# Differential Equations

MAT 22B – Fall 2009

Midterm 1 – Solutions

Name: .......................................................

I.D Number.......................................................

Signature.......................................................

Keep you student's i.d. visible in front of you.
No books, no calculators or electronic devices allowed.
Turn your cell phone off.
No communication among the students will be tolerated.

| Problem | Total Points | Student's Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| 6 | 20 | |
| Total | 100 | |

Problem 1. [10 pts]
The half-life of a radioactive material is the time required for an amount of this material to decay to one-half of its original value. Show that for any radioactive material that decays according to the equation

$$Q' = -rQ\,,$$

where $Q = Q(t)$ is the amount left at time $t$ and $r$ is a positive constant, the half-life $\tau$ and the decay rate $r$ satisfy the equation

$$r\tau = \ln 2\,.$$

Solution: Let $Q_0$ be the amount of material at $t = 0$. The solution of the initial value problem is

$$Q(t) = Q_0 \mathrm{e}^{-rt}\,, \qquad t \in \mathbb{R}\,.$$

By definition, $\tau$ is such that $Q(\tau) = 1/2\, Q_0$. The above solution therefore yields an equation for the half-life,

$$Q_0 \mathrm{e}^{-r\tau} = \frac{1}{2} Q_0\,,$$

so that $-r\tau = -\ln 2$.

*Remark.* Since the equation is autonomous, its solutions are invariant under time translation. The half-life is therefore independent of the choice of initial time (here $t = 0$).

Problem 2. [10pts]
i. Find the solution of the differential equation

$$\frac{dy}{dx} = xy^2$$

that passes through the point $(0, 8)$.
ii. What is the maximal domain (i.e. the range of $x$-values) of this branch of solution?

Solution:
i. The equation is separable. Its solution is given implicitly by the equation

$$\int_8^y \frac{d\eta}{\eta^2} = \int_0^x \xi \, d\xi \quad \Longleftrightarrow \quad -\frac{1}{y} + \frac{1}{8} = \frac{x^2}{2} .$$

Hence,

$$y(x) = \frac{8}{-4x^2 + 1} .$$

*Remark.* One could also proceed by indefinite integration to obtain $-1/y = x^2/2 + C$ and choose the constant to fit the initial condition.
ii. The solution has singularities at $x = \pm 1/2$. The domain containing the initial condition $x = 0$ is therefore $(-1/2, 1/2)$.

Problem 3. [20pts]
Consider the differential equation

$$\frac{du}{dt} = -ku + T_0 \cos \omega t \,, \qquad k > 0 \,, T_0 > 0 \,.$$

i. Find its general solution;
*Hint:* $e^{\alpha t} \cos \beta t = \frac{\alpha}{\beta^2+\alpha^2} \frac{d}{dt}\big((\cos \beta t + \frac{\beta}{\alpha} \sin \beta t)\, e^{\alpha t}\big)$;

ii. Is there an equilibrium solution?

iii. Does the initial condition influence the long time behaviour of the solution?

Solution: i. Since the equation is of first order and linear, we shall use an integrating factor $\mu(t)$ to solve it. For the l.h.s. to be the derivative of $\mu(t)y$, $\mu$ must satisfy the ODE $k\mu = \mu'$ so that we can choose

$$\mu(t) = e^{kt} \,.$$

The direct integration of the original differential equation yields

$$e^{kt} u = T_0 \int e^{kt} \cos \omega t \, dt = \frac{kT_0}{\omega^2 + k^2} \left( (\cos \omega t + \frac{\omega}{k} \sin \omega t)\, e^{kt} \right) + C \,.$$

Hence,

$$u(t) = \frac{kT_0}{\omega^2 + k^2} \left( \cos \omega t + \frac{\omega}{k} \sin \omega t \right) + C e^{-kt} \,.$$

ii. No. This can be seen directly from the equation itself, namely by noting that there is no value $\widetilde{u}$ such that $-k\widetilde{u} + T_0 \cos \omega t = 0$ for all $t$.
iii. No. The initial condition does only influence the integration constant $C$ and therefore only the last term of the solution. As $t \to +\infty$, the latter decays exponentially to zero and the imprint of the initial condition vanishes. In fact, although there is no equilibrium solution, the periodic solution consisting of only the first term is a so-called stationary solution: the system converges to this oscillating behaviour independently of the initial condition.

