Name: Solutions

Student ID# and Section: ____________

Midterm Exam 2

Monday, March 1

MAT 21B, Temple, Winter 2021

Print names and ID's clearly. Write the solutions clearly and legibly. Do not write near the edge of the paper. Show your work on every problem. Be organized and use notation appropriately.

You are NOT allowed to consult the internet, Piazza, your classmates, friends or family members, tutors, or any other outside sources, etc. during the exam. You may NOT provide or receive any assistance from another student taking this exam. You may NOT use any electronic devices to look up hints or solutions for this exam. Show all work. Correct answers with insufficient or incorrect justification will receive little or no credit. By signing above you acknowledge that you have read and will abide by these conditions. Your exam will not be graded without your signature.

Signature: ____________

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A rod placed on the x-axis between $x = 1$ and $x = 2$ has a density of $\delta(x) = x^2 \frac{kg}{m}$.

(a) Find the total mass of the rod.

$$M = \int_1^2 \delta(x)dx = \int_1^2 x^2 dx = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\boxed{M = \frac{7}{3} kg}$$

(b) Find the x-coordinate of the center of mass of the rod.

$$\bar{x} = \frac{M_y}{M}$$

$$M_y = \int_1^2 x\,\delta(x)dx = \int_1^2 x^3 dx = \frac{x^4}{4}\Big]_1^2 = \frac{16}{4} - \frac{1}{4}$$

$$= \frac{15}{4}$$

$$M = 7/3$$

$$\bar{x} = \frac{15}{4} \cdot \frac{3}{7} = \frac{45}{28}$$

Problem #2 (20pts): Recall that the anti-derivative of x^n is given by $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, which holds for every positive and negative integer except $n = -1$. Recall that we defined $\ln x = \int_1^x \frac{dt}{t}$ to be the missing anti-derivative.

(a) By what theorem is $\frac{d}{dx} \ln x = \frac{1}{x}$? Fundamental Theorem of Calculus

(b) Use the fact that $\frac{d}{dx} \ln x = \frac{1}{x}$ to prove that $\ln bx = \ln b + \ln x$.

$\frac{d}{dx} \ln bx = \frac{1}{bx} \cdot b = \frac{1}{x}$ (Chain rule)

$\frac{d}{dx}(\ln b + \ln x) = \frac{1}{x} \Rightarrow \ln bx = \ln b + \ln x + \text{const}$

set $x = 0$ to conclude Const $= 0$.

(c) Recall that we defined $y = e^x$ to be the inverse of $y = \ln x$, so

$$e^{\ln x} = x = \ln e^x.$$

Use the chain rule to show that we must have $\frac{d}{dy} e^y = e^y$ if $y = \ln x$.

$e^{\ln x} = x$

diff both sides to get

$\frac{d}{dx} e^{\ln x} = 1$

$\frac{d}{dx} e^{\ln x} = \frac{d}{dy} e^y \frac{d}{dx} \ln x = \frac{d}{dy} e^y \cdot \frac{1}{x} = 1$

$\therefore \frac{d}{dy} e^y = x = e^y$ ✓

Problem #3 (20pts): **(a)** Recall that the principal grows like $P = P_0 e^{rt}$ where r is the interest rate. If the interest rate is 20 percent per year, find the time it takes for your money to double.

$P = P_0 e^{.2t}$. The doubling time satisfies

$$2\cancel{P_0} = \cancel{P_0} e^{.2t} \Rightarrow 2 = e^{.2t} \Rightarrow \ln 2 = .2t$$

$$\boxed{t = \frac{.2}{\ln 2}}$$

(b) Use the method of separation of variables to find the solution $y(t)$ of the initial value problem $y' = t^3 y$, $y(0) = 2$.

$$\frac{dy}{dt} = t^3 y \Leftrightarrow dy = t^3 y\, dt \Leftrightarrow \frac{dy}{y} = t^3 dt$$

$$\int_{y_0}^{y(t)} \frac{dy}{y} = \int_0^t \underbrace{t^3 dt}$$

or other dummy variable

$$\ln y \Big]_{y_0}^{y(t)} = \frac{t^4}{4}\Big]_0^t = \frac{t^4}{4}$$

$$\ln \frac{y(t)}{y_0} = \frac{t^4}{4} \Rightarrow y(t) = \underbrace{y_0}_{y_0 = 2} e^{t^4/4} = 2e^{(t^4/4)}$$

Problem #4 (20pts): Evaluate the following integrals:

(a) $\int_0^{\pi/4} \frac{\sin x}{\cos x} dx = \int_{x=0}^{x=\pi/4} \frac{du}{u} = \ln|\cos x| \Big]_0^{\pi/4}$

$u = \cos x$

$du = \sin x\, dy$

$= \ln|\cos \pi/4| - \ln|\cancel{\cos(0)}^{\,1}|$

0

$= \boxed{\ln\left|\frac{\sqrt{2}}{2}\right|}$

(b) $\int xe^x dx = uv - \int v\, du = xe^x - \int e^x dx$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

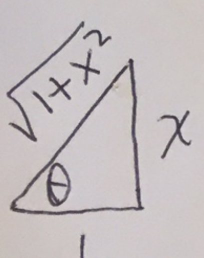
$= \boxed{xe^x - e^x + C}$

(c) $\int \frac{x}{4+x^2} dx = \int \frac{2\tan\theta}{4+4\tan^2\theta} 2\sec^2\theta\, d\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$= \int \frac{4\tan\theta \cancel{\sec^2\theta}}{4(\cancel{1+\tan^2\theta})} d\theta$



$= \int \tan\theta\, d\theta = -\int \frac{du}{u} = -\ln|u| + C$

$u = \cos\theta$

$du = \sin\theta\, d\theta$

$= -\ln|\cos\theta| + C$

$\frac{\sin x}{\cos x}$

$= -\ln \frac{1}{\sqrt{1+x^2}} + C$

Problem #5 (20pts):

(a) Use the method of partial fractions to evaluate the anti-derivative:

$\int \frac{2x^2+x+1}{(x-2)(x+1)(x-3)} =$ $\int \frac{A}{X-2} + \frac{B}{X+1} + \frac{C}{X-3}\,dx$

use trick
mult by $(x-r)$
& set $x=r$

$$A = \left.\frac{2x^2+x+1}{(x+1)(x-3)}\right|_{x=2} = \frac{2\cdot 2^2+2+1}{(2+1)(2-3)} = \frac{11}{-3}$$

$$B = \left.\frac{2x^2+x+1}{(x-2)(x-3)}\right|_{x=-1} = \frac{2(-1)^2-1+1}{(-1-2)(-1-3)} = \frac{2}{12} = \frac{1}{6}$$

$$C = \left.\frac{2x^2+x+1}{(x-2)(x+1)}\right|_{x=3} = \frac{2\cdot(3)^2+3+1}{(3-2)(3+1)} = \frac{22}{4} = \frac{11}{2}$$

$$\int \frac{2x^2+x+1}{(x-2)(x+1)(x-3)}dx = A\ln|x-2| + B\ln|x+1| + C\ln|x-3| + C$$

(b) Write the partial fractions expansion for the following integral. (Do not solve for the constants.)

$\int \frac{x+1}{(x-1)(x+2)^2(x^2+x+1)^3)}dx =$ $\int \frac{A}{X-1} + \frac{B}{X+2} + \frac{C}{(X+2)^2} + \frac{Dx+E}{x^2+x+1}$

$$+ \frac{Fx+G}{(x^2+x+1)^2} + \frac{Hx+K}{(x^2+x+1)^3}\,dx$$