Name: Solutions

Student ID#: ______

Section: ______

Midterm Exam 1

Monday, October 18

MAT 21D, Temple, Fall 2021

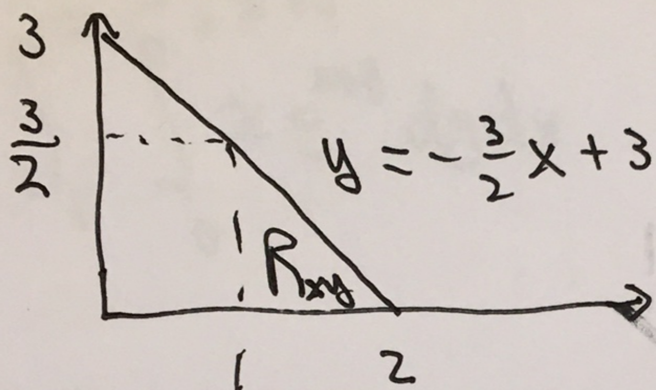
Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_1^2 \int_0^{-\frac{3}{2}x+3} dydx. \tag{1}$$

The region is $0 \leq y \leq -\frac{3}{2}x+3$, $1 \leq x \leq 2$



(b) Write (1) as an iterated integral with limits reversed.

Solve $y = -\frac{3}{2}x+3$ for x $\Rightarrow$ $x = -\frac{2}{3}x+2$

$$\int_0^{3/2} \int_1^{-\frac{2}{3}x+2} dx\, dy$$

Problem #2 (20pts): For (1) of Problem #1, assume $\mathbf{R}_{xy}$ is a metal plate of density $\delta(x, y) = e^{x+y}$.

(a) Write the coordinates of the center of mass in terms of iterated integrals. (Do not evaluate.)

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M} \qquad M = \int_1^2 \int_0^{-\frac{3}{2}x+3} e^{x+y} \, dy \, dx$$

$$M_y = \int_1^2 \int_0^{-\frac{3}{2}x+3} x e^{x+y} \, dy \, dx \qquad M_x = \int_1^2 \int_0^{-\frac{3}{2}x+3} y e^{x+y} \, dy \, dx$$

(b) Write the Kinetic Energy of rotation of R_{xy} about the axis $x = 1$ in terms of iterated integrals. (Do not evaluate.)

$$KE = \frac{1}{2} I_L \omega^2$$

$$I_L = \iint_{R_{xy}} r^2 \delta \, dA = \int_1^2 \int_0^{-\frac{3}{2}x+3} (x-1)^2 e^{x+y} \, dy \, dx$$

$$r^2 = (x-1)^2$$

Problem #3a (10pts): Find the *amplification factor* (or *Jacobian*) J for the volume change

$$dA_{xyz} = J\, dA_{uvw}$$

if the coordinate transformation is the linear transformation $x = 2u - v + w$, $y = u + v - 3w$, $z = u - w$. Does this preserve orientation? Explain.

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} -\nabla x- \\ -\nabla y- \\ -\nabla z- \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -3 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\underset{\substack{\uparrow \\ \text{3rd row}}}{=} 1 \cdot \underset{3-1}{\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix}} + (-1) \underset{2+1}{\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}} = 1 \cdot 2 - (1) \cdot 3 = -1 < 0$$

$\Rightarrow$ "orientation reversing"

Problem #3b (10pts): Derive the formula for J in the case of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

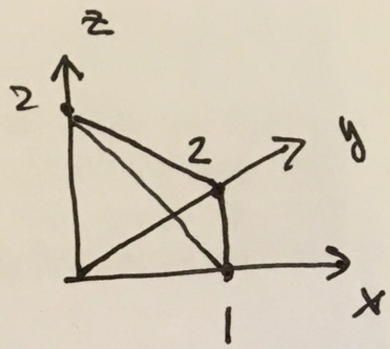
$$J = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

Problem #4 (20pts): Use iterated integration to determine the mass of the tetrahedron D_{xyz} in the first octant, bounded on the bottom and sides by the coordinate planes, and on top by the plane $z = -2x - y + 4$, assuming a constant density $\delta = \delta_0 = constant$. Sketch the region D_{xyz} of integration.

Attached

Solution #4: $z = -2x - y + 2$



$$M = \int_0^1 \int_0^{-2x+2} \underbrace{\int_0^{-2x-y+2} \delta_0 \, dz}_{\delta_0(-2x-y+2)} \, dy \, dx$$

$$= \delta_0 \int_0^1 \underbrace{\int_0^{-2x+2} (-2x-y+2) \, dy}_{} \, dx$$

$$-2xy - \frac{y^2}{2} + 2y \Big]_0^{-2x+2}$$

$$-2x(-2x+2) - \frac{(-2x+2)^2}{2} + 2(-2x+2)$$

$$4x^2 - 4x - \frac{\overset{2}{\cancel{4}}x^2 - \overset{4}{\cancel{8}}x + \overset{2}{\cancel{4}}}{\cancel{2}} - 4x + 4$$

$$2x^2 - 4x + 2$$

$$= \delta_0 \int_0^1 2x^2 - 4x + 2 \, dx = \delta_0 \left(\frac{2x^3}{3} - \frac{4x^2}{2} + 2x \right) \Big]_0^1 = \delta_0 \left(\frac{2}{3} - 2 + 2 \right)$$

$$\boxed{\frac{2}{3}\delta_0}$$

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Problem #5 (20pts): (a) Use spherical coordinates to set up the iterated integral for the volume of the region D cut from the solid sphere $\rho \leq 3cm$ by restricting $0 \leq \phi \leq \frac{\pi}{4}$. Then evaluate the iterated integral. What units apply to this volume?

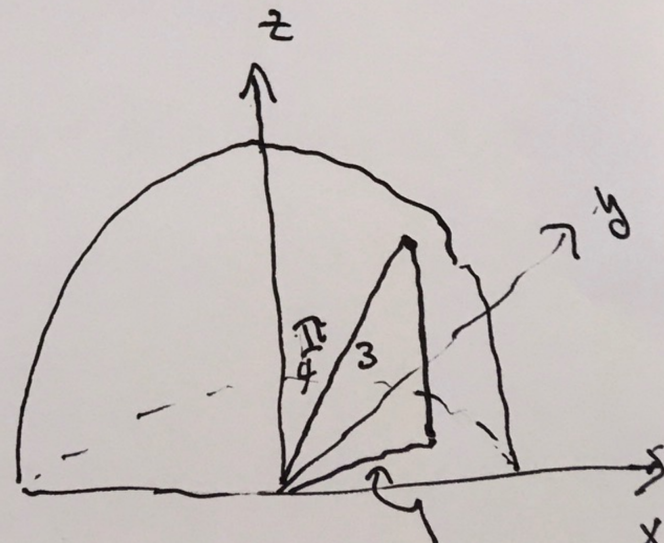
$$I \int_0^{2\pi} \int_0^{\pi/4} \int_0^{3} d\rho \, d\phi \, d\theta = \frac{3}{2}\pi^2 \text{ cm}^3$$

Problem #5b: Set up the integral in cylindrical coordinates, but do not evaluate.

$$\rho^2 = 9 = x^2 + y^2 + z^2 \Rightarrow z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$$

Region sits over circle of radius $\frac{3\sqrt{2}}{2} \Rightarrow$

$$= \int_0^{2\pi} \int_0^{\frac{3\sqrt{2}}{2}} \int_0^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta$$



$r = 3\sin\frac{\pi}{4}$

$= \frac{3\sqrt{2}}{2}$