Name: Solutions

Student ID#:

Section: Temple

Midterm Exam 1

Friday, February 2

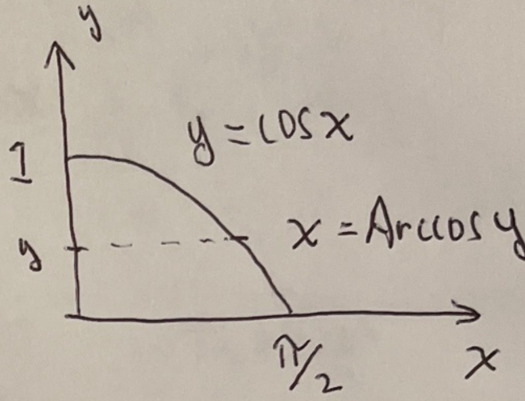
MAT 21D, Temple, Winter 2024

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): (a) Draw the region of integration and evaluate:

$$\int_0^{\pi/2}\int_0^{\cos x} \sin x \, dy dx$$



$$= \int_0^{\pi/2} \sin x \; y\Big]_{y=0}^{y=\cos x} dy\, dx$$

$$= \int_0^{\pi/2} \sin x \cos x \, dx \qquad u = \sin x \quad du = \cos x \, dx$$

$$\int_{x=0}^{x=\pi/2} u \, du = \frac{u^2}{2}\Big]_{x=0}^{x=\pi/2} = \frac{\sin^2 x}{2}\Big]_0^{\pi/2} = \frac{1}{2}$$

(b) Write with limits of integration reversed. (Do not evaluate.)

$$\int_0^1 \int_0^{\text{Arccos}\, y} \sin x \, dx \, dy$$

Problem #2 (20pts): Use polar coordinates to evaluate the integral $\int_0^\infty e^{-2x^2}\,dx$..
Explain whether this is orientation preserving or not. Justify every step.

$$I = \int_0^\infty e^{-2x^2}\,dx \qquad I^2 = \int_0^\infty e^{-2x^2}\,dx \int_0^\infty e^{-2y^2}\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-2(x^2+y^2)}\,dx\,dy \qquad r = \sqrt{x^2+y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$= \int_0^{\pi/2} \int_0^\infty r e^{-2r^2}\,dr\,d\theta$$

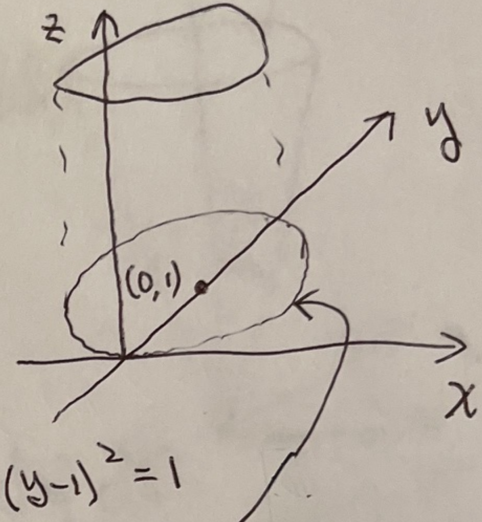
$$u = -2r^2 \quad du = -4r\,dr$$

$$= \int_0^{\pi/2} \int_0^{-\infty} \left(\frac{1}{-4}\right) e^{u}\,du\,d\theta$$

$$= \int_0^{\pi/2} \left(-\frac{1}{4}\right) e^{u}\Big]_0^{-\infty}\,d\theta = \int_0^{\pi/2} \left(-\frac{1}{4}\right)(-1)\,d\theta$$

$$= \frac{1}{4}\cdot\frac{\pi}{2} \Rightarrow I = \sqrt{\frac{\pi}{8}} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{\sqrt{2\pi}}{8}$$

Problem #3 (20pts): Set up the integral in cylindrical coordinates for the Kinetic Energy of rotation of a solid $\mathcal{D}$ rotated about the z-axis at angular velocity ω, assuming density $\delta = x^2 + y^2 + z^2$, and assuming $\mathcal{D}$ is the region bounded by the plane $z = 0$, laterally by the circular cylinder $x^2+(y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$. (Recall: $KE = \frac{1}{2}I_z\omega^2$.)



$$I_z = \iiint_D \underbrace{R^2}_{r^2}\ \underbrace{\delta(x,y,z)}_{r^2+z^2}\ dV$$

$$= \int_0^{\pi}\int_0^{2\cancel{\cos\theta}\ \sin(\theta)}\int_0^{r^2} r^2(r^2+z^2)\ dz\,dr\,d\theta$$

$x^2+(y-1)^2=1$

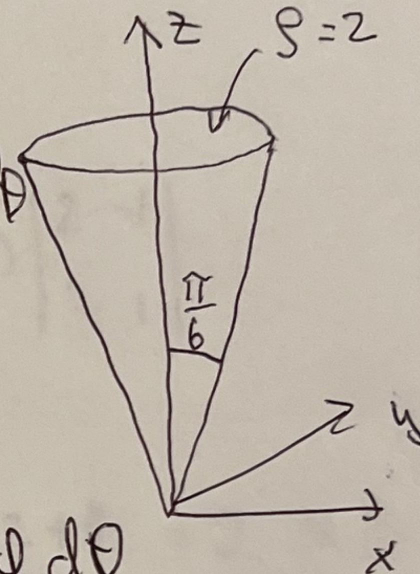
$x^2+y^2-2y+1=1$

$r^2 = 2y = 2r\cancel{\cos\theta}\ \sin(\theta)$

$r = 2\cancel{\cos\theta}\ \sin(\theta)$

Problem #4 (20pts): Find the total mass of the ice-cream shaped cone $\mathcal{D}$ cut from $\rho \leq 2$ by $\phi = \pi/6$, assuming the density is $\delta = \cos\phi$. Draw the region, set up the iterated integral, and evaluate it.

$$M = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \underbrace{\cos\varphi}_{\delta}\, \underbrace{\rho^2 \sin\varphi}_{J}\, d\rho\, d\varphi\, d\theta$$



$$= \int_0^{2\pi}\int_0^{\pi/6} \left.\frac{\rho^3}{3}\right]_0^2 \sin\varphi \cos\varphi\, d\varphi\, d\theta$$

$$= \frac{8}{3}\cdot 2\pi \int_0^{\pi/6} \sin\varphi \cos\varphi\, d\varphi$$

$u = \sin\varphi$

$du = \cos\varphi\, d\varphi$

$$= \frac{16}{3}\pi \left.\frac{u^2}{2}\right]_{\varphi=0}^{\varphi=\pi/6} = \frac{16}{3}\pi \left.\frac{\sin^2\varphi}{2}\right]_0^{\pi/6}$$

$$= \frac{16\pi}{3\cdot 8} = \frac{2\pi}{3}$$

Problem J.

$$J = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 2(3+1) - 1(0+2) + (0-2)$$
$$= 8-2-2$$
$$= 4$$

$$\iiint_D y\, dV = \iiint_D y(u,v,w)\, dV$$
$$= \int_2^3 \int_1^2 \int_0^1 (v-w)\, du\, dv\, dw$$
$$= \int_2^3 \int_1^2 (1-0)(v-w)\, dv\, dw$$
$$= \int_2^3 \left. \frac{v^2}{2} - vw \right]_1^2 dw$$
$$= \int_2^3 \frac{3}{2} - w\, dw$$
$$= \left. \frac{3}{2}w - \frac{w^2}{2} \right]_2^3$$
$$= \frac{3}{2} - \frac{9-4}{2} = -1$$