \langle f_l, f_n\rangle\langle f_n, f_k\rangle\Big). \tag{8}$$

By assumption $\langle f_k, f_l\rangle = \pm i\sqrt{\frac{1}{n-1}}$ for all $k, l = 1, \dots, n$ with $k \neq l$. Hence

$$(G^*G)_{k,l} = \frac{n-1}{n}\Big(\pm i\sqrt{\frac{1}{n-1}} \pm \frac{1}{n-1}\Big), \tag{9}$$

and therefore

$$|(G^*G)_{k,l}| = \frac{1}{\sqrt{2d}}, \tag{10}$$

for all $k, l = 1, \dots, n-1; k \neq l$, which completes the proof. ❏

Remark: Since the off-diagonal entries of the Gram matrix of a $(2d, d)$-ETF associated with a skew-symmetric conference matrix always satisfy condition (5), Theorem 1.1 proves Renes' conjecture.

Corollary 1.2 *A necessary condition for the Gram matrix of an* (n, d)*-ETF to satisfy*

$$\langle f_k, f_l\rangle = \pm i\sqrt{\frac{n-d}{d(n-1)}}, \qquad \text{for all } k \neq l, \tag{11}$$

is that $n = 2d$.

Proof: We repeat the steps of the proof of Theorem 1.1 for arbitrary $n \in \mathbb{N}$ with $n > d$. Denoting $\alpha = n/d$, we have

$$\langle f_k, f_l \rangle = \pm i \sqrt{\frac{d(\alpha-1)}{d(\alpha d - 1)}}, \qquad \text{for all } k \neq l. \tag{12}$$

The right-hand side of equation (10) now becomes

$$\frac{\sqrt{\alpha d(\alpha-1) - \alpha + 2}}{\alpha d}. \tag{13}$$

Equating (13) with (4) and solving for α gives as only feasible solution $\alpha = 2$.

❑

Corollary 1.3 *If $\{f_k\}_{k=1}^n$ is a real-valued (n, d)-ETF, then the canonical tight frame associated with the frame obtained by removing an arbitrary element from $\{f_k\}_{k=1}^n$ can never be equiangular, except for the trivial case $n = d + 1$.*

Proof: Let R denote the Gram matrix of a real-valued (n, d)-ETF. We claim that the entries $R_{k,l}, k \neq l$ cannot all have the same sign unless $n = d + 1$. To see this we first recall that R has d eigenvalues that are equal to n/d and $n - d$ eigenvalues equal to 0, cf. (10).

Now assume that $\text{sign}(R_{k,l}) = -1$ for all $k \neq l$. In this case R is a circulant matrix and its eigenvalues are given by the Discrete Fourier Transform $\hat{r}$ of the first column r of R. Since $r = [1, -c, -c, \ldots, -c]^T$, where $c = \sqrt{(n-d)/(dn-d)}$, it follows that

$$\hat{r} = [-\sqrt{n}c + (1+c)/\sqrt{n}, (1+c)/\sqrt{n}, (1+c)/\sqrt{n}, \ldots, (1+c)/\sqrt{n}]^T.$$

Clearly, $\hat{r}$ can have at most its first entry equal to zero, and this can happen only if $n = d+1$. In case $\text{sign}(R_{k,l}) = 1$ for all $k \neq l$, $\hat{r}$ would have only strictly positive entries, contradicting the fact that R must have $n - d > 0$ eigenvalues equal to zero.

We now repeat the steps of the proof of Theorem 1.1 for real-valued frames, and with $n = \alpha d$ for some $\alpha > 1$ such that $n \in \mathbb{N}$. Equation (9) becomes

$$(G^*G)_{k,l} = \frac{n-1}{n}\Big(\pm \sqrt{\frac{\alpha-1}{n-1}} \pm \frac{\alpha-1}{n-1}\Big). \tag{14}$$

From above we know that the off-diagonal entries of G^*G cannot all have the same sign, except if $n = d+1$, which leads to the trivial case $G^*G = I_d$. Thus, in order to have an equiangular frame for $n > d + 1$ we must have

$$\left|1 + \sqrt{\frac{\alpha - 1}{n - 1}}\right| = \left|1 - \sqrt{\frac{\alpha - 1}{n - 1}}\right| \tag{15}$$

which is not possible. This completes the proof. ❑

References

[1] D.M. Appleby. SIC-POVMs and the extended Clifford group. *Math. Phys.*, 46:052107, 2005.

[2] O. Christensen. *An introduction to frames and Riesz bases.* Birkhäuser, Boston, 2003.

[3] J.H. Conway, R.H. Hardin, and N.J.A. Sloane. Packing lines, planes, etc.: packings in Grassmannian spaces. *Experiment. Math.*, 5(2):139–159, 1996.

[4] P. Delsarte, J. M. Goethals, and J. J. Seidel. Bounds for systems of lines and Jacobi poynomials. *Philips Res. Repts*, 30(3):91*–105*, 1975. Issue in honour of C.J. Bouwkamp.

[5] G.H. Golub and C.F. van Loan. *Matrix Computations.* Johns Hopkins, Baltimore, third edition, 1996.

[6] R.B. Holmes and V.I. Paulsen. Optimal frames for erasures. *Linear Algebra Appl.*, 377:31–51, 2004.

[7] D. Kalra. Complex equiangular cyclic frames and erasures. *Linear Algebra Appl.*, 419(1):373–399, 2007.

[8] P. W. H. Lemmens and J. J. Seidel. Equiangular lines. *J. Algebra*, 24:494–512, 1973.

[9] J. Renes. Equiangular tight frames from Paley tournaments. *Linear Algebra Appl.*, 426(2-3):497–501, 2007.

[10] T. Strohmer and R. Heath. Grassmannian frames with applications to coding and communications. *Appl. Comp. Harm. Anal.*, 14(3):257–275, 2003.

[11] M. Sustik, J.A. Tropp, I. Dhillon, and R.W. Heath Jr. On the existence of equiangular tight frames. *Linear Algebra Appl.*, 426(2–3):619–635, 2007.

[12] J. H. van Lint and J. J. Seidel. Equilateral point sets in elliptic geometry. *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, 28:335–348, 1966.

[13] G. Zauner. *Quantum Designs – Foundations of a Non-Commutative Theory of Designs.* PhD thesis, University of Vienna, 1993.