Name: Solutions

Student ID#: ____________

Section: ____________

Final Exam

Friday December 10, 1-3pm

MAT 21D, Temple, Fall 2021

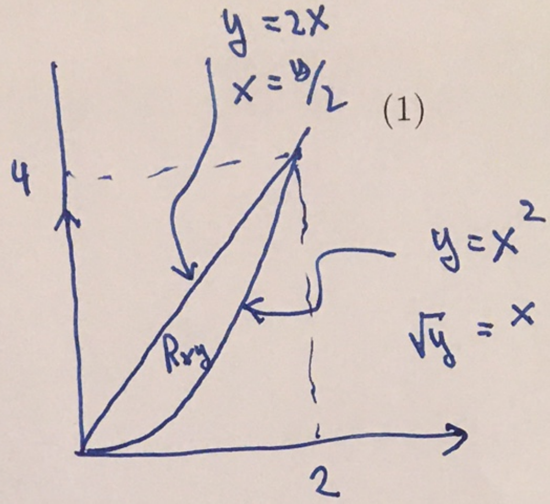
Print name and ID's clearly. Have student ID ready. Write solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Correct answers with no supporting work will not receive full credit. No calculators, notes, books, cellphones...allowed.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
Total		200

Problem #1 (20pts): **(a)** Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$x^2 = 2x \Rightarrow x = 2$

$$\int_0^2 \int_{x^2}^{2x} dy dx. \quad (1)$$



$$\int_0^2 \int_{x^2}^{2x} dy\,dx = \int_0^2 y\Big]_{x^2}^{2x} dx$$

$$= \int_0^2 (2x - x^2)\,dx = \frac{2x^2}{2} - \frac{x^3}{3}\Big]_0^2$$

$$= \frac{\cancel{2}(2)^2}{\cancel{2}} - \frac{2^3}{3} = 4 - \frac{8}{3} = \frac{12-8}{3} = \frac{4}{3}$$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

$$\int_0^4 \int_{y/2}^{\sqrt{y}} dx\,dy$$

Problem #2 (20pts): Assume the region $\mathbf{R}_{xy}$ of Problem 1 is a metal plate with density $\delta(x, y) = x \sin y$. Set up iterated integrals for the following: (You need not evaluate.)

(a) The total mass M.

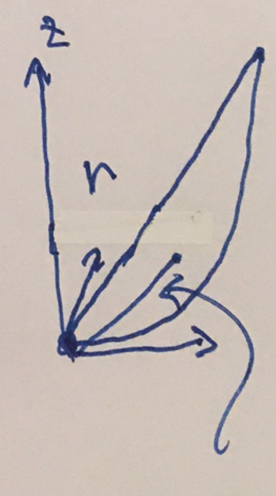
$$M = \int_0^2 \int_{x^2}^{2x} x \sin y \, dy \, dx$$

(b) The coordinates of the center of mass.

$$\bar{x} = \frac{1}{M} \iint_{R_{xy}} x \delta \, dA = \frac{1}{M} \int_0^2 \int_{x^2}^{2x} x^2 \sin y \, dy \, dx$$

$$\bar{y} = \frac{1}{M} \iint_{R_{xy}} y \delta \, dA = \frac{1}{M} \int_0^2 \int_{x^2}^{2x} xy \sin y \, dy \, dx$$

(c) The kinetic energy of rotation about the z-axis.



$$KE = \frac{1}{2} I_z \omega^2 \qquad I_z = \iint_{R_{xy}} r^2 \delta \, dA$$

$$KE = \frac{1}{2} \omega^2 \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) x \sin y \, dy \, dx$$

$$r = \sqrt{x^2 + y^2}$$

Problem #3 (20pts): Let $\mathcal{R}_{xy}$ be the circle of radius $a > 0$. Evaluate the following integral: (Hint: polar coordinates.)

$$\int\int_{\mathcal{R}_{xy}} \cos\left(\frac{x^2+y^2}{a^2}\right) dA.$$

$x = a\cos\theta$
$y = a\sin\theta$

$$= \iint_{R_{r\theta}} \cos \frac{a^2\cos^2\theta + a^2\sin^2\theta}{a^2} \, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{a} \cos(r^2)\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_{r=0}^{r=a} \cos u \, du \, d\theta$$

