Solutions

Name: ______________________

Student ID# and Section: ______________________

Midterm Exam 1

Wednesday, February 3

MAT 21B, Temple, Winter 2021

Print names and ID's clearly. Write the solutions clearly and legibly. Do not write near the edge of the paper. Show your work on every problem. Be organized and use notation appropriately.

You are NOT allowed to consult the internet, Piazza, your classmates, friends or family members, tutors, or any other outside sources, etc. during the exam. You may NOT provide or receive any assistance from another student taking this exam. You may NOT use any electronic devices to look up hints or solutions for this exam. Show all work. Correct answers with insufficient or incorrect justification will receive little or no credit. By signing above you acknowledge that you have read and will abide by these conditions. Your exam will not be graded without your signature.

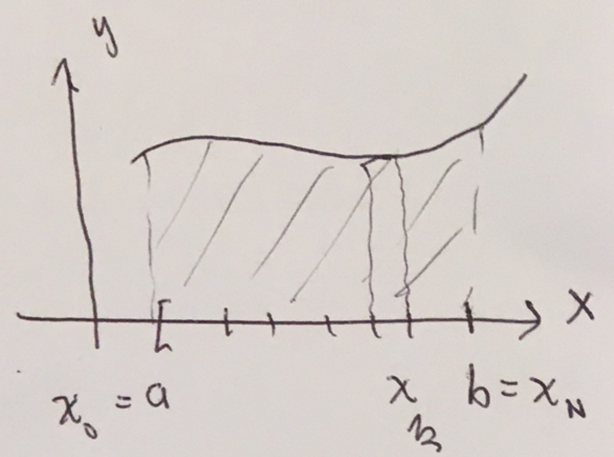
Signature: ______________________

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Let $y = f(x)$ be a continuous function for $a \le x \le b$.

(a) State the Area Problem and define the Riemann integral over $[a, b]$ precisely as the limit of a Riemann Sum. (Draw the correct picture, and define and label grid points correctly.)

Area problem: find the area under graph of f between $x=a$ & $x=b$



$x_k = a + k\Delta x$

$\Delta x = \frac{b-a}{N}$

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \underbrace{\sum_{k=1}^{N} f(x_k)\Delta x}_{\text{Riemann Sum}}$$

(b) State the Fundamental Theorem of Calculus, and explain in your own words why it is so important.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

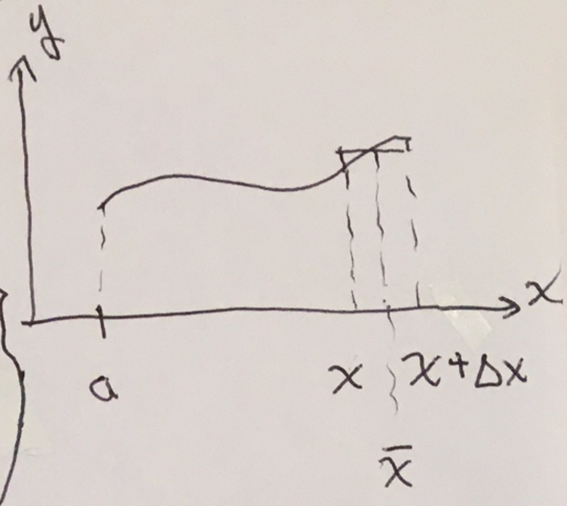
where $F'(x) = f(x)$

This is important because it tells you that the Area Problem can be evaluated using the theory of derivatives

Ie $$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

The LHS is defined by ~~a~~ Riemann Sums and the RHS is any anti-derivative of f.

Problem #2 (20pts): Recall that $F(x) = \int_a^x f(t)dt$ denotes the area under the graph of f between a and x. Assuming f is a continuous function, draw this area in a diagram, and prove that $F'(x) = f(x)$.



We prove $F'(x) = f(x)$

- $$F'(x) = \lim_{\Delta x \to 0} \left(\frac{\left\{ \int_a^{x+\Delta x} f(x)dx - \int_a^{x} f(x)dx \right\}}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} \underbrace{\int_x^{x+\Delta x} f(x)\,dx}_{= f(\bar{x})\Delta x} = \frac{1}{\cancel{\Delta x}} f(\bar{x}) \cancel{\Delta x} \right] = f(x)$$

by mean value thm for integrals

f continuous $\Rightarrow$ $\lim_{\bar{x} \to x} f(\bar{x}) = f(x)$

Problem #3 (20pts): Use the substitution method to evaluate the following anti-derivatives: (Show every step.)

