Name: Solutions

Student ID#: ________

Section: ________

Midterm Exam 2

Monday, November 20

MAT 185A, Temple, Fall 2023

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts):(a) Use $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$ to define $\cos z$ and $\sin z$ for $z \in \mathbb{C}$.

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Define: $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\qquad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

(b) Find the real and imaginary parts of $s \cos z = u + iv$ and verify the CR equations $u_x = v_y$, $u_y = -v_x$.

$$\cos z = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y} e^{ix} + e^{y} e^{-ix}}{2}$$

$$= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos(-x) + i \sin(-x))}{2}$$

$$= \underbrace{\frac{e^{y} + e^{-y}}{2} \cos x}_{u} - i \underbrace{\frac{e^{y} - e^{-y}}{2} \sin x}_{v}.$$

$u = \cosh y \cos x$
$-i \sinh y \sin$

$u_x = -\cosh y \sin x = V_y$, 2-3 $u_y = \sinh y \cos x = -V_y$

Problem #2 (20pts):(a) Find all w such that $w^5 = 1 + i$.

$w = re^{i\theta}$ $\quad w^5 = r^5 e^{i\theta 5} = (1+i) = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2\pi n\right)}$

$r = (\sqrt{2})^{1/5} = 2^{1/10}$ $\quad 5\theta = \frac{\pi}{4} + 2\pi n$ $\quad n = 0,1,2,3,4$

$\theta = \frac{\pi}{20} + \frac{2n\pi}{5} = \underset{n=0}{\frac{\pi}{20}},\ \underset{n=1}{\frac{\pi}{20} + \frac{2\pi}{5} = \frac{9\pi}{20}},\ \underset{n=2}{\frac{\pi}{20} + \frac{4\pi}{5} = \frac{17\pi}{20}},\ \underset{n=3}{\frac{25\pi}{20}},$

$\underset{n=4}{\frac{33\pi}{20}}$ ✓

(b) Find all w such that $w^\pi = 1 + i$.

$e^{\pi \log w} = \sqrt{2}\, e^{i\pi/4}$

$e^{\pi(\log r + i(\theta + 2\pi n))} = \sqrt{2}\, e^{i\pi/4}$

$e^{\pi} e^{i\pi(\theta + 2\pi n)} = \sqrt{2}\, e^{i\pi/4}$

$\Rightarrow \pi(\theta + 2\pi n) = \frac{\pi}{4}$

$\boxed{\theta = \frac{1}{4} + 2\pi n}$ $\quad n \in \mathbb{Z}$

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Problem #3 (20pts): Assume $f(z) = u + iv$ is entire, by which we mean analytic everywhere.

(a) Prove that $\Delta v = v_{xx} + v_{yy} = 0$.

C-R $u_x = v_y$, $u_y = -v_x$

$u_{xx} = v_{yx}$ $\quad u_{yy} = -v_{xy}$

$u_{xx} + u_{yy} = 0$ ✓

(b) Recall the Cauchy Integral Formula on the circle C_R of radius R

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w-z}dw. \tag{1}$$

Using the notation $\mathbf{x} = (x, y)$, and assuming $f(z) = u + iv$ is entire (analytic everywhere), prove that $v(\mathbf{x}_0)$ is given by its average on C_R for every $R > 0$, that is

$$v(\mathbf{x}_0) = \frac{1}{2\pi}\int_0^{2\pi} v(\mathbf{x}_0 + Re^{it})dt, \quad \mathbf{x}_0 = (x_0, y_0). \tag{2}$$

(Note slight abuse of notation $\mathbf{x}(t) \equiv Re^{it}$).

$$f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w-z_0}dw = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(w(t))}{\cancel{Re^{it}}} \cancel{iRe^{it}}\, dt$$

$w(t) = z_0 + Re^{it}$ $\quad 0 \le t \le 2\pi$

$dw = iRe^{it}$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(w(t))dt = \frac{1}{2\pi}\int_0^{2\pi} u(w(t))dt + \frac{i}{2\pi}\int_0^{2\pi} v(w(t))dt$$

$$\therefore v(\underset{\sim}{x}_0) = \frac{1}{2\pi}\int_0^{2\pi} v(\underset{\sim}{x}_0 + Re^{it})\, dt$$ ✓

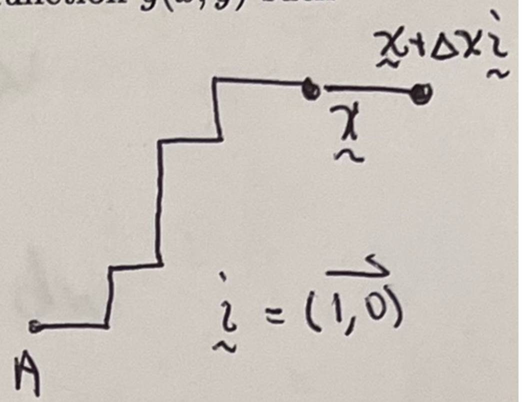
(c) Write a Riemann sum for the integral in (2) of part (b), and show that $v(\mathbf{x}_0)$ is the limit of the average of v evaluated at N points evenly spaced around the circle of radius R.

