Solutions Midterm I
Math 21C Winter 2020 (Temple)

#1 Definition of limit:

$\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n > N \ |a_n - L| < \varepsilon$

(You can use equivalent words, including
"if $n > N$ then $|a_n - L| < \varepsilon$"
"$n > N$ implies $|a_n - L| < \varepsilon$")

- To prove $a_n = 2 + \frac{(-1)^n}{n} \longrightarrow 2$.

Fix $\varepsilon > 0$. We find N st $n > N$ implies $|a_n - 2| < \varepsilon$.

We need $\left|2 + \frac{(-1)^n}{n} - 2\right| = \frac{1}{n} < \varepsilon$ so $n > \frac{1}{\varepsilon}$ suffices.

Thus choose $N = \frac{1}{\varepsilon}$. Then $n > N$ implies

$$|a_n - 2| = \left|\frac{1}{n}\right| < \frac{1}{N} < \varepsilon$$

#2 Assume $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

(a) $$.888\ldots = \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \cdots$$

$$= \frac{8}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

$$= \frac{8}{10} \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k = \frac{8}{10} \cdot \underbrace{\frac{1}{1-\frac{1}{10}}}_{10/9} = \frac{8}{9}$$

(b) Let $S_n = \sum_{k=N}^{n} r^k = r^N + r^{N+1} + \cdots + r^n$

$(n \geq N)$

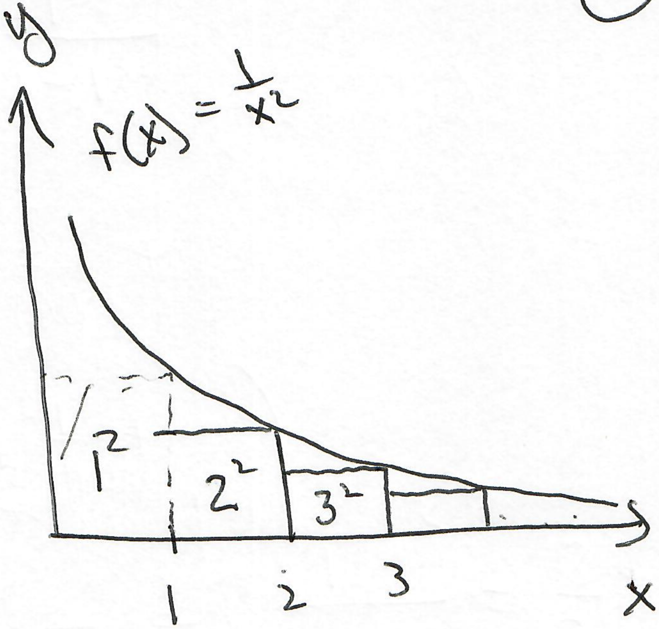
$$r S_n = r^{N+1} + r^{N+2} + \cdots + r^{n+1}$$

$$(1-r) S_n = S_n - r S_n = r^N - r^{n+1} \Rightarrow S_n = \frac{r^N - r^{n+1}}{1-r}$$

$$\sum_{k=N}^{n} r^n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{r^N - r^{n+1}}{1-r} = \frac{r^N}{1-r}$$

