Name: SOLUTIONS

Student ID# and Section: ______________

Midterm Exam 1

Wednesday, February 3

MAT 185A, Temple, Winter 2021

Print names and ID's clearly. Write the solutions clearly and legibly. Do not write near the edge of the paper. Show your work on every problem. Be organized and use notation appropriately.

You are NOT allowed to consult the internet, Piazza, your classmates, friends or family members, tutors, or any other outside sources, etc. during the exam. You may NOT provide or receive any assistance from another student taking this exam. You may NOT use any electronic devices to look up hints or solutions for this exam. Show all work. Correct answers with insufficient or incorrect justification will receive little or no credit. By signing above you acknowledge that you have read and will abide by these conditions. Your exam will not be graded without your signature.

Signature: ______________

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Let a, b be real numbers. For $z = a + ib$, recall $\bar{z} = a - ib$.

(a) Let $z = a + ib$, $w = c + id$. Prove: $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.

$$\overline{z+w} = \overline{a+ib+c+id} = \overline{(a+c)+i(b+d)} = (a+c)-i(b+d)$$

$$\bar{z}+\bar{w} = a-ib+c-id = (a+c)-i(b+d) \checkmark$$

$$\overline{zw} = \overline{(a+ib)(c+id)} = \overline{ac-bd+i(bc+ad)}$$

$$= (ac-bd)-i(bc+ad)$$

$$\bar{z}\bar{w} = (a-ib)(c-id) = (ac-bd)-i(bc+ad) \checkmark$$

(b) Let $z = x + iy$ denote an arbitrary complex number. Find a formula, in terms of x and y, for the multiplicative inverse z^{-1} of z, and demonstrate directly that $zz^{-1} = 1$.

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy}\frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

$$zz^{-1} = (x+iy)\left(\frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}\right)$$

$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + i\left(\frac{yx}{x^2+y^2} - \frac{xy}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2}{x^2+y^2} = 1 \quad \checkmark$$

Problem #2 (20pts): Let $f(z) = z^3$.

(a) Derive a formula for $f'(z)$ directly from the limit definition of derivative.

$$\frac{d}{dz} z^3 = \lim_{\Delta z \to 0} \left[\frac{(z+\Delta z)^3 - z^3}{\Delta z} = \frac{\cancel{z^3} + 3z^2 \Delta z + 3z \Delta z^2 + \Delta z^3 - \cancel{z^3}}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[\frac{3z^2 \cancel{\Delta z} + 3z \Delta z^{\cancel{2}} + \Delta z^{\cancel{3}\,2}}{\cancel{\Delta z}} = \frac{3z^2 + 3z\Delta z + \Delta z^2}{1} \right]$$

$$= 3z^2$$

(b) Find the real and imaginary parts of $f(z) = z^3$.

$$(x+iy)^3 = [(x+iy)(x+iy)](x+iy)$$

$$= [x^2 - y^2 + i\underbrace{(xy+yx)}_{2xy}](x+iy)$$

$$= \underbrace{x^3 - xy^2 - (2xy^2)}_{u(x,y)} + i\underbrace{(2x^2y + x^2y - y^3)}_{v(x,y)}$$

(c) Verify that the function $f(z) = z^3$ satisfies the Cauchy-Riemann equations.

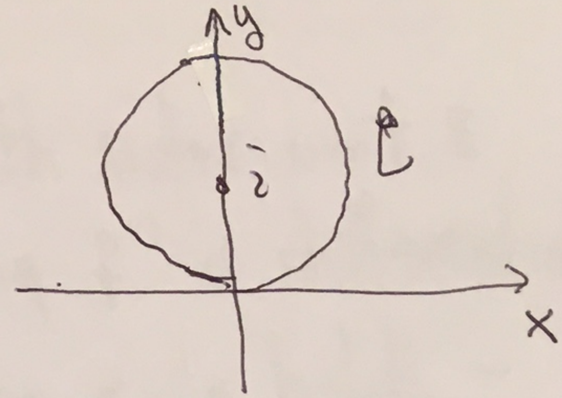
$$u_x = 3x^2 - y^2 - 2y^2 = 3(x^2 - y^2)$$

$$v_y = 2x^2 + x^2 - 3y^2 = 3(x^2 - y^2) = u_x \checkmark$$

$$u_y = -2xy - 4xy = -6xy$$

$$v_x = 4xy + 2xy = 6xy = -u_y \checkmark$$

Problem #3 (20pts): Let C denote the simple closed curve given by the unit circle centered at $z = i$, oriented counterclockwise around $z = i$. Find an explicit parameterization of this curve, and use it to evaluate $\int_C \frac{dz}{z-i}$ by the method of explicit parameterization.



