Name: ______________________________

Student ID# and Section: ______________________

Final Exam

Tuesday, March 16, 8-10am

MAT 21B, Temple, Winter 2021

Print names and ID's clearly. Write the solutions clearly and legibly. Do not write near the edge of the paper. Show your work on every problem. Be organized and use notation appropriately.

You are NOT allowed to consult the internet, Piazza, your classmates, friends or family members, tutors, or any other outside sources, etc. during the exam. You may NOT provide or receive any assistance from another student taking this exam. You may NOT use any electronic devices to look up hints or solutions for this exam. Show all work. Correct answers with insufficient or incorrect justification will receive little or no credit. By signing above you acknowledge that you have read and will abide by these conditions. Your exam will not be graded without your signature.

Signature: ______________________

Final Exam

Tuesday, March 16, 8-10am

MAT 21B, Temple, Winter 2021

Name: ____________________

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Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
Total		200

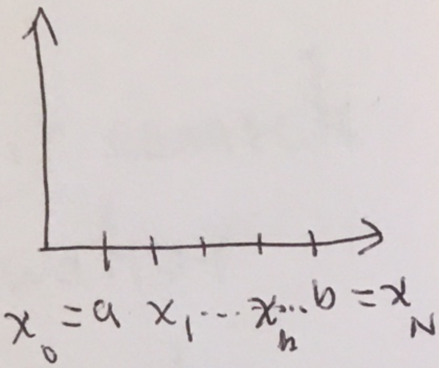
Problem #1 (20pts): Assume $f(x)$ is a continuous function for $a \leq x \leq b$, and assume $F'(x) = f(x)$.

(a) Define the Riemann sum for f over $[a, b]$ determined by $\Delta x_1, \ldots, \Delta x_N$, assuming

$$\Delta x_1 + \cdots + \Delta x_N = b - a.$$

(Draw the correct picture, and define and label grid points correctly.)

$x_k = a + \Delta x_1 + \cdots + \Delta x_k$ $\qquad$ $x_k - x_{k-1} = \Delta x_k$



$$RS = \sum_{k=1}^{N} f(x_k)\Delta x_k$$

(b) Use **(a)** to define the Riemann Integral $\int_a^b f(x)dx$.

$$\int_a^b f(x)dx = \lim_{N \to \infty} \sum_{k=1}^{N} f(x)dx$$

(c) State the Fundamental Theorem of Calculus, and explain in your own words why it is so important.

Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The FTC is important because it connects the area problem to the derivative problem, by telling us area can be computed by finding an anti-derivative.

(Or some words saying FTC connects the area problem to the derivative problem in an intelligent way)

Problem #2 (20pts): Evaluate the following anti-derivatives:

(a) $\int \frac{x^2}{(2x^3-1)^{1/3}}dx$ $= \frac{1}{6}\int \frac{du}{u^{1/3}} = \frac{1}{6}u^{2/3} + C$

$u = 2x^3 - 1$

$du = 6x^2$

$= \frac{1}{6}(2x^3-1)^{2/3} + C$

(b) $\int \cos^2 x \sin^3 x dx$ $= \int \cos^2 x(1-\cos^2 x)\sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$= -\int u^2(1-u^2)du = -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

(c) $\int \cos^2 x dx$ $= \frac{1}{2}\int 1+\cos 2x\, dx = \frac{1}{2}x + \frac{1}{4}\sin x + C$

$\cos^2 x - \sin^2 x = \cos 2x$

$\cos^2 x + \sin^2 x = 1$

$2\cos^2 x = 1 + \cos 2x$

Problem #3 (20pts): Evaluate the following indefinite integrals:

(b) $\int \frac{x}{4-x^2}dx =$

$$\int \frac{2\cos\theta}{4-4\cos^2\theta}(-2\sin\theta)\,d\theta = -\int \frac{\cos\theta}{\sin^2\theta}\sin\theta\,d\theta$$

$x = 2\cos\theta$

$dx = -2\sin\theta\,d\theta$

$$= -\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|u| + C$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \ln|\sin\theta| + C$$

$$= \ln\left|\sqrt{1-\cos^2\theta}\right| + C = \ln\left|\sqrt{1-\frac{x^2}{4}}\right| + C$$

