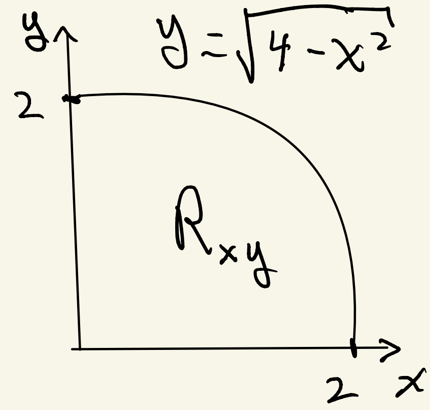
#1

(a) $$\iint_{R_{xy}} xy\,dA = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$



$$= \int_0^2 \int_0^{\sqrt{4-y^2}} xy\,dx\,dy$$

(b) $$\iint_{R_{xy}} xy\,dA = \int_0^{\pi/2} \int_0^2 r\cos\theta\, r\sin\theta \cdot r\,dr\,d\theta$$

$$= \frac{2^4}{4} \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

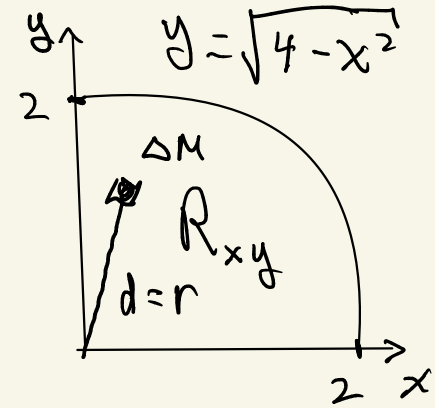
$$= 4\ \frac{u^2}{2}\Big]_{\theta=0}^{\theta=\pi/2} = 2\sin^2\theta\Big]_0^{\pi/2} = \boxed{2}$$

#2 (a) $\delta(x,y) = xy \Rightarrow \text{Mass} = \iint_{R_{xy}} xy \, dA$

$$M = \int_0^2 \int_0^{\sqrt{4-x^2}} xy \, dy \, dx$$

(b) $$\bar{x} = \frac{1}{M} \int_0^2 \int_0^{\sqrt{4-x^2}} x^2 y \, dy \, dx$$

$$\bar{y} = \frac{1}{M} \int_0^2 \int_0^{\sqrt{4-x^2}} x y^2 \, dy \, dx$$



(c) $$KE = \frac{1}{2} I_z \omega^2$$

$$I_z = \int_0^2 \int_x^{\sqrt{4-x^2}} \underbrace{(x^2+y^2)}_{r^2} xy \, dy \, dx$$

#3 If $\|\vec{v}(t)\| = \text{const}$, then $\|\vec{v}(t)\|^2 = \vec{v}\cdot\vec{v} = \text{const}^2$

$$\Rightarrow 0 = \frac{d}{dt}(\vec{v}\cdot\vec{v}) = \vec{v}'\cdot\vec{v} + \vec{v}\cdot\vec{v}' = 2\,\vec{a}\cdot\vec{v}$$

$$\Rightarrow \vec{a}\cdot\vec{v} = 0 \quad \checkmark$$

#4

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi n} \vec{F}\cdot\vec{v}\,dt$$

$$\vec{F} = \overrightarrow{\left(-\frac{y}{r^2}, \frac{x}{r^2}\right)}$$

$$\vec{r}(t) = \cos t\,\underset{\sim}{i} + \sin t\,\underset{\sim}{j} + ct\,\underset{\sim}{k}$$

