Name: Solutions

Student ID#: ____________

Section: ____________

Midterm Exam 1

Wednesday, October 25

MAT 185A, Temple, Fall 2023

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts):

(a) Find the real and imaginary parts u and v, respectively, of $f(z) = z^2$.

$$z^2 = (x+iy)^2 = (x+iy)(x+iy) = \underbrace{x^2-y^2}_{\substack{u(x,y)\\ \text{real}}} + i\underbrace{(2xy)}_{\substack{v(x,y)\\ \text{imag}}}$$

(b) Prove that $\frac{d}{dz}z^2 = 2z$, independent of how $\Delta z \to 0$.

$$\frac{d}{dz}z^2 = \lim_{\Delta z \to 0}\left[\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{\cancel{z^2} + 2z\cancel{\Delta z} + \Delta z^{\cancel{2}} - \cancel{z^2}}{\cancel{\Delta z}} = 2z + \Delta z\right]$$

$$= 2z$$

(c) Evaluate $\int_C z^2 dz$ where C is a curve in the plane taking $A = -2$ to $B = 3i$. Simplify.

$$\int_C z^2 dz = \frac{1}{3}z^3\Big]_A^B = \frac{1}{3}(-2)^3 - \frac{1}{3}(3i)^3$$

$$= \frac{1}{3}(-8) - 9i^2 i$$

$$= -\frac{8}{3} + 9i$$

Problem #2 (20pts): Let $w = f(z)$ where $z = x+iy$, $w = u(x,y)+iv(x,y)$.

State the Cauchy-Riemann equations, and prove that if $f'(z) \cancel{= g'(z)}$ 0, (so f ~~and g~~ satisf~~y~~ies the Cauchy-Riemann equations), then $f(z) = \cancel{g(z) +}$ *const*.

Cauchy-Riemann: $u_x = v_y$, $u_y = -v_x$

Assume $f'(z) = 0$. Then $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

exists indept of how $\Delta z \to 0$. Taking $\Delta z = \Delta x$,

$$f'(z) = u_x + i v_x = 0. \text{ So } u_x = 0 = v_x.$$

But CR $\Rightarrow$ $u_y = -v_x = 0$ and $v_y = u_x = 0$, so

$\nabla u = 0 = \nabla v$. Thus $u = \text{const}$, $v = \text{const}$ $\Rightarrow$

$f(z) = u + iv = \text{const}$.

Problem #3 (20pts):. Assume $f(u) = u + iv$ satisfies the Cauchy-Riemann equations.

(a) Find vector fields $\mathbf{G}_1$ and $\mathbf{G}_2$ such that

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} \mathbf{G}_1 \cdot \mathbf{T}\,ds + i\int_{\mathcal{C}} \mathbf{G}_2 \cdot \mathbf{T}\,ds.$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \int_C \overrightarrow{(u,-v)} \cdot \vec{T}\,ds + i\int_C \overrightarrow{(v,u)} \cdot \vec{T}\,ds$$

$$\vec{G}_1 = \overrightarrow{(u,-v)} \qquad \vec{G}_2 = \overrightarrow{(v,u)}$$

(b) Use Cauchy-Riemann to prove $Curl\ \mathbf{G}_1 = 0 = Curl\ \mathbf{G_2}$.

$$|\mathrm{Curl}\,\vec{G}_1| = |N_x - M_y| = |(-v)_x - (u)_y| = |-v_x - u_y| = 0$$

$u_y = -v_x$

CR-1

$$|\mathrm{Curl}\,\vec{G}_2| = |N_x - M_y| = |u_x - v_y| = 0$$

$u_x = v_y$ CR-2

Problem #4 (20pts): Recall $Log(z) = \ln r + i\theta(z)$, $z \in \mathbb{C}\backslash\{\text{neg real axis}\}$, where $r = |z| = \sqrt{x^2+y^2}$, and $\theta(z)$ is the angle $\overrightarrow{(x,y)}$ makes with the x-axis.

(a) Find $\frac{\partial}{\partial x}\ln r$ and $\frac{\partial}{\partial x}\theta = \frac{\partial}{\partial x}\text{Arctan}(y/x)$, and show that they agree with the real and imaginary parts of $f(z) = 1/z$. Explain why.

$$\frac{\partial}{\partial x}\ln r = \frac{-1}{r} r_x = \frac{-1}{r}\frac{x}{r} = -\frac{x}{r^2}$$

$$\frac{\partial}{\partial x}\text{Arctan}(y/x) = \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = -\frac{y}{r^2}$$

$$\frac{1}{z} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \underbrace{\frac{x}{r^2}}_{\text{real}} - i\underbrace{\frac{y}{r^2}}_{\text{imag}}$$

True because $\frac{d}{dz}\log z = u_x + i v_x = \frac{1}{z}$

(b) Evaluate $\int_{\mathcal{C}} \frac{dz}{z-1}$ by direct parameterization, where $\mathcal{C}$ is the unit circle centered at $z=1$.

$$\int_C \frac{dz}{z-1} = \int_0^{2\pi} \frac{ir\cancel{e^{i\theta}}}{r\cancel{e^{i\theta}}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

$z(t) = 1 + re^{i\theta}$

$0 \le \theta \le 2\pi$

$dz = ire^{i\theta}$

Problem #5 (20pts): Recall that $e^{ia} = \cos a + i \sin a$.

(a) Prove that $e^{i(a+b)} = e^{ia}e^{ib}$, and use this to prove (by induction) that $e^{inz} = \left(e^{ia}\right)^n$ for $n = 1, 2, 3, \ldots$.

$$e^{ia}e^{ib} = (\cos a + i\sin a)(\cos b + i \sin b)$$

$$= \underbrace{\cos a \cos b - \sin a \sin b}_{\cos(a+b)} + i(\underbrace{\sin a \sin b + \sin b \sin a}_{\sin(a+b)}) = e^{i(a+b)}$$

$n=1$ $\quad e^{i2a} = (e^{ia})^2$

assume $n=1$ $\quad e^{ina} = e^{i(a+(n-1)a)} \underset{\substack{\uparrow \\ n=2}}{=} e^{ia}e^{i(n-1)a} \underset{\substack{\uparrow \\ \text{case} \\ n-1}}{=} e^{ia}(e^{ia})^{n-1} = e^{ina}$ ✓

(b) Show that if $f'(z) = z$ and its inverse $f^{-1}(z)$ exists and is analytic, then $\left(f^{-1}\right)'(z) = 1/z$.

$f^{-1}(f(z)) = z \quad$ diff both sides

$$\frac{d}{dz} f^{-1}(f(z)) \underset{\substack{\uparrow \\ w=f(z) \\ \text{chain rule}}}{=} 1 \Rightarrow (f^{-1})'(w)\frac{dw}{dz} = 1$$

$$= (f^{-1})'(w) = \frac{1}{dw/dz} \underset{\substack{\uparrow \\ \frac{dw}{dz} = f'(z) = f(z) = w}}{=} \frac{1}{w} \Leftrightarrow (f^{-1})'(z) = \frac{1}{z}$$ ✓