#3 (a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

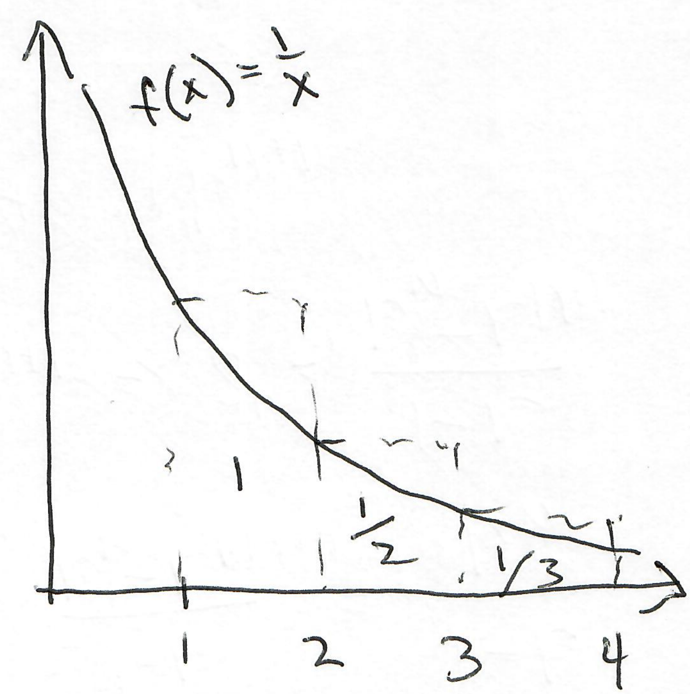
Since rectangles representing $\frac{1}{n^2}$ lie below graph of $\frac{1}{x^2}$,



$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le \int_1^{\infty} \frac{dx}{x^2} + 1 = \left.\frac{x^{-1}}{-1}\right]_1^{\infty} + 1 = 2$$

(b) Since rectangles representing $\frac{1}{n}$ lie above graph of $\frac{1}{x}$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \ge \int_1^{\infty} \frac{dx}{x} = \infty$$



#4 (a) $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$,

if we estimate $|f^{(n+1)}(c)| \leq 1$, we get

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Thus: $\cos x = P_{2n}(x) + R_{2n}(x)$

with $|R_{2n}(x)| \leq \dfrac{|x|^{2n+1}}{(2n+1)!}.$

(b) By alternating series test, the error is estimated by the next nonzero term $\Rightarrow$

$$|R_{2n}(x)| \leq \frac{x^{2(n+1)}}{2(n+1)!} = \frac{x^{2n+2}}{(2n+2)!}.$$

(c) (b) is smaller because $|x|<1$.

#5 (a) $\sum_{n=1}^{\infty} \frac{(5x-3)^n}{n^2}$

Root Test: If $\sqrt[n]{|a_n|} \to \rho < 1$, then series converges, $\to \rho > 1$ diverges.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(5x-3)^n}{n^2}\right|} = \frac{|5x-3|}{(n^2)^{1/n}} = \frac{|5x-3|}{(\sqrt[n]{n})^2}$$

$$\to |5x-3| < 1 \iff -1 < 5x-3 < 1$$

$$2 < 5x < 4$$

$$\boxed{2/5 < x < 4/5}$$

#5 b $\displaystyle\sum_{n=0}^{\infty} \frac{n^n (x-2)^n}{n!} = \sum_{n=0}^{\infty} a_n$

Ratio Test: If $\left|\frac{a_{n+1}}{a_n}\right| \to \rho < 1$ converges

$\left|\frac{a_{n+1}}{a_n}\right| \to \rho > 1$ diverges

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}(x-2)^{\cancel{n+1}}}{(n+1)\cancel{!}} \frac{\cancel{n!}}{n^n \cancel{(x-2)^n}}\right|$$

$$= \left|\frac{\cancel{(n+1)}(n+1)^n (x-2)}{\cancel{n+1}\ n^n}\right| = \left(\frac{n+1}{n}\right)^n |x-2|$$

$$= \left(1+\frac{1}{n}\right)^n |x-2| \longrightarrow e|x-2| < 1$$

$\Leftrightarrow |x-2| < \frac{1}{e} \Leftrightarrow -\frac{1}{e} < x-2 < \frac{1}{e}$

$\Leftrightarrow \boxed{2-\frac{1}{e} < x < 2+\frac{1}{e}}$ $\quad R = \frac{1}{e}$

#5 Alternatively: Write

$$\sum_{n=0}^{\infty} \frac{n^n (x-2)^n}{n!} = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = e = \frac{1}{R} \checkmark$$