Solutions

Name: ____________________

Student ID# and Section: ____________________

Midterm Exam 2

Monday, March 1

MAT 185A, Temple, Winter 2021

Print names and ID's clearly. Write the solutions clearly and legibly. Do not write near the edge of the paper. Show your work on every problem. Be organized and use notation appropriately.

You are NOT allowed to consult the internet, Piazza, your classmates, friends or family members, tutors, or any other outside sources, etc. during the exam. You may NOT provide or receive any assistance from another student taking this exam. You may NOT use any electronic devices to look up hints or solutions for this exam. Show all work. Correct answers with insufficient or incorrect justification will receive little or no credit. By signing above you acknowledge that you have read and will abide by these conditions. Your exam will not be graded without your signature.

Signature: ____________________

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): **(a)** Find the real and imaginary parts of e^z and show that they satisfy the Cauchy-Riemann equations.

$$e^z = e^x e^{iy} = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$$

$$u_x = e^x \cos y = v_y$$

$$u_y = -e^x \sin x = -v_x \quad ✓$$

(b) Prove that $|e^z| \leq 3^{|z|}$.

$$|e^z| = |e^x e^{iy}| = |e^x| \leq |e^{\sqrt{x^2+y^2}}|$$

$$\leq |3^{\sqrt{x^2+y^2}}| = 3^{|z|} \quad ✓$$

Problem #2 (20pts): Find the $7'th$ roots of $z = 1 + i$. That is, find all w such that $w^7 = 1+i$.

$$1+i = \sqrt{2}\, e^{i\pi/4}$$

$$w^7 = \sqrt{2}\, e^{i\pi/4} \Rightarrow w = (\sqrt{2})^{1/7} e^{\frac{i(\pi/4 + 2\pi k)}{7}}$$

$$k = 0, 1, 2, 3, 4, 5, 6$$

Problem #3(20pts): Assume $f : \mathbb{C} \to \mathbb{C}$ is continuous, and $\int_C f(z)dz = 0$ for every smooth simple closed curve C. Prove directly that f has an anti-derivative.

Define $F(z) = \int_{z_0}^{z} f(w)\,dw$ which is indept of curve from $0 \to z$ because $\oint_C f(z)dz = 0$

or any base point

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(w)\,dw$$ wlog take straight line from z to $z+\Delta z$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z) + o(\Delta z)\,dz$$ f cont

$$= \underbrace{\frac{1}{\Delta z} f(z) \int_{z}^{z+\Delta z} dz}_{f(z)} + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} o(\Delta z)\,dz$$

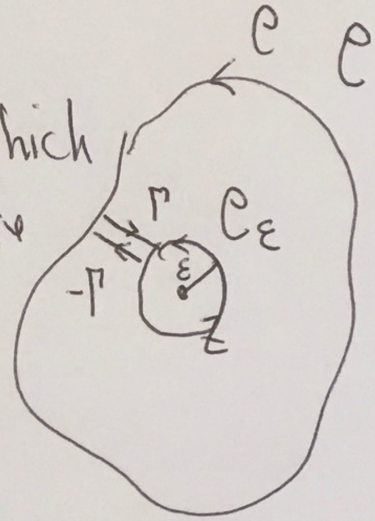
$$\left| \frac{1}{\Delta z}\int_{z}^{z+\Delta z} o(\Delta z)\,dz \right| \leq \frac{1}{|\Delta z|} \cdot |\Delta z| \cdot o(\Delta z) \xrightarrow[\Delta z \to 0]{} 0$$

$\therefore F'(z) = f(z)$ ✓

Problem #4(20pts): Assume the Cauchy-Goursat Theorem, that for every analytic function in a simply connected domain $\mathcal{D}$, we have $\int_C f(z)dz = 0$ for every simple closed curve C in $\mathcal{D}$. Prove the Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$$

for every simple closed curve $\mathcal{C}$ which contains z. Draw the appropriate picture.



$\mathcal{C} + \Gamma - \mathcal{C}_\varepsilon - \Gamma$ is a simple closed curve inside of which $\frac{f(w)}{w-z}$ is analytic, where $\mathcal{C}_\varepsilon$ is the circle center z, radius ε. Then

$$0 = \int_{\mathcal{C}+\Gamma-\mathcal{C}_\varepsilon-\Gamma} \frac{f(w)}{w-z}dw = \int_{\mathcal{C}} + \cancel{\int_\Gamma} - \int_{\mathcal{C}_\varepsilon} - \cancel{\int_\Gamma} \Rightarrow$$

$$\int_{\mathcal{C}} \frac{f(w)}{w-z}dw = \int_{\mathcal{C}_\varepsilon} \frac{f(w)}{w-z}dw = \int_{\mathcal{C}_\varepsilon} \frac{f(z)+O(\varepsilon)}{w-z}dw$$

$$= f(z)\int_{\mathcal{C}_\varepsilon} \frac{dw}{w-z} + \int_{\mathcal{C}_\varepsilon} \frac{O(\varepsilon)}{w-z}dw = 2\pi i f(z)$$

Problem #5 (20pts): (a) Assume f is an entire function satisfying $|f(z)| < |z|$. Prove f is a linear function, i.e., $f(z) = az + b$ for some complex numbers a and b.

Solution: By Cauchy's Theorem, $f''(z) = \frac{2!}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z)^3}dw$ where C_R is the circle of radius R center z. Thus we can estimate

$$|f''(z)| = \left|\frac{2!}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z)^3}dw\right| \le \frac{1}{\pi}2\pi R\frac{M}{R^3} = \frac{2}{R^2}(|z| + R)$$

where $|f(w)| \le |w| \le (|z| + R) = M$ on C_R. Therefore

$$|f''(z)| \le \frac{2}{\pi R^2}(|z| + R) \to 0$$

as $R \to \infty$ so $f''(z) = 0$ for every z. Therefore $f'(z) = constant$, and so f is linear.

(b) Assuming part (a), prove that if f is non-constant, entire, and satisfies $|f(z)| < |z|$, then there must exist a z_0 at which $f(z_0) = 0$.

By part (a) $f(z) = az+b$. Moreover, f non-constant $\Rightarrow a \neq 0 \Rightarrow f(z_0) = 0$ when $0 = az_0 + b \Rightarrow z_0 = \frac{-b}{a}$ ✓