$u = r^2$

$du = 2r\,dr$

$$= \frac{1}{2}\int_0^{2\pi} \sin u \Big]_{r=0}^{r=a} da = \frac{1}{2} \cdot 2\pi \sin a^2 = \boxed{\pi \sin a^2}$$

Problem #4 (20pts): Assume

$$\vec{\mathbf{F}}(x,y,z) = (ze^{xz})\,\mathbf{i} + z\,\mathbf{j} + (xe^{xz} + y + 2)\,\mathbf{k}.$$

(a) Find Div $\vec{\mathbf{F}}$.

$$\text{Div}\,\vec{F} = M_x + N_y + P_z = z^2 e^{xz} + 0 + x^2 e^{xz} = (z^2 + x^2)\,e^{xz}$$

(b) Find Curl $\vec{\mathbf{F}}$.

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ ze^{xz} & z & xe^{xz} + y + 2 \end{vmatrix}$$

$$= \underset{\sim}{i}(1-1) - \underset{\sim}{j}\left(\frac{\partial}{\partial x}(xe^{xz}) - \frac{\partial}{\partial z}(ze^{xz})\right) + \underset{\sim}{k}(0-0)$$

$$= -\underset{\sim}{j}\left(e^{xz} + xze^{xz} - e^{xz} - xze^{xz}\right) = 0$$

(c) Use the method of partial integration to find an f such that $\mathbf{F} = \nabla f$.

$$f(x,y,z) = \int_x z e^{xz}\, dx = e^{xz} + g(y,z)$$

$$\frac{\partial f}{\partial y} = z = \frac{\partial g}{\partial y} \Rightarrow g = zy + h(z)$$

$$\frac{\partial f}{\partial z} = x e^{xz} + y + 2 = x e^{xz} + y + h'(z) \Rightarrow h(z) = 2z + C$$

$$\boxed{f(x,y,z) = e^{xz} + zy + 2z}$$

(d) Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ along any smooth curve C taking $A = (1, -3, 1)$ to $B = (-1, 2, -1)$.

$$\int_C \vec{F} \cdot \vec{T}\, ds = f(B) - f(A) = f(-1,2,-1) - f(1,-3,1)$$

$$= \cancel{e^{(-1)(-1)}} + (-1)(2) + 2\underbrace{(-1) - \cancel{e^{1\cdot 1}} - (1)(-3) - 2(1)}_{-4}$$

$$= -2 + 3 - 4 = -6 + 3 = -3$$

Problem #5 (20pts): Let $\vec{\mathbf{v}} \equiv \vec{\mathbf{F}} = xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$ be the velocity field of a moving fluid.

(a) Find the unit vector in the direction of the axis of maximal circulation per area at point $P = (1, -1, 2)$.

$$\vec{n} = \frac{\operatorname{Curl} \vec{v}}{\| \operatorname{Curl} \vec{v} \|} \qquad \operatorname{Curl} \vec{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ xz & xy & yz \end{vmatrix} = z\underset{\sim}{i} + x\underset{\sim}{j} + y\underset{\sim}{k}$$

$$\operatorname{Curl} \vec{v}\,(1,-1,2) = \overrightarrow{(2,1,-1)} \qquad \vec{n} = \frac{\overrightarrow{(2,1,-1)}}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}\overrightarrow{(2,1,-1)}$$

(b) Find the maximal circulation per area at point $P = (1, -1, 2)$.

$$\text{Max Circ/Area} = \| \operatorname{Curl} \vec{v} \| = \sqrt{4+1+1} = \sqrt{6}$$

(c) Find the circulation per area around axis $\vec{\mathbf{w}} = \overrightarrow{(1,1,-1)}$ at point $P = (1,-1,2)$.

$$\vec{n} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{\overrightarrow{(1,1,-1)}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}\overrightarrow{(1,1,-1)}$$

$$\frac{\text{Circ}}{\text{Area}} \text{ around } \vec{n} = \text{Curl}\,\vec{v} \cdot \vec{n} = \overrightarrow{(2,1,-1)} \cdot \overrightarrow{(1,1,-1)}\frac{1}{\sqrt{3}}$$

$$= (2+1+1)\frac{1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

(d) Describe all axes $\vec{\mathbf{n}}$ around which there is zero circulation per area at point $P = (1,-1,2)$.