Parameterization:

$$z(t) = i + (\cos t + i \sin t)$$

$$dz = -\sin t + i \cos t \qquad 0 \le t \le 2\pi$$

$$\int_C \frac{dz}{z-i} = \int_0^{2\pi} \frac{-\sin t + i \cos t}{\cos t + i \sin t}\, dt$$

$$= \int_0^{2\pi} \frac{(-\sin t + i\cos t)(\cos t - i \sin t)}{\cos^2 t + \sin^2 t}\, dt$$

$$= \cancel{\int_0^{2\pi} \frac{-\sin t \cos t + \cos t \sin t}{1}\, dt} + i \int_0^{2\pi} \frac{\cos^2 t + \sin^2 t}{1}\, dt$$

$$0$$

$$= i \int_0^{2\pi} dt = i\, t \Big]_0^{2\pi} = \boxed{2\pi i}$$

Problem #4 (20pts): Recall that the exponential function $f(z) = e^z = e^x e^{iy}$ is invertible for $-\pi < y < \pi$, so $f^{-1}(f(z)) = z$. Assuming the inverse is differentiable, prove that the inverse f^{-1} of the exponential *must* be the missing anti-derivative of the function $h(z) = 1/z$.

$f^{-1}(f(z)) = z$ diff both sides wrt z assuming f^{-1} is differentiable & chain rule holds –

$$\frac{d}{dz} f^{-1}(f(z)) = 1$$

$$f^{-1\prime}(f(z)) \cdot f'(z) = 1$$

$$f^{-1\prime}(f(z)) = \frac{1}{f'(z)}$$

Let $w = e^z = f(z)$

$f'(z) = e^z = w$

$$\boxed{f^{-1\prime}(w) = \frac{1}{w}}$$

f^{-1} is an anti-deriv of $g(w) = \frac{1}{w}$ ✓

Problem #5 (20pts): Assume $F(z) = U + iV$ and $f(u) = u + iv$ ~~are entire functions~~, and assume F satisfies the Cauchy Riemann equations with $F'(z) = f(z)$, on a non-simply connected domain.

(a) Use $dz = dx + idy$ to derive formulas for two vector fields $\vec{\mathbf{G}}_1$ and $\vec{\mathbf{G}}_2$ such that

$$\int_C f(z)dz = \int_C \vec{\mathbf{G}}_1 \cdot \mathbf{T}ds + i\int_C \vec{\mathbf{G}}_2 \cdot \mathbf{T}ds.$$

(Make sure to use Leibniz notation correctly.)

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \int_C \vec{G}_1 \cdot \vec{T}\,ds + i\int_C \vec{G}_2 \cdot \vec{T}\,ds$$

where $\vec{G}_1 = \overrightarrow{(u,-v)}$, $\vec{G}_2 = \overrightarrow{(v,u)}$

(b) Show that f satisfies the Cauchy-Riemann equations.

Since $F'(z) = f(z)$, & $F'(z) = U_x + iV_x = u + iv$,

$(\Delta z = \Delta z)$ ✓

$u_x = U_{xx}$, $v_y = V_{xy} = (V_y)_x = U_{xx}$,

$u_y = U_{xy} = (V_y)_y = V_{yy}$,

$\boxed{U_x = V_y \quad U_y = -V_x \quad \text{C-R for } F}$

$V_x = V_{xx} = -U_{yx} = -u_y$ ✓

(c) Show that $\vec{\mathbf{G}}_1$ and $\vec{\mathbf{G}}_2$ are conservative.

$$\vec{G}_1 = \overrightarrow{(u,-v)} = \overrightarrow{(U_x, -V_x)} \underset{\substack{\uparrow \\ \text{C-R}}}{=} \overrightarrow{(U_x, U_y)} = \nabla U$$

$$\vec{G}_2 = \overrightarrow{(V,u)} = \overrightarrow{(V_x, U_x)} \underset{\text{C-R}}{=} \overrightarrow{(V_x, V_y)} = \nabla V$$

Note: D not simply connected $\Rightarrow$ Curl free does not guarantee conservative?

(d) State the FTC in terms of F & f on D.

$$\int_{\mathcal{C}} f(z)\,dz = F(B) - F(A)$$

I.e., FTC holds because F exists, even on a non-simply connected domain.