Name: Solutions

Student ID#: __________

Section: __________

Midterm Exam 2

Friday, November 19

MAT 21D, Temple, Fall 2021

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Let $\vec{\mathbf{r}} = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$. Find:

(**a**) The velocity $\vec{\mathbf{v}}(t)$.

$$\vec{v}(t) = \underset{\sim}{i} - \sin t\,\underset{\sim}{j} + \cos t\,\underset{\sim}{k}$$

(**b**) The acceleration $\vec{\mathbf{a}}(t)$.

$$\vec{a} = -\cos t\,\underset{\sim}{j} - \sin t\,\underset{\sim}{k}$$

(**c**) The unit tangent vector $\vec{\mathbf{T}}(t)$.

$$\|\vec{v}\| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}$$

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}}\left(\underset{\sim}{i} - \sin t\,\underset{\sim}{j} + \cos t\,\underset{\sim}{k}\right)$$

(d) The principle normal vector $\vec{\mathbf{N}}(t)$.

$$\vec{N} = \frac{\frac{d}{dt}\vec{T}(t)}{\|\frac{d}{dt}\vec{T}(t)\|} = \frac{\frac{1}{\sqrt{2}}(-\cos t\,\underset{\sim}{j} - \sin t\,\underset{\sim}{k})}{1/\sqrt{2}} = -(\cos t\,\underset{\sim}{j} + \sin$$

(e) The speed $\frac{ds}{dt}$.

$$\frac{ds}{dt} = \|\vec{v}\| = \sqrt{2}$$

(f) The arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.

$$\Delta s = \int_0^{\pi/2} \frac{ds}{dt}\,dt = \int_0^{\pi/2} \sqrt{2}\,dt = \frac{\pi}{2\sqrt{2}}$$

Problem #2 (20pts): Let $\overrightarrow{\mathbf{F}} = 2xyz\mathbf{i} + (x^2z + 2z)\mathbf{j} + (x^2y + 2y + 2z)\mathbf{k}$.

(a) Show that $\text{Curl}(\overrightarrow{\mathbf{F}}) = 0$ (note that $\text{Curl}(\overrightarrow{\mathbf{F}})$ may also be written as $\nabla \times \overrightarrow{\mathbf{F}}$).

$$\text{Curl}\,\tilde{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & x^2z+2z & x^2y+2y+2z \end{vmatrix}$$

$$= \underset{\sim}{i}\left[\frac{\partial}{\partial y}(x^2y+2y+2z) - \frac{\partial}{\partial z}(x^2z+2z)\right]$$

$$- \underset{\sim}{j}\left[\frac{\partial}{\partial x}(x^2y+2y+2z) - \frac{\partial}{\partial z}(2xyz)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x^2z+2z) - \frac{\partial}{\partial y}(2xyz)\right]$$

$$= \underset{\sim}{i}\left[(x^2+2) - (x^2+2)\right] - \underset{\sim}{j}\left[(2xy) - (2xy)\right] + \underset{\sim}{k}\left[2xz - 2xz\right]$$

$$= 0 \checkmark$$

(b) Use the method of partial integration to find an f such that $\vec{\mathbf{F}} = \nabla f$.

$$f(x,y,z) = \int_x 2xyz \, dx = x^2yz + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2z + \frac{\partial g}{\partial y} = x^2z + 2z \Rightarrow \frac{\partial g}{\partial y} = 2z$$

$$g = 2zy + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2yz + 2zy + h(z) \right) = x^2y + 2y + 2z$$

$$\cancel{x^2y} + \cancel{2y} + h(z) = \cancel{x^2y} + \cancel{2y} + 2z$$

$$h(z) = z^2$$

$$\boxed{f(z) = x^2yz + 2zy + z^2}$$

Problem #3 (20pts): Recall Green's Theorem:

$$\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int\int_{\mathcal{R}} N_x - M_y \, dA$$

Find a function $N(x, y)$ such that the vector field $\mathbf{F} = 2xy^2\mathbf{i} + N(x, y)\mathbf{j}$ has the property that $\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ is equal to Area($\mathcal{R}$), the area enclosed by $\mathcal{C}$.

Need $N_x - M_y = 1$ $\quad M = 2xy^2$

$N_x - (2xy^2)_y = 1$

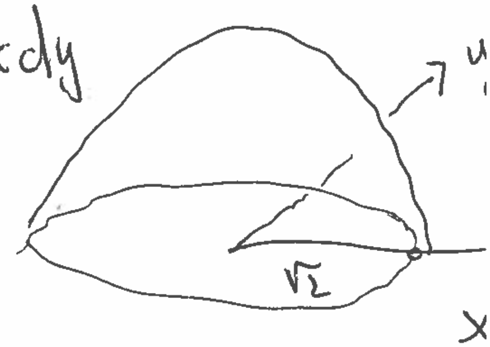
$N_x = 4xy + 1$

$N = \frac{4x^2y}{2} + x + f(y)$ does it

OR $N = 2x^2y + x$ is one choice.