(a) $\int_0^\pi x\sin x\,dx =$ $\quad -x\cos x + \int \cos x\,dx$

$u = x \quad du = dx$

$dv = \sin x\,dx \quad v = -\cos x$

$$= -x\cos x + \sin x + C$$

Problem #4 (20pts): Evaluate the following definite integrals:

(a) $\int_{\pi/6}^{\pi/4} \frac{\cos x}{\sin x} dx = \int_{x=\pi/6}^{x=\pi/4} \frac{du}{u} = \ln|u| \Big]_{x=\pi/6}^{x=\pi/4}$

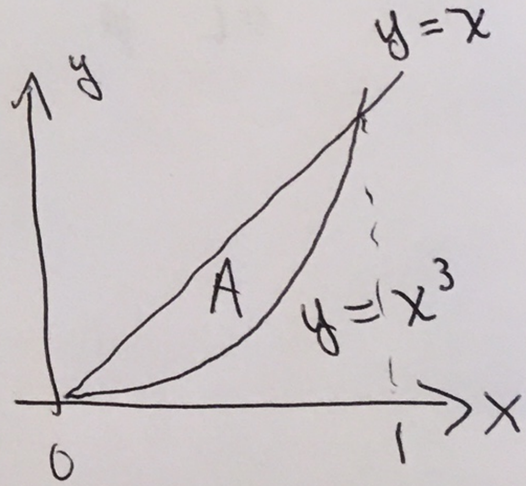
$u = \sin x$

$du = \cos x \, dx$

$= \ln|\sin x| \Big]_{\pi/6}^{\pi/4}$

$= \ln|\sin \pi/4| - \ln|\sin \pi/6| = \ln \frac{\sqrt{2}}{2} - \ln \frac{1}{2} = \ln(\sqrt{2})$

(b) Find the area between the graphs of $f(x) = x^3$ and $f(x) = x$ between $x = 0$ and $x = 1$.



$\text{Area} = A = \int_0^1 x - x^3 \, dx$

$= \frac{x^2}{2} - \frac{x^4}{4} \Big]_0^1$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Problem #5 (20pts): A rod placed on the y-axis between $y = 0$ and $y = 3$ has a density of $\delta(y) = y^2 \frac{kg}{m}$.

(a) Find the total mass of the rod. (Include units.)

$$M = \int_0^3 \delta(y)\,dy = \int_0^3 y^2\,dy = \frac{y^3}{3}\Big]_0^3 = \frac{3^3}{3} = 3^2$$

$$= 9\,kg$$

(b) Find the distance from the x-axis to the center of mass of the rod.

$$M\bar{y} = \int_0^3 y\,\delta(y)\,dy = \int_0^3 y^3\,dy = \frac{y^4}{4}\Big]_0^3$$

$$= \frac{3^4}{4}$$

$$\bar{y} = \frac{1}{9} \cdot \frac{3^4}{4} = \frac{3^2}{4} = \frac{9}{4}.$$

The distance from $y=0$ to $\bar{y}$ is $\frac{9}{4}$ meters

Problem #6 (20pts): Assume that we started with the definition of $f(x) = e^x$ as the unique function satisfying $\frac{d}{dx}f(x) = f(x)$, and assume we showed that $f(x)$ increases from zero to infinity as x increases from zero to infinity. Then we could define $\ln x$ as the inverse $\ln x = f^{-1}(x)$, so $\ln f(x) = x$ for $x \in \mathcal{R}$, and $f(\ln x) = x$ for $x > 0$. Assuming only this

(a) Show that $\frac{d}{dx}\ln x = \frac{1}{x}$ Let $x = f(y)$. Then

$$\ln(f(y)) = y$$

So $\frac{d}{dy}\ln(f(y)) = \frac{d}{dy}y = 1 \underset{\substack{\uparrow \\ \text{chn} \\ \text{rule}}}{\Rightarrow} \left(\frac{d}{dx}\ln x\right)\frac{d}{dy}f(y) = 1$

Thus $\frac{d}{dx}\ln x = \frac{1}{f'(y)} = \frac{1}{f(y)} = \frac{1}{x}$ ✓

(b) What further information do we need to know about $f(x)$ to conclude that $\ln x = \int_1^x \frac{dt}{t}$? Explain.