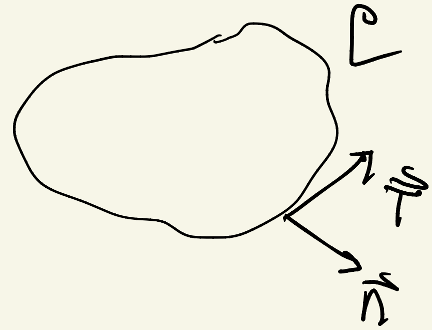
$$\vec{v} = \vec{r}\,'(t) = -\sin t\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j}$$

$$= \int_0^{2\pi n} \overrightarrow{(-y, x)} \cdot \overrightarrow{(-\sin t, \cos t)}\,dt$$

$$= \int_0^{2\pi n} \overrightarrow{(-\sin t, \cos t)} \cdot \overrightarrow{(-\sin t, \cos t)}\,dt$$

$$= \int_0^{2\pi n} \sin^2 t + \cos^2 t\,dt = 2\pi n$$

#5 $\vec{F} = \overrightarrow{(M, N)}$ define $\vec{F}_\perp = \overrightarrow{(N, -M)}$



(a) $\vec{T} = \overrightarrow{(T_x, T_y)} \Rightarrow \vec{T}_\perp = \overrightarrow{(T_y, T_x)}$

so $\vec{T}_\perp = \vec{n}$. Thus

$$\vec{F} \cdot \vec{T} = \vec{F}_\perp \cdot \vec{T}_\perp = MT_x + NT_y$$

thus: $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \int_{\mathcal{C}} \vec{F}_\perp \cdot \vec{n}\, ds =$ Flux of $\vec{F}_\perp$ out thru $\mathcal{C}$

(b) $\delta(x,y) = xy^2$. Green's Thm says

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \iint_R N_x - M_y \, dA = \iint_R (N_x - x)\, dA$$

$M = xy$

Thus if $N_x - x = xy^2$, we have $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = M$

Need: $N_x = xy^2 + x = x(y^2+1)$

Choose: $N = \frac{x^2}{2}(y^2+1)$ ✓

#6 (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$. Then $\nabla f = \overrightarrow{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)}$

Let $\vec{F} = \overrightarrow{(M, N, P)}$. Then

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \underset{\sim}{i}(P_y - N_z) - \underset{\sim}{j}(P_x - M_z) + \underset{\sim}{k}(N_x - M_y)$$

$$\text{Div}\,\vec{F} = M_x + N_y + P_z$$

(b) $$\text{Div}(\text{Curl}\,\vec{F}) = \text{Div}\,\overrightarrow{(P_y - N_z, -(P_x - M_z), N_x - M_y)}$$

$$= (P_y - N_z)_x - (P_x - M_z)_y + (N_x - M_y)_z$$

$$= \cancel{P}_{yx} - \cancel{N}_{zx} - \cancel{P}_{xy} + \cancel{M}_{zy} + \cancel{N}_{xz} - \cancel{M}_{yz}$$

$$= 0$$ ✓

(#7) $\vec{v} = \vec{F} = \overrightarrow{(x, 0, xyz)}$

(a) Max'al Circulation per area is $\|\operatorname{Curl}\vec{F}\|$ around axis $\operatorname{Curl}\vec{F}/\|\operatorname{Curl}\vec{F}\|$ @ $P = (1, -1, 2)$

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ x & 0 & xyz \end{vmatrix} = \underset{\sim}{i}(xz - 0) - \underset{\sim}{j}(yz - 0) + \underset{\sim}{k}(0 - 0)$$

$$= \overrightarrow{(xz, -yz, 0)}$$

$\operatorname{Curl}\vec{F}(1, -1, 2) = \overrightarrow{(-1, 2, 0)}$ $\quad \|\overrightarrow{(-1,2,0)}\| = \sqrt{1+4} = \sqrt{5}$

Axis $\vec{n} = \frac{1}{\sqrt{5}}\overrightarrow{(-1, 2, 0)}$, Max Circ/area $= \sqrt{5}$

(b) $\operatorname{Curl}\vec{F} \cdot \vec{n} = 0 \Rightarrow \overrightarrow{(-1, 2, 0)} \cdot \vec{n} = 0$

