Name: ______________________

Student ID#: ______________________

Section: ______________________

Final Exam

Tuesday March 21, 10:30-12:30, 1020 TLC

MAT 21D, Temple, Winter 2023

Print name and ID's clearly. Have student ID ready. Write solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Correct answers with no supporting work will not receive full credit. No calculators, notes, books, cellphones...allowed.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
Total		200

Problem #1 (20pts): (a) Sketch the region of integration $\mathcal{R}_{xy}$ which is the quarter circle of radius $r = 2$, above the x-axis and to the right of the y-axis. Iterate the integral $\int\int_{\mathcal{R}_{xy}} xy\, dA$ in order $dydx$, and order $dxdy$. (Do not evaluate.)

(b) Evaluate the integral of Part (a) using polar coordinates.

Problem #2 (20pts): Assume the region $\mathbf{R}_{xy}$ of Problem 1 is a metal plate with density $\delta(x, y) = xy$. Set up an iterated integral in x and y for the following: (You need not evaluate.)

(a) The total mass M.

(b) The coordinates of the center of mass.

(c) The kinetic energy of rotation for angular rotation rate ω about z-axis.

Problem #3 (20pts): Assume a mass m moves along a path $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, through a conservative force field $\mathbf{F} = \nabla f$.

(**a**) Show that $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = f(B) - f(A)$ for any curve taking A to B.

(**b**) Show that if further, $\mathbf{F} = m\mathbf{a}$, then also $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$. (Here $v_A = \|v(a)\|^2$, $v_B = \|v(b)\|^2$.)

Problem #4 (20pts): Let

$$\vec{\mathbf{F}} = -\frac{y}{r^2}\,\mathbf{i} + \frac{x}{r^2}\,\mathbf{j}, \quad r = \sqrt{x^2 + y^2},$$

and let $\mathcal{C}$ denote the helix $\vec{\mathbf{r}}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + ct(t - 2\pi n)\,\mathbf{k}$ restricted to $0 \le t \le 2\pi n$, where $c > 0$ is constant, and $n > 0$ is an integer. Show that $\mathcal{C}$ is closed, and evaluate $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

Problem #5 (20pts): (a) Let $\mathcal{C}$ be a positively oriented simple closed curve in the plane, and let $\vec{\mathbf{F}} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. Manipulate $\vec{\mathbf{F}}$ and $\vec{\mathbf{T}}$ to rewrite the line integral $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ as a flux integral. Define your terms carefully.

(b) Assume an infinite plate of density $\delta(x,y) = xy^2$. Find $N(x,y)$ such that the vector field $\vec{\mathbf{F}} = xy\,\mathbf{i} + N(x,y)\,\mathbf{j}$ has the property that its line integral around every simple closed curve gives the total mass inside that curve. (Hint: Green's Theorem)

Problem #6 (20pts): (a) Define ∇f, $\mathbf{Curl}\,\vec{\mathbf{F}}$ and $\mathbf{Div}\,\vec{\mathbf{F}}$.

(b) Show that $\mathbf{Div}(\mathbf{Curl}\,\vec{\mathbf{F}}) = 0$.

Problem #7 (20pts): Let $\vec{\mathbf{v}} \equiv \vec{\mathbf{F}} = x\mathbf{i} + xyz\mathbf{k}$ be the velocity field of a moving fluid.

(a) Find the maximal circulation per area, and the axis around which it is maximal, at $P = (1, -1, 2)$.

(b) Describe all axes $\vec{\mathbf{n}}$ around which there is zero circulation per area at point $P = (1, -1, 2)$.

(c) Find the circulation per area around axis $\vec{\mathbf{w}} = \overrightarrow{(2,-2,1)}$ at point P.

(d) Find the frequency ω and period T for a bead rotating with $\vec{\mathbf{v}}$ around a circle of radius ϵ, center P, around axis $\vec{\mathbf{w}} = \overrightarrow{(2,-2,1)}$, (in the limit $\epsilon \to 0$.)

Problem #8 (20pts): Consider Kepler's Laws under the simplifying assumption that the planets move in circular orbits with the sun at the center, (not a terrible approximation). In this case, each planet moves around a circle with position vector

$$\vec{\mathbf{r}}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}, \tag{1}$$

where R and ω are constants which depend on the planet, and t is the time. Assuming Newton's inverse square force law in the form

$$\overrightarrow{\mathbf{a}} = -G\frac{1}{r^2}\frac{\overrightarrow{\mathbf{r}}}{r}, \tag{2}$$

(where G is Newton's gravitational constant), derive Kepler's third law

$$\frac{T^2}{R^3} = K, \tag{3}$$

where T is the period of the planet's rotation and K is a constant independent of the planet.

Problem #9 (20pts): Let $\mathcal{S}$ denote the upper hemisphere $x^2+y^2+z^2=9$, $z \geq 0$, let $\mathcal{C}$ denote its boundary circle $x^2+y^2=9$ in the (x,y)-plane, and let $\vec{\mathbf{F}} = -\frac{y}{2}\,\mathbf{i} + \frac{x}{2}\,\mathbf{j}$. Verify Stokes Theorem $\int\int_{\mathcal{S}} Curl\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ as follows:

(a) Evaluate $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ directly by parameterization.

(b) Find the unit normal $\vec{\mathbf{n}}$ on $\mathcal{S}$.

(c) Find $J(x,y)$ so that $dS = J(x,y)\,dxdy$ on $\mathcal{S}$.

(d) Use (b) and (c) to evaluate $\int\int_{\mathcal{S}} Curl\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$ using (x,y) as a coordinate system.

Problem #10 (20pts): Let $\mathcal{V}_{xyz}$ denote the volume enclosed by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, for a,b,c positive, let $\mathcal{S}$ denote the surface which is its closed boundary, and let $\vec{\mathbf{F}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Evaluate the flux $\int\int_{\mathcal{S}}\vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,dS=0$. (Hint: Use the Divergence Theorem, the substitution $x=au, y=bv, z=cw$, and volume of sphere is $\frac{4}{3}\pi r^3$.)