By FTC, $\frac{d}{dx}\ln x = \frac{1}{x} \Rightarrow \ln x = \int_1^x \frac{dx}{x} + C$

To conclude $C = 0$ we need $\ln(1) = 0$

which requires $\boxed{1 = f(0)}$

Problem #7 (20pts):

(a) Use the method of partial fractions to evaluate the anti-derivative:

$\int \frac{\cancel{2x^2}+x+1}{(x^2-1)(x-3)} = \int \frac{\cancel{2x^2}+x+1}{(x-1)(x+1)(x-3)} = \int \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-3}\,dx$

Mult by $(x-1)$ & set $x=1 \Rightarrow \frac{1+1}{(1+1)(1-3)} = \frac{2}{-4} = -\frac{1}{2} = A$

Mult by $x+1$ & set $x=-1 \Rightarrow 0 = B$

Mult by $x-3$ & set $x=3 \Rightarrow \frac{3+1}{(3-1)(3+1)} = \frac{\cancel{4}}{2\cdot\cancel{4}} = \frac{1}{2} = C$

$$\int \frac{x+1}{(x^2-1)(x-3)}dx = A\ln|x-1| + B\cancel{\ln}|x+1| + C\ln|x-3| + C$$
$$= -\frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x-3| + C$$

(b) Write the partial fractions expansion for the following integral. (Do not solve for the constants.)

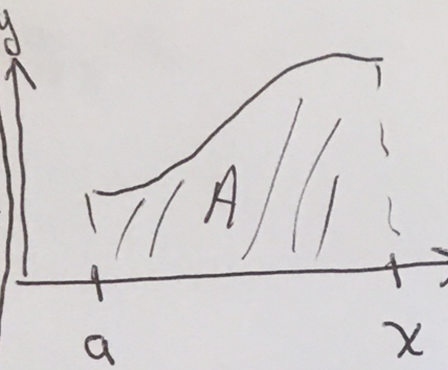
$\int \frac{x^2-1}{(x-1)(x+2)^3(x^2+1)^2}dx =$

$$\int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} + \frac{E}{x^2+1} + \frac{F}{(x^2+2)^2}\,dx$$

Problem #8 (20pts): (a) Recall that $F(x) = \int_a^x f(t)dt$ denotes the area under the graph of f between a and x. Assuming f is a continuous function, draw this area in a diagram, and prove that $F'(x) = f(x)$.

$$A = F(x) = \int_a^x f(t)\,dt$$

$$F'(x) = \lim_{\Delta x \to 0} \left[\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)\,dt \right]$$



$$= \lim_{\Delta x \to 0} \left[\frac{1}{\cancel{\Delta x}} \left(f(x) + o(1) \right) \cancel{\Delta x} \right] = f(x), \quad o(1) \to 0 \text{ as } x \to 0.$$

(b) Find $F'(x)$ if $F(x) = \int_1^{\sin x} \cos t\, dt$.

$$F'(x) = \frac{d}{dx} \int_1^{(\sin x)} \cos t\, dt = \frac{d}{dx} F(u(x))$$

$$u(x) = \sin x$$

$$= F'(u) \cdot \frac{du}{dx} = f(u) \cdot \cos x$$

$$= \cos(\sin x) \cdot \cos x.$$

Problem #9 (20pts): Let $a > 0$, and let $f(x)$ denote the parabola defined for $-a \leq 0 \leq a$, such that $f(-a) = y_1$, $f(0) = y_2$ and $f(a) = y_3$.