All axes $\perp$ Curl $\vec{v}$ $\Rightarrow$ $\vec{n} : \vec{n} \cdot \text{Curl}\,\vec{v} = 0$

$$\overrightarrow{(n_1, n_2, n_3)} \cdot \overrightarrow{(2,1,-1)} = 0$$

$$\boxed{2n_1 + n_2 - n_3 = 0}$$

Problem #6 (20pts): **(a)** Let $F = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$, let C be a smooth curve that takes A to B, and let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Use Leibniz's substitution principle to show the following are equal:

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\,ds = \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{v}}\,dt = \int_C M dx + N dy + P dz.$$

Let $\vec{r}(t)$, $t_A \le t \le t_B$ be parameterization of C

Liebniz Notation Gives: $d\vec{r} = \vec{v}\,dt = \vec{T}\,ds$

$\frac{d\vec{r}}{dt} = \vec{V} = \frac{ds}{dt}\vec{T}$ so: $\vec{F}\cdot\vec{T}\,ds = \vec{F}\cdot d\vec{r} = \vec{F}\cdot\vec{v}\,dt$ gives 1st three.

Since $\vec{F} = \overrightarrow{(M,N,P)}$, $\vec{F}\cdot d\vec{r} = (M,N,P)\cdot\overrightarrow{(dx,dy,dz)} = Mdx+Ndy+Pdz$ ✓

(b) Find $M(x,y)$ such that the vector field $\vec{\mathbf{F}} = \overrightarrow{(M, xy)}$ has the property that the flux of $\vec{\mathbf{F}}$ through $\mathcal{C}$ equals the area $\mathcal{A}$ enclosed by $\mathcal{C}$ for every simple closed curve $\mathcal{C}$.

(Hint: Recall the divergence form of Green's Theorem: $\int_{\mathcal{C}} \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,ds = \int\int_{\mathcal{A}} Div(\vec{\mathbf{F}})dA$.)

Need: $Div(\vec{F}) = M_x + N_y = 1$. Since $N = xy$,

we need $M_x + x = 1$ or $\frac{\partial M}{\partial x}(x,y) = 1 - x$

$\boxed{M = x - \frac{x^2}{2}}$ suffice

Problem #7 (20pts): (a) The vector field $\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3}$ is Newton's inverse square force field with all constants set equal to one. Recall $\vec{\mathbf{r}} = \overrightarrow{(x, y, z)}$, and $r = \|\vec{\mathbf{r}}\| = \sqrt{x^2 + y^2 + z^2}$.

(a) Calculate $Curl(\vec{\mathbf{F}})$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{x}{r^3} & -\frac{y}{r^3} & -\frac{z}{r^3} \end{vmatrix}$$

use $\frac{\partial}{\partial x} r = \frac{x}{r}$

$\frac{\partial}{\partial y} r = \frac{y}{r}$

$\frac{\partial}{\partial z} r = \frac{z}{r}$

$$= \underset{\sim}{i}\left(-\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right) - \underset{\sim}{j}\left(\frac{\partial}{\partial x}\left(-\frac{z}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right)$$

$$+ \underset{\sim}{k}\left(\frac{\partial}{\partial x}\left(-\frac{y}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right)$$

$$= \underset{\sim}{i}\left((-1)(-3)\frac{z}{r^4}\frac{y}{r} + (-3)\frac{y}{r^4}\frac{z}{r}\right)$$

$$- \underset{\sim}{j}\left(+3\frac{z}{r^4}\frac{x}{r} - 3\frac{x}{r^4}\frac{z}{r}\right)$$

$$+ \underset{\sim}{k}\left(3\frac{y}{r^4}\frac{x}{r} - 3\frac{x}{r^4}\frac{y}{r}\right)$$

$$= 0$$

(**b**) Calculate $\nabla\left(\frac{1}{r}\right)$

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\frac{x}{r}, \quad \frac{\partial}{\partial y}\frac{1}{r} = -\frac{1}{r^2}\frac{y}{r}, \quad \frac{\partial}{\partial z}\frac{1}{r} = -\frac{1}{r^2}\frac{z}{r}$$

So $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ with $\vec{r} = \overrightarrow{(x, y, z)}$

(**c**) Show that $\int_C \vec{\mathbf{F}} \cdot \vec{T} ds = \frac{1}{\|B\|} - \frac{1}{\|A\|}$ for any smooth curve C taking A to B. (You may use any theorem you can state correctly.)

