Extra Homework: Math 116 S10 Temple

#1 Let $A^i_{\cdot j}$ be a $\binom{1}{1}$-tensor. Prove that

$$A_{ij} = g_{ik} A^k_{\cdot j}$$

transforms like a $\binom{0}{2}$-tensor.

#2 Show that if $A^i_{\cdot j} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a $\binom{1}{1}$-tensor, and $\frac{\partial}{\partial y^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$ where $B^i_\alpha = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, then A is symmetric but $\bar{A} = \bar{A}^\alpha_\beta$ is not. (I.e., symmetry is not a property of $\binom{1}{1}$-tensors, but $\binom{0}{2}$-tensors)

~~**#3** Assume $A^i_{\cdot j} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ is symmetric wrt inner product g, has complex eigenvalues, but eigenvectors have zero length wrt the complex inner~~

#3 Assume $g \equiv g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (the metric of special relativity). Show that

$$A = A^i_j = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

is symmetric wrt inner product g, has complex eigenvalues, but eigenvectors have zero length wrt the complex inner product.

#4 Let $g = g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, so that $\lambda = 1$ & 2 are eigenvalues. Find a change of basis $\frac{\partial}{\partial y_\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$ such that in the $\underset{\sim}{y}$-basis, $\bar{g} = \bar{g}_{\alpha\beta}$ does not have eigenvalues $\lambda = 1, 2$. (I.e., eigenvalues are not properties of $\binom{0}{2}$-tensors. Hint $\bar{g} = B^T g B$.)

#5 Assume that in $\underset{\sim}{x}$-coordinates, $g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ at every point P, so that M is Euclidean space in $\underset{\sim}{x}$-coordinates. Let $\underset{\sim}{y}$ denote polar coordinates, so

$$y^1 = r, \quad y^2 = \theta$$

and

$$x^1 = r\cos\theta, \quad x^2 = r\sin\theta.$$

(1) Find the 2x2 matrix $B^i_\alpha = \dfrac{\partial x^i}{\partial y^\alpha}$ row colm

(2) Using this, find $\bar{g}_{\alpha\beta}$, the metric in $\underset{\sim}{y}$-coordinates at $\underset{\sim}{y}(P) = (r, \theta)$.

#6 Prove that $A^i_i = \bar{A}^\alpha_\alpha$, ie the contraction is independent of coordinates