PROBLEM 4. [20 pts]
i. Give a linear differential equation with constant coefficients (the order is arbitrary), for which the function

$$y(t) = \sin(2t+1)$$

is a solution.
ii. Show that

$$x(s) = \int_1^s e^{\left(t^3+t^2/2\right)}\, dt$$

is a solution of the initial value problem

$$\begin{cases} e^{-s^3}(x'' - sx') = 3s^2 e^{\frac{1}{2}s^2} \\ x(1) = 0 \end{cases} .$$

SOLUTION: i. By taking successive derivatives, one notes that $y' = 2\cos(2t+1)$ and $y'' = -4\sin(2t+1)$. The simplest equation (with constant coefficients) is therefore

$$y'' = -4y\,.$$

ii. The derivatives of the proposed solution read

$$\begin{aligned} x' &= e^{s^3+s^2/2}\,, \qquad \text{by the fundamental theorem of calculus,} \\ x'' &= (3s^2+s)e^{s^3+s^2/2}\,. \end{aligned}$$

Hence,

$$e^{-s^3}(x'' - sx') = e^{-s^3}((3s^2+s) - s)e^{s^3+s^2/2} = 3s^2 e^{s^2/2}$$

which proves that it is indeed a solution of the ODE.
Moreover, $x(1) = \int_1^1 e^{\left(t^3+t^2/2\right)}\, dt = 0$. The initial condition is therefore satisfied, too.

Problem 5. [20pts]
Consider the differential equation

$$\frac{dy}{dt} = 3y\left(y^2 - y - 2\right).$$

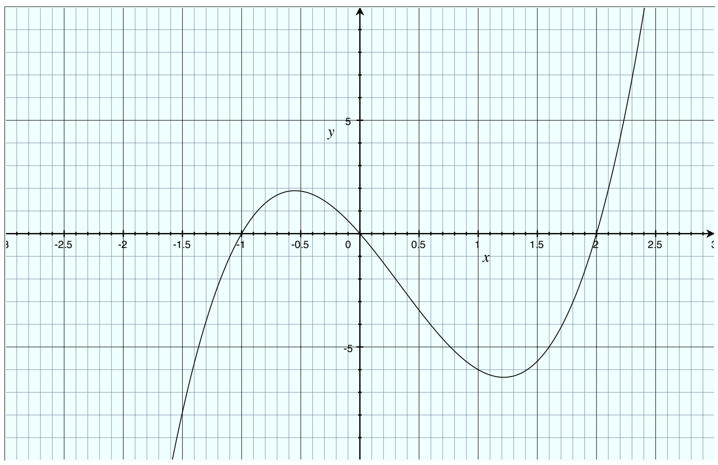
i. Find the equilibrium solutions;

ii. Draw the phase line;

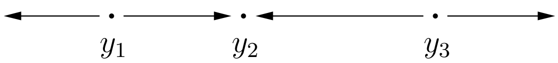
iii. Discuss the stability of the critical points;

iv. Sketch the solutions in the $(t, y)$-plane. You do not need to provide the complete analysis of concavity/convexity.

Solution: i. The ODE is an autonomous equation $y' = f(y)$, for $f(y) = 3y(y-2)(y+1)$. Its equilibrium solutions correspond to the critical points $y_1 = -1$, $y_2 = 0$, and $y_3 = 2$.
ii. From the sign of $f$ between the critical points,