<u>Thm</u>: if $\vec{F} = \nabla f$, then $\int_C \vec{F} \cdot \vec{T} ds = f(B) - f(A)$

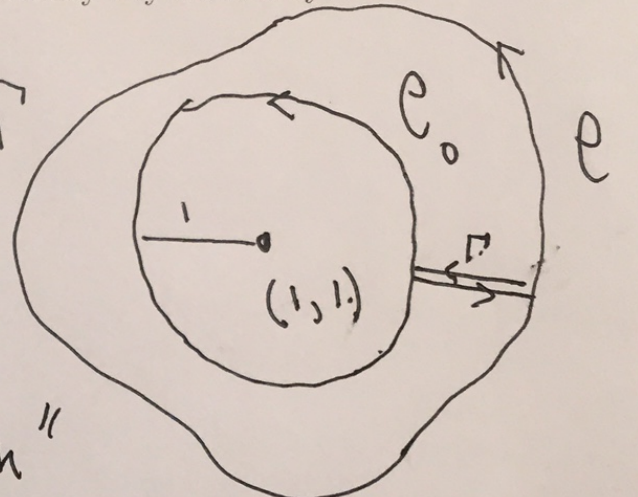
So $\int_C \vec{F} \cdot \vec{T} ds = \frac{1}{r(B)} - \frac{1}{r(A)} = \frac{1}{\|B\|} - \frac{1}{\|A\|}$

$\vec{F} = -\frac{\vec{r}}{r^3}$ $\quad$ $f(\underset{\sim}{x}) = \frac{1}{r(\underset{\sim}{x})}$

Problem #8 (20pts): Let $\vec{\mathbf{F}} = \overrightarrow{(M(x,y), N(x,y))}$ denote a vector field in the plane such that its components M, N are smooth functions, and assume

$$Curl(\vec{\mathbf{F}}) = 0 \text{ for all } (x,y) \neq (1,1).$$

Assume further that $\int_{C_0} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \pi$ where C_0 is the positively oriented unit circle centered at $(1,1)$. Give an argument demonstrating that $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \pi$ for any smooth, positively oriented simple closed curve C surrounding the point $(1,1)$. Draw a picture, and state correctly any theorem you use.



The curve $C - \Gamma - C_0 + \Gamma$ is a simple closed curve whose interior is the region "between" C_0 & C, which excludes $(1,1)$, and excludes Γ. Since this is simply connected, and $Curl\,\vec{F} = 0$ inside, Our theorem says

$$0 = \int_{C-\Gamma-C_0+\Gamma} \vec{F} \cdot \vec{T}\, ds = \int_C - \int_\Gamma - \int_{C_0} + \int_\Gamma$$

so $\displaystyle\int_C \vec{F} \cdot \vec{T}\, ds = \int_{C_0} \vec{F} \cdot \vec{T}\, ds$ ✓

Problem #9 (20pts): Let $\mathcal{S}$ denote the upper hemisphere $x^2+y^2+z^2=4$, $z \geq 0$, let $\mathcal{C}$ denote its boundary circle $x^2+y^2=4$ in the (x,y)-plane, and let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$. Verify Stokes Theorem $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int\int_{\mathcal{S}} Curl\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma$ as follows:

(a) Evaluate $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ directly by parameterization.

Set $x = 2\cos t$, $y = 2\sin t$, $\vec{r}(t) = \overrightarrow{(2\cos t, 2\sin t)}$, $\vec{v}(t) = 2\overrightarrow{(-\sin t, \cos t)}$

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, dt = \int_0^{2\pi} 2\overrightarrow{(\sin t, \cos t)} \cdot 2\overrightarrow{(-\sin t, \cos t)}\, dt$$

$$= -4 \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = -8\pi$$

(b) Find the unit normal $\vec{\mathbf{n}}$ on $\mathcal{S}$.

$$\vec{n}(x,y,z) = \overrightarrow{(x,y,z)} \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{\|\vec{r}\|} \text{ by geometry}$$