(a) $\int x^2 \sin x^3 dx$. $= \frac{1}{3}\int \sin u \, du = -\frac{1}{3}\cos u + C$

$u = x^3 \quad du = 3x^2 dx$

$= -\frac{1}{3}\cos(x^3)$

(b) $\int (3x-1)^5 5dx$. $= \frac{5}{3}\int u^5 du = \frac{5}{3}\frac{u^6}{6} + C$

$u = 3x - 1$

$du = 3dx$

$= \frac{5}{18}(3x-1)^6 + C$

continuously

~~**(c)** $\int_1^2 F'(\sin(2x))\sin(x)\cos(x)\,dx$. (Here F can be any differentiable function.)~~

$\cancel{u = \sin 2x}$

$\cancel{du = 2\sin x \cos x}$

$\cancel{= \frac{1}{2}\int_{x=1}^{x=2} F'(u)\,du = \frac{1}{2}F(u)\Big]_{x=1}^{x=2}}$

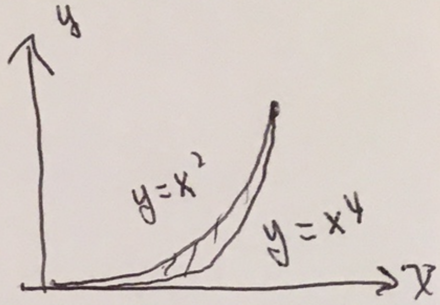
$\cancel{= \frac{1}{2}F(\sin 2x)\Big]_1^2}$

Ill-composed problem-
Everyone gets full credit!

$\cancel{= \frac{1}{2}\left[F(\sin(4)) - F(\sin(2))\right]}$

Problem #4 (20pts): Let A denote the region between the graphs of $y = x^2$ and $y = x^4$ for $0 \le x \le 1$.

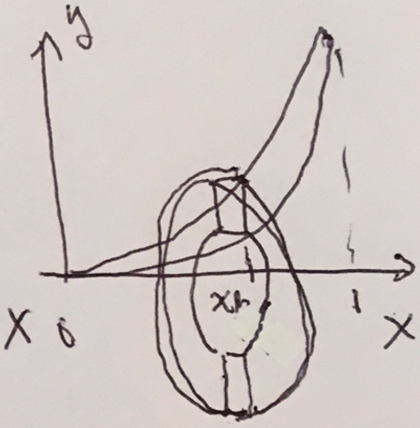
(a) Draw the region A and evaluate its area.



$$A = \int_0^1 x^2 - x^4 \, dx = \frac{x^3}{3} - \frac{x^5}{5} \Big]_0^1$$

$$= \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15}$$

(b) Find the volume of the region obtained by rotating A about the x-axis, based on an integral with respect to x. (State which method you use, disk, washer or shell, and argue starting with a Riemann sum.)



$$\text{Vol} = \lim_{\Delta x \to 0} \sum_{k=1}^{N} \Delta V_k$$ Washer Method

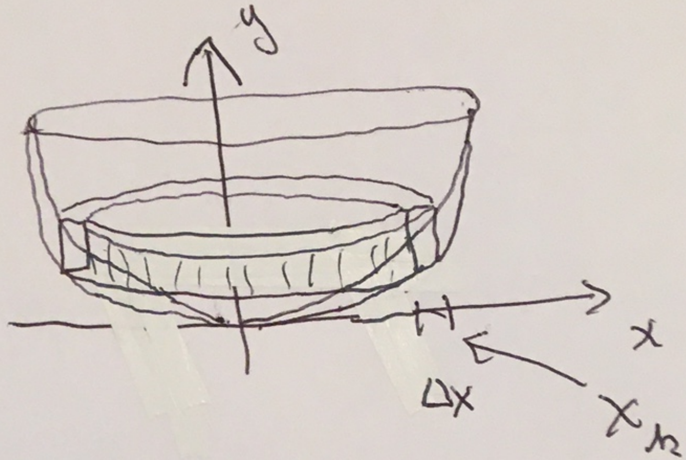
$$\Delta V_k = \pi (R_k^2 - r_k^2) \Delta x = \pi (x^4 - x^8) \Delta x$$

$$\text{Vol} = \lim_{\Delta x \to 0} \underbrace{\sum_{k=1}^{N} \pi (x^4 - x^8) \Delta x}_{\text{Riemann Sum}} = \int_0^1 \pi (x^4 - x^8) \, dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{x^9}{9} \right)_0^1 = \pi \left(\frac{1}{5} - \frac{1}{9} \right) = \pi \frac{4}{45}$$

(c) Find the volume of the region obtained by rotating A about the y-axis, based on an integral with respect to x. (State which method you use, disk, washer or shell, and argue starting with a Riemann sum.)