Problem #4 (20pts): Let $\mathcal{S}$ be the portion of the surface $z = 2 - x^2 - y^2$ within the cylinder $x^2 + y^2 \le 2$. If $\mathcal{S}$ has density $\delta = \frac{r^2}{\sqrt{1+4r^2}} \frac{kg}{m^2}$, where $r = \sqrt{x^2+y^2}$, find the mass of the surface.

$$\text{Mass} = \iint_{D} \delta \, dS = \iint_{D} \delta \, \alpha(x,y)\, dx\, dy$$



$$\alpha = \|f\| \qquad f = z - 2 + x^2 + y^2$$

$$\overrightarrow{\nabla f} = (1, 2x, 2y) \qquad \|f\| = \sqrt{1 + 4(x^2+y^2)}$$

$$= \sqrt{1+4r^2}$$

$$\text{Mass} = \iint_{x^2+y^2 \le 2} \frac{r^2}{\cancel{\sqrt{1+4r^2}}} \cdot \cancel{\sqrt{1+4r^2}} \, dx\, dy$$

$$dx\,dy = r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} r^2 \, r \, dr\, d\theta = \int_0^{2\pi} \left.\frac{r^4}{4}\right]_0^{\sqrt{2}} d\theta$$

$$= \cancel{2}\pi \cdot \frac{(\sqrt{2})^4}{\cancel{4}_{\,2}} = \pi\, 2^2 = \boxed{4\pi}$$

Problem #5 (20pts): Recall Newton's inverse square gravitational force law $\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3}$, where $\vec{\mathbf{r}} = (x, y, z)$, $r = \sqrt{x^2 + y^2 + z^2}$ and the constants have been set equal to one.

(a) Show $\nabla\left(\frac{1}{r}\right) = \vec{\mathbf{F}}$.

$$\frac{\partial}{\partial x} r = \frac{x}{r} \Rightarrow \frac{\partial}{\partial x} r^{-1} = (-1) r^{-2} \frac{x}{r} = -\frac{x}{r^3}$$

$$\frac{\partial}{\partial y} r = \frac{y}{r} \Rightarrow \frac{\partial}{\partial y} r^{-1} = -\frac{y}{r^3} \; ; \; \frac{\partial}{\partial z} r^{-1} = -\frac{z}{r^3}$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\overrightarrow{(x,y,z)}}{r^3} = -\frac{\vec{r}}{r^3} \checkmark$$

(b) Use (a) to evaluate the work $\int_C \vec{\mathbf{F}} \cdot \mathbf{T}\, ds$ done by the gravitational force for motion along a curve C taking point A to point B. Hint: When writing the answer, use $r(A) = ||A||$.

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \nabla\left(\frac{1}{r}\right) \cdot \vec{T}\, ds = \frac{1}{r(B)} - \frac{1}{r(A)}$$

$\vec{F}$ is conservative

(c) Assume a parameterization $\overrightarrow{\mathbf{r}}(t)$ of C, in which $\overrightarrow{\mathbf{r}}(t_A) = A$, $\overrightarrow{\mathbf{r}}(t_B) = B$, $t_A \leq t \leq t_B$ and assume $\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}}$, where $\overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{r}}''$ is the acceleration. Show that:

$$\int_C \overrightarrow{\mathbf{F}} \cdot \mathbf{T}\, ds = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2.$$

Hint:

$$\int \overrightarrow{\mathbf{r}}''(t) \cdot \overrightarrow{\mathbf{r}}'(t)\, dt = \frac{1}{2}||\overrightarrow{\mathbf{r}}'(t)||^2 + \text{constant}$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_{t_A}^{t_B} m\, \vec{r}''(t) \cdot \vec{v}(t)\, dt$$

$$= \int_{t_A}^{t_B} m \frac{1}{2} \frac{d}{dt}\, \vec{v} \cdot \vec{v}\, dt = \frac{m}{2} V_B^2 - \frac{m}{2} V_A^2 \checkmark$$

(d) Use parts (b) and (c) to show $\mathrm{E}(t) = \frac{1}{2}mv(t)^2 - \frac{1}{r(t)}$ is constant all along the orbit, that is, we have a notion of conservation of energy for planetary motion. Hint: Consider the indefinite integrals of (b) and (c).

<u>d</u>