

MIDTERM EXAM I
Math 16A
Temple-Winter 2010

–Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

–Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

#5

#6

#7

#8

TOTAL:

1. Determine the following limits (simplify answer):

(a) (6 pts) $\lim_{x \rightarrow 2} \sqrt{\frac{x^2-4}{x-2}}$ (Hint: Factor)

$$\begin{aligned}\lim_{x \rightarrow 2} \sqrt{\frac{x^2-4}{x-2}} &= \lim_{x \rightarrow 2} \sqrt{\frac{(x-2)(x+2)}{x-2}} \\ &= \lim_{x \rightarrow 2} \sqrt{\frac{(x+2)}{1}} = \sqrt{4} = 2\end{aligned}$$

(b) (6 pts) $\lim_{x \rightarrow 0} \frac{\sqrt{x+2}-\sqrt{2}}{4x}$ (Hint: Conjugate)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+2}-\sqrt{2}}{4x} &= \lim_{x \rightarrow 0} \left\{ \frac{\sqrt{x+2}-\sqrt{2}}{4x} \cdot \left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}} \right) \right\} \\ &= \lim_{x \rightarrow 0} \left\{ \frac{x+2-2}{4x(\sqrt{x+2}+\sqrt{2})} \right\} \\ &= \lim_{x \rightarrow 0} \left\{ \frac{1}{4(\sqrt{x+2}+\sqrt{2})} \right\} \\ &= \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}\end{aligned}$$

(No simplification required past $\frac{1}{5\sqrt{2}}$.)

2. Determine the following limits:

(c) (6 pts) $\lim_{x \rightarrow \infty} \frac{2x^5 + 3x - 4}{3x^5 - 7}$ (Hint: Divide by highest power)

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{2x^5 + 3x - 4}{3x^5 - 7} &= \lim_{x \rightarrow +\infty} \left\{ \frac{2x^5 + 3x - 4}{3x^5 - 7} \left(\frac{1/x^5}{1/x^5} \right) \right\} \\ &= \lim_{x \rightarrow +\infty} \left\{ \frac{2 + \frac{3}{x^4} - \frac{4}{x^5}}{3 - \frac{7}{x^5}} \right\} \\ &= \frac{2}{3} \end{aligned}$$

(d) (6 pts) $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\tan^2 x \cos x}$ (Hint: Simplify)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\tan^2 x \cos x} &= \lim_{x \rightarrow 0} \left\{ \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x} \cos x} \right\} \\ &= \lim_{x \rightarrow 0} \{ \cos x \} \\ &= 1 \end{aligned}$$

3. Let $f(x) = \sqrt{4 - x^2}$, $g(x) = 2 \sin x$.

(a) (5 pts) Find the Domain of f .

$$\text{Domain}(f) = \{x : -2 \leq x \leq 2\}.$$

(b) (5 pts) Find the (precise) Range of g .

$$\text{Range}(g) = \{x : -2 \leq x \leq 2\}.$$

(c) (6 pts) Find $(f \circ g)(x)$.

$$(f \circ g)(x) = \sqrt{4 - 4 \sin^2 x}.$$

(d) (5 pts) Prove that $(f \circ g)(x)$ is defined for every real number x . (I.e, show the Domain of $f \circ g$ is all of \mathcal{R} .)

- Since the domain of g is all real numbers and the range of g lies within the domain of f , it follows that the domain of $f \circ g = f(g(x))$ equals the domain of g equals all of \mathcal{R} .

- Or...Let $x \in \mathcal{R}$. Then $-2 \leq g(x) \leq 2$, so $4 - g(x)^2 \geq 0$, and thus $f(g(x)) = \sqrt{4 - g(x)^2}$ is well defined.

4. Consider the function $f(x) = \frac{2x+1}{x-3}$ with Domain $x \neq 3$.

(a) (7 pts) Find a formula for $f^{-1}(x)$.

Set $y = \frac{2x+1}{x-3}$, solve for x and then switch x and y :

$$xy - 3y = 2x + 1$$

$$x(y - 2) = 3y + 1$$

$$x = \frac{3y + 1}{y - 2}$$

So

$$f^{-1}(x) = \frac{3x + 1}{x - 2}$$

(b) (4 pts) Find the Domain of f^{-1} .

$$\text{Domain}(f^{-1}) = \{x \in \mathcal{R} : x \neq 2\}$$

(c) (4 pts) Evaluate $f^{-1}(1)$

$$f^{-1}(1) = \frac{3 \cdot 1 + 1}{1 - 2} = -4$$

5. Find all vertical and horizontal asymptotes (you needn't graph the functions):

(a) (3 pts) $y = \sin x$

No vertical or horizontal asymptotes

(b) (6 pts) $y = \frac{2x^2}{(x^2-1)(x+2)}$

Vertical asymptotes at $x = -1, 1, -2$; Horizontal asymptote $y = 0$.

(c) (6 pts) $y = \tan x$

Since $y = \tan x = \frac{\sin x}{\cos x}$, vertical asymptotes are the zeros of $\cos x$ which are the lines $x = n\pi$ for $n \in \mathcal{R}$. No limits at $x \rightarrow \pm\infty$, so no horizontal asymptotes.

6. Given a function $f(x)$:

(a) (4 pts) State the *definition* of the derivative $f'(x)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(b) (7 pts) Directly from the definition, derive the value of $f'(2)$ if $f(x) = x^3$. (Hint: $(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$).

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left\{ \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \right. \\ &= \left. \frac{3hx^2 + 3h^2x + h^3}{h} = 3x^2 + 3hx - h^2 \right\} = 3x^2 \end{aligned}$$

7. (7 pts) Draw the graph of a function continuous except at $x = 2$, such that $f(2) = 1$, $\lim_{x \rightarrow 2^-} f(x) = 0$, and $\lim_{x \rightarrow 2^+} f(x) = 4$. (Use closed and open dots correctly at $x = 2$.)

8. (7 pts) Find the equation of the line *perpendicular* to the graph of $y = 2x^2 - 3$ at the point $(1, -1)$.

Calculate $f'(x) = 4x$, so $f'(1) = 4$, thus the perpendicular slope is $m = -\frac{1}{4}$, and the line is

$$\frac{y + 1}{x - 1} = -\frac{1}{4}$$

or

$$y = -\frac{1}{4}x - \frac{3}{4}$$