$$\text{Vol} = \lim_{\Delta x \to 0} \sum_{k=1}^{N} \Delta V_k$$



$$\Delta V_k = 2\pi R_k h_k \Delta x$$

(Shell)

$$\text{Vol} = \lim_{\Delta x \to 0} \underbrace{\sum_{k=1}^{N} 2\pi x_k (x_k^2 - x_k^4) \Delta x}_{\text{Riemann Sum}}$$

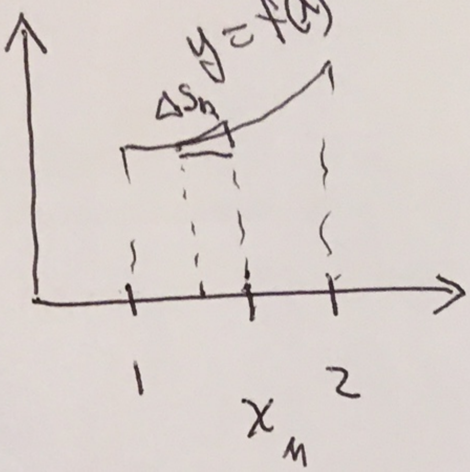
$$= \int_0^1 2\pi x(x^2 - x^4)\, dx = 2\pi \int_0^1 x^3 - x^5\, dx$$

$$= 2\pi\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Bigg]_0^1 = 2\pi\left(\frac{1}{4} - \frac{1}{6}\right) = \boxed{\frac{\pi}{6}}$$

Problem #5 (20pts): Consider the curve defined by the graph of the function $y = f(x) = \sin(x^2)$ between $x = 1$ and $x = 2$.

(a) Derive an integral formula for the arclength of the curve, starting with a Riemann sum. You need not evaluate the integral. (Draw a picture.)

$$L = \lim_{\Delta x \to 0} \sum_{k=1}^{N} \Delta S_k$$



$$\Delta S_n = \sqrt{\Delta x_n^2 + \Delta y_n^2}$$

$$= \sqrt{1 + \left(\frac{\Delta y_n}{\Delta x_n}\right)^2}\,\Delta x_n$$

$$\approx \sqrt{1 + f'(x_n)^2}\,\Delta x$$

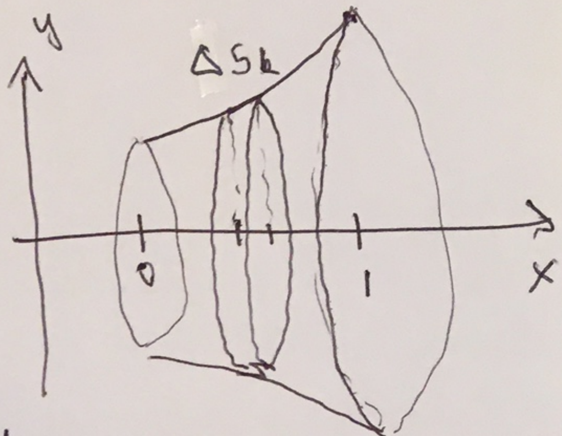
$$L = \lim_{\Delta x \to 0} \underbrace{\sum_{k=1}^{N} \sqrt{1 + f'(x_n)^2}\,\Delta x}_{\text{Riemann Sum}} = \int_1^2 \sqrt{1 + f'(x)^2}\,dx$$

$$= \int_1^2 \sqrt{1 + [2x\cos(x^2)]^2}\,dx$$

Note: you can use $ds = \sqrt{dx^2 + dy^2}$ and go from there as well.

(b) Derive an integral formula for the area of the region obtained by rotating the curve about the x-axis, starting with a Riemann sum. You need not evaluate the resulting integral. (Draw a picture.)

$$\text{Sur Area} = \lim_{\Delta x \to 0} \Delta S'_k$$



$$\Delta S'_k = 2\pi R_k \cdot \Delta S_k$$

$$= 2\pi f(x_k)\sqrt{1+f'(x_k)^2}\,\Delta x$$

$$\text{Surface Area} = \lim_{\Delta x \to 0} \underbrace{\sum_{k=1}^{N} 2\pi f(x_k)\sqrt{1+f'(x_k)^2}\,\Delta x}_{\text{Riemann Sum}}$$

$$= \int_1^2 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

$$= \int_1^2 2\pi \sin(x^2)\sqrt{1+(2x\cos(x^2))^2}\,dx$$

Again – OK to use $ds = \sqrt{dx^2+dy^2} = \sqrt{1+f'(x)^2}\,dx$ if incorporated into a Riemann Sum correctly.