(a) Determine the area associated with the Riemann Sum when $\Delta x = a$.

$$A = f(0)\Delta x + f(a)\Delta x = y_2 \cdot a + y_3 \cdot a$$

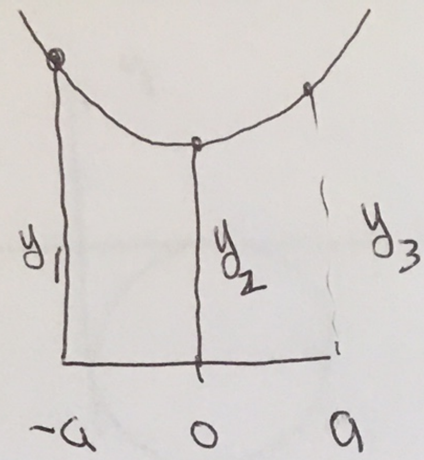
(b) Determine the area associated with the Trapezoidal Rule when $\Delta x = a$.

$$A = \frac{1}{2}\left(f(-a) + f(0)\right)\Delta x + \frac{1}{2}\left(f(0) + f(a)\right)\Delta x$$

$$= \frac{1}{2}(y_1 + y_2) \cdot a + \frac{1}{2}(y_2 + y_3)a$$

$$= \left(\frac{1}{2}y_1 + y_2 + \frac{1}{2}y_3\right)a$$

(c) *Derive* a formula for $\int_{-a}^{a} f(x)dx$. (Hint: Think the derivation of Cramer's Rule.)



$$y = Ax^2 + Bx + C$$

$$\text{Area} = \int_{-a}^{a} Ax^2 + Bx + C \, dx$$

$$= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big]_{-a}^{a}$$

$$= \frac{2Aa^3}{3} + 2aC = \frac{2a}{3}\left(Aa^2 + 3C\right)$$

$$\left.\begin{aligned} y_1 &= Aa^2 + B(-a) + C \\ y_2 &= C \\ y_3 &= Aa^2 + Ba + C \end{aligned}\right\} \qquad \begin{aligned} 2Ba &= y_3 - y_1 \\ 2Aa^2 + 2C &= y_1 + y_3 \\ Aa^2 &= \frac{y_1 + y_3}{2} - y_2 \end{aligned}$$

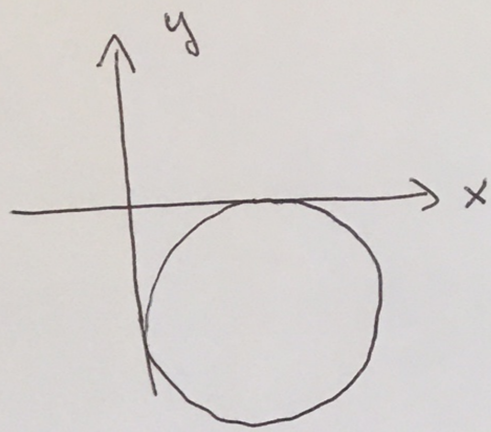
$$\therefore \text{Area} = \frac{2a}{3}\left(\frac{y_1 + y_3}{2} - y_2 + 3y_2\right)$$

$$= \frac{a}{3}\left(y_1 + y_3 + 4y_2\right)$$

Problem #10 (20pts): Consider the curve defined parametrically by the graph of $x = \cos 2t + 1$, $y = \sin 2t - 1$, $0 \le t \le \pi$.

(a) Find the arclength of the curve.

$$ds = \sqrt{f'(t)^2 + g'(t)^2}\,dt \qquad f(t) = \cos 2t + 1$$
$$g(t) = \sin 2t - 1$$



$$= \sqrt{2^2\cos^2 2t + 2^2\sin^2 2t}\,dt$$

$$= 2dt$$

$$\text{Length} = \int_0^{\pi} 2\,dt = \boxed{2\pi}$$

(b) Find the surface area of the region obtained by rotating the curve about the y-axis.

$$\text{Area} = \int_0^{\pi} x\,ds = \int_0^{\pi} (\cos 2t + 1)\,2\,dt$$

$$= \left(\frac{1}{2}\sin 2t + t\right)2 \Big]_0^{\pi}$$

$$= \frac{1}{2}\sin 2\pi + \pi = \boxed{\pi}$$