$\Rightarrow \vec{n}$ lies in the plane $\perp$ $\overrightarrow{(-1, 2, 0)}$

(c) Circ/area around $\vec{w} = \overrightarrow{(2, -2, 1)}$ @ P is

$$\operatorname{Curl}\vec{F} \cdot \frac{\vec{w}}{\|\vec{w}\|} = \overrightarrow{(-1, 2, 0)} \cdot (2, -2, 1) \cdot \frac{1}{\sqrt{2^2 + 2^2 + 1^2}}$$

$$= (-2 - 4 + 0)\frac{1}{\sqrt{9}} = \frac{-6}{3} = -2 \checkmark$$

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(#8) Differentiating (1) twice gives

$$\vec{a} = \vec{r}\,''(t) = -\omega^2 \vec{r},$$

and setting this equal to $\vec{a}$ in (2) gives

$$-\omega^2 = -\frac{G}{r^3} = -\frac{G}{R^3},$$

where we use that $r = R$ on circle of radius R. Now the period T of (1) satisfies $\omega T = 2\pi$, so $T = 2\pi/\omega$ and $\omega^2 = 4\pi^2/T^2$. Putting this into (4) gives

$$\frac{4\pi^2}{T^2} = \frac{G}{R^3},$$

or

$$\frac{T^2}{R^3} = \frac{4\pi^2}{G}$$

where the RHS is independent of planet. ✓

#9

(a) $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,dt = \int_0^{2\pi} \overrightarrow{\left(-\frac{y}{2}, \frac{x}{2}\right)} \cdot 3\overrightarrow{(-\sin t, \cos t)}\,dt$

$\vec{r}(t) = 3\overrightarrow{(\cos t, \sin t)}$

$\vec{v}(t) = 3\overrightarrow{(-\sin t, \cos t)}$

$= \frac{3}{2}\int_0^{2\pi} 3\overrightarrow{(-\sin t, \cos t)} \cdot \overrightarrow{(-\sin t, \cos t)}\,dt$

$= \frac{9}{2}\cdot 2\pi = 9\pi$

(b) Unit normal on sphere of radius 3 is

$3\vec{n} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \Rightarrow n = \frac{1}{3}\overrightarrow{\left(x, y, \sqrt{9-x^2-y^2}\right)}$

(c) Use (x,y) as coordinates.

$\vec{r}(x,y) = \overrightarrow{\left(x, y, \sqrt{9-x^2-y^2}\right)} \qquad J = \frac{1}{\vec{n}\cdot\underset{\sim}{k}} = \frac{3}{\sqrt{9-x^2-y^2}}$

(d) $\iint_{\mathcal{S}} \mathrm{Curl}\vec{F}\cdot\vec{n}\,dS = \iint_{x^2+y^2\le 3} \mathrm{Curl}\vec{F}\cdot\vec{n}\,J\,dx\,dy$

$\mathrm{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{vmatrix} = \underset{\sim}{k}\left(\frac{1}{2}+\frac{1}{2}\right) = \overrightarrow{(0,0,1)}$

$= \iint_{x^2+y^2\le 3} \overrightarrow{(0,0,1)}\,\frac{1}{3}\left(x, y, \sqrt{9-x^2-y^2}\right)\cdot\frac{3}{\sqrt{9-x^2-y^2}}\,dx\,dy = \pi\cdot 3^2 = 9\pi$

#10 $\mathcal{V}_{xyz}$ is vol inside ellipsoid $\mathcal{S}$ given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \text{and } \vec{F}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$$

Find: $\displaystyle\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS$

Soln: Divergence Thm Says:

$$\iiint_{\mathcal{V}_{xyz}} \mathrm{Div}\vec{F}\,dV = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS$$

But $\mathrm{Div}\vec{F}=3 \Rightarrow \displaystyle\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = 3\iiint_{\mathcal{V}} dV$

Setting $x=au$, $y=bv$, $z=cw$, $dx\,dy\,dz = J\,du\,dv\,dw$

Thus: $\displaystyle\iiint_{\mathcal{V}_{xyz}} dV = \iiint_{\mathcal{V}_{uvw}} abc\,du\,dv\,dw$, $J=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

unit sphere has volume $\frac{4}{3}\pi$

$$\therefore \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,dS = 3abc\left(\tfrac{4}{3}\pi\right) = 4\pi abc \checkmark$$