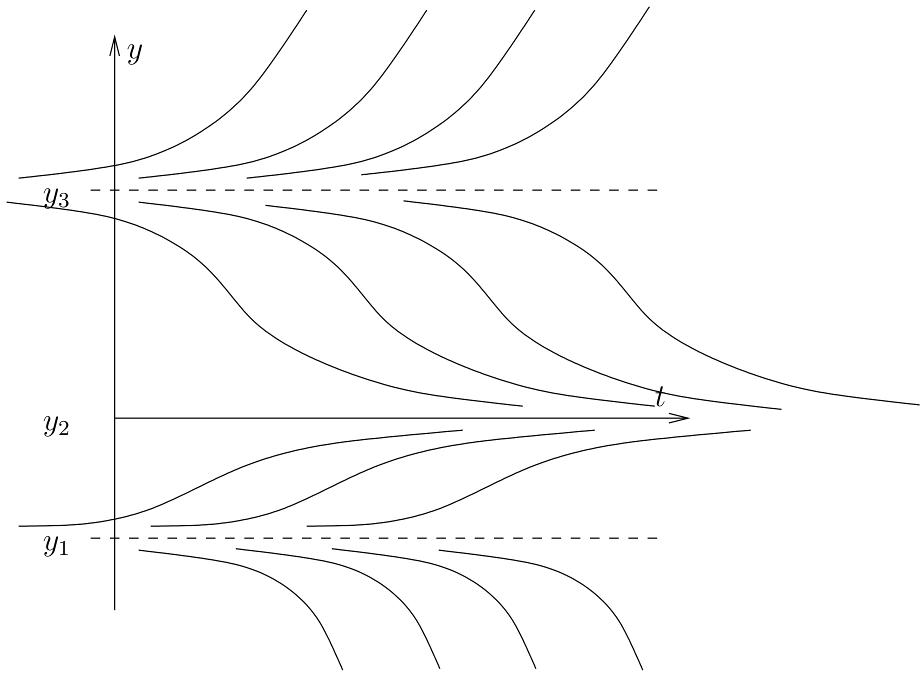

one obtains the phase line




iii. It follows that $y_1$ and $y_3$ are unstable, $y_2$ is asymptotically stable.
*Remark.* The same conclusion can be obtained by computing the sign of $f' = 3(3y^2 - 2y - 2)$ at the critical points,

$$\begin{aligned} f'(-1) = 9 > 0 \quad &\Longrightarrow y_1 \text{ unstable} \\ f'(0) = -6 < 0 \quad &\Longrightarrow y_2 \text{ stable} \\ f'(2) = 18 > 0 \quad &\Longrightarrow y_3 \text{ unstable} \end{aligned}$$

iv. The asymptotic analysis leads to the following sketch of solutions



PROBLEM 6. [20pts]
Consider the initial value problem

$$\begin{cases} 2yy' + \frac{2}{t}y^2 = \frac{1}{t^3} \\ y(1) = -2 \end{cases}$$

i. What can you say about existence and uniqueness of the solution?

ii. Solve the IVP
*Hint:* Find a suitable integrating factor.

SOLUTION: i. The equation is of the form $y' = f(t, y)$ for

$$f(t, y) = -\frac{y}{t} + \frac{1}{2yt^3},$$

and is therefore nonlinear. Both $f$ and its partial derivative $\partial f/\partial y = -1/t - 1/(2y^2t^3)$ are discontinuous along the lines $\{y = 0\}$ and $\{t = 0\}$. The assumptions of the existence and uniqueness theorem are therefore valid in any rectangle $R = [a, b] \times [\alpha, \beta]$ such that $(1, -2) \in R$ and $b > a > 0$, $\alpha < \beta < 0$. The theorem then ensures that a unique solution exists in some time interval $[t_1, t_2] \ni 1$ with $t_1 > 0$.
ii. After multiplication of the ODE by the yet unknown function $\mu(t)$, one observes that the l.h.s. is the derivative of $\mu(t)y^2$ if

$$\mu' = \frac{2\mu}{t},$$

since $(\mu y^2)' = 2\mu yy' + \mu' y^2$. A solution of this auxiliary equation is $\mu(t) = \exp(\int(2/t)dt) = t^2$, whence

$$t^2y^2 = \int \frac{1}{t}\,dt = \ln t + C.$$

Of the two branches of solutions

$$y(t) = \pm\sqrt{\frac{\ln t}{t^2} + \frac{C}{t^2}},$$

the negative one passes through the point $(1, -2)$, namely for $C = 4$. The solution of the IVP is therefore

$$y(t) = -\frac{\sqrt{\ln t + 4}}{t}.$$