Letting $\Delta t = \frac{2\pi}{N}$, $\underset{\sim}{x}_n = \underset{\sim}{x}_0 + Re^{it_n}$

$t_n = n\Delta t = \frac{2\pi n}{N}$ $n = 0, \dots, N-1$

$$\frac{1}{2\pi}\int_0^{2\pi} v(\underset{\sim}{x}_0 + Re^{it})\,dt = \lim_{\Delta t \to 0} \underbrace{\frac{1}{N\cancel{\Delta t}} \sum_{k=1}^{N} v(\underset{\sim}{x}_n)\cancel{\Delta t}}_{\text{Average}}$$

Problem #4 (20pts): Assume $\vec{G} = \overrightarrow{(M,N)}$ is a smooth vector field in the plane which satisfies the property that its line integral around every rectangle (with sides parallel to x, y-axes) is zero. Prove there exists a function $g(x,y)$ such that $\nabla g = \vec{G}$.



$\underline{\text{Define}}$ $g(\underset{\sim}{x}) = \displaystyle\int_A^{\underset{\sim}{x}} \vec{G}\cdot\vec{T}\, ds$

$\underset{\sim}{i} = \overrightarrow{(1,0)}$

where the integral is taken along any cont curve from A to $\underset{\sim}{x}$ formed by sides of rectangles.

$$\left.\frac{\partial g}{\partial x}\right|_{\underset{\sim}{x}} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left\{ \int_A^{\underset{\sim}{x}+\Delta x \underset{\sim}{i}} \vec{G}\cdot\vec{T}\, ds - \int_A^{\underset{\sim}{x}} \vec{G}\cdot\vec{T}\, ds \right\}$$

Now take same curve from $A \to \underset{\sim}{x}$ and horizontal line from $\underset{\sim}{x} \to \underset{\sim}{x} + \Delta x \underset{\sim}{i}$ $\Rightarrow$

$$\left.\frac{\partial g}{\partial x}\right|_{\underset{\sim}{x}} = \lim_{\Delta x \to 0} \left\{ \int_{\underset{\sim}{x}}^{\underset{\sim}{x}+\Delta x \underset{\sim}{i}} \vec{G}\cdot\vec{T}\, ds = \int_0^{\Delta x} \vec{G}\cdot\vec{v}\, dt = \int_0^{\Delta t} M\, dt \right\}$$

$$\vec{r}(t) = \underset{\sim}{x} + t\underset{\sim}{i}$$
$$\vec{r}\,'(t) = \vec{v} = \underset{\sim}{i}$$

Since M cont, $\left.\dfrac{\partial g}{\partial x}\right|_{\underset{\sim}{x}} = \displaystyle\lim_{\Delta x \to 0} \frac{1}{\Delta t}\int_0^{\Delta t} M(\underset{\sim}{x}) + o(1)\, dt = M(\underset{\sim}{x})$

Similarly, $\dfrac{\partial g}{\partial y} = N \Rightarrow \nabla g = \vec{F}$ ✓

Problem #5 (20pts):(a). Derive Cauchy's estimate for the third derivatives of an analytic function directly from the Cauchy Integral Formula given in equation (1) in problem (**3b**) above.

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w-z}\, dw$$

$$f'(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{(w-z)^2}\, dw$$

$$\vdots$$

$$f^{(3)}(z) = \frac{3!}{2\pi i} \int_{C_R} \frac{f(w)}{(w-z)^4}\, dw$$

$$|f^{(3)}(z)| \leq \frac{3\cdot 2}{2\pi} \underbrace{|C_R|}_{2\pi R} \frac{M}{R^4} \qquad M = \max_{w \in B_R(z_0)} |f|$$

$$= \frac{3M}{R^3}$$

(b) Assume f is an entire function which satisfies the condition

$$|f(z)| \leq A + B|z|^2$$

for some constants A and B. Prove that $f(z) = a + bz + cz^2$ for some constants a, b, c. (Hint: It suffices to prove that $f'''(z) = const.$, because this alone implies f is a quadratic function of z.)

$|f(z)| \leq A + B|z|^2 \Rightarrow |f(z)| \leq A + BR^2$ on $\overline{B_R(z_0)}$. Thus by part (a)

$$|f^{(3)}(z)| \leq \frac{3M}{R^3} \leq \frac{3(A+BR^2)}{R^3}$$

$M \leq A + BR^2$

$$\leq \frac{3A}{R^3} + \frac{3B}{R} \xrightarrow[R\to\infty]{} 0$$

$\therefore f^{(3)}(z) = 0 \quad \forall z \in \mathbb{C}$

$\therefore f$ is quadratic ✓