OR $\vec{r}(x,y) = \overrightarrow{(x,y,z)}$, $z = \sqrt{4-x^2-y^2}$ $\quad \sqrt{x^2+y^2} \leq 2$

$$\vec{r}_x = \overrightarrow{\left(1, 0, \frac{-x}{z}\right)}, \quad \vec{r}_y = \overrightarrow{\left(0, 1, \frac{-y}{z}\right)}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{x}{z} \\ 0 & 1 & -\frac{y}{z} \end{vmatrix} = \overrightarrow{\left(\frac{x}{z}, \frac{y}{z}, 1\right)}, \quad \|\vec{r}_x \times \vec{r}_y\| = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} = \frac{\sqrt{x^2+y^2+z^2}}{z}$$

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \checkmark$$

(c) Find $\alpha(x, y)$ so that $d\sigma = \alpha(x, y)\, dxdy$ on $\mathcal{S}$.

$$\alpha(x,y) = \|\vec{r}_x \times \vec{r}_y\| = \frac{\sqrt{x^2+y^2+z^2}}{z} \quad \text{with } z = \sqrt{4-x^2-y^2}$$

(d) Use (b) and (c) to evaluate $\int\int_S Curl\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma$ using (x, y) as a coordinate system.

$$Curl\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = -2\underset{\sim}{k}$$

$$\iint_S Curl\vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{xy}} \overrightarrow{(0,0,-2)} \cdot (\vec{r}_x \times \vec{r}_y)\, dxdy$$

circle radius 2

$$= \iint_{R_{xy}} -2\, dxdy = -2\pi(2)^2 = -8\pi$$ ✓

Problem #10 (20pts): Recall the Divergence Theorem:

$$\int\!\!\int_{\mathcal{S}} \vec{\mathbf{F}}\cdot\vec{\mathbf{n}}\,d\sigma = \int\!\!\int\!\!\int_{\mathcal{V}} Div(\vec{\mathbf{F}})\,dV \qquad (2)$$

where $\mathcal{S}$ is a closed surface bounding the volume $\mathcal{V}$. Let $\mathcal{S}$ be the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Assuming $\vec{\mathbf{F}}=\rho^2(\mathbf{i}+\mathbf{j}+\mathbf{k})$ with $\rho=\sqrt{x^2+y^2+z^2}$, use the Divergence Theorem to evaluate the flux of $\vec{\mathbf{F}}$ through $\mathcal{S}$ (on the left hand side) by evaluating the triple integral on the right hand side. (Hint: Use the substitution $u=ax, v=by, w=cz$.)

Set $x = au,\ y = bv,\ z = cw$ $\quad \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$ gives

$$\iiint_{V_{xyz}} Div\,\vec{F}\,dV = \iiint_{V_{uvw}} Div\,F\,(abc)\,du\,dv\,dw = abc\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{1} Div\,F\,\bar{\rho}^2\sin\varphi\,d\bar{\rho}\,d\varphi\,d\theta$$

unit ball radius $\bar{\rho}=1$

$u = \bar{\rho}\sin\varphi\cos\theta$

$v = \bar{\rho}\sin\varphi\sin\theta$

$w = \bar{\rho}\cos\varphi$

$dV_{uvw} = \bar{\rho}^2\sin\varphi\,dV_{\bar{\rho}\varphi\theta}$

Now $Div\,\vec{F} = 2\rho(\rho_x+\rho_y+\rho_z)$

$= 2\rho\left(\frac{x}{\rho}+\frac{y}{\rho}+\frac{z}{\rho}\right)$

$= 2(au+bv+cz)$

$$\iiint_V div\,\vec{F}\,dV = 2abc\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{1} a\bar{\rho}^3\sin^2\varphi\cos\theta + b\bar{\rho}^3\sin^2\varphi\sin\theta + c\bar{\rho}^3\sin\varphi\cos\varphi\;d\bar{\rho}\,d\varphi\,d\theta$$

__But__: each integral is zero by reversing order of int

i.e. $\int_0^{2\pi}\cos\theta\,d\theta = 0 = \int_0^{2\pi}\sin\theta\,d\theta$ & $\int_0^{\pi}\sin\varphi\cos\varphi\,d\varphi = 0$. $\therefore$ Ans 0