§16.7

Meaning of Curl--Final Exam Problem

Stokes Thm: $\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \iint_{\mathcal{S}} \operatorname{Curl}\vec{F}\cdot\vec{n}\,d\sigma$

Div Thm: $\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma = \iiint_{\mathcal{D}} \operatorname{div}\vec{F}\,dV$

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \underset{\sim}{i}(P_y - N_z) - \underset{\sim}{j}(P_x - M_z) + \underset{\sim}{k}(N_x - M_y)$$

$\operatorname{div}\vec{F} = M_x + N_y + P_z$

Ex Find $\operatorname{Curl}\vec{F}$, $\operatorname{div}\vec{F}$ at $\underset{\sim}{x}_0 = (1,-1,z)$ if $\vec{F} = x\underset{\sim}{i} + xyz\underset{\sim}{k}$

Q: what does $\operatorname{Curl}\vec{F}$ & $\operatorname{div}\vec{F}$ measure about $\vec{F}$ @ $\underset{\sim}{x}_0$?

- LHS we understand physically –

$\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ = circulation in $\vec{F}$ around $\mathcal{C}$

$\iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma$ = Flux of $\vec{F}$ thru $\mathcal{S}$

$= \frac{\text{mass}}{\text{time}}$ out of $\mathcal{S}$ if $\vec{F} = \delta\vec{v}$

Q: What do Curl $\vec{F}$ and div $\vec{F}$ measure?

Ans: we can use Stokes & Div to get physical meaning of Curl $\vec{F}$ & div $\vec{F}$:

Before. $\text{div}\,\vec{F} = \dfrac{\text{flux of } \vec{F}}{\text{volume}}$

What about Curl $\vec{F}$?

Claim: Curl $\vec{F}$ has a length $|\text{Curl}\,\vec{F}|$ and direction $\dfrac{\text{Curl}\,\vec{F}}{|\text{Curl}\,\vec{F}|}$

① Curl $\vec{F}$ points in direction of the axis around which the fluid has the max circulation per area

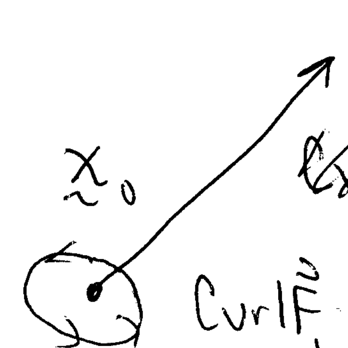
② $|\text{Curl}\,\vec{F}|$ = magnitude of the maximum circulation per area

(3) $\operatorname{Curl}\vec{F}\cdot\vec{n}$ = Circulation per area around the axis $\vec{n}$ at $\underset{\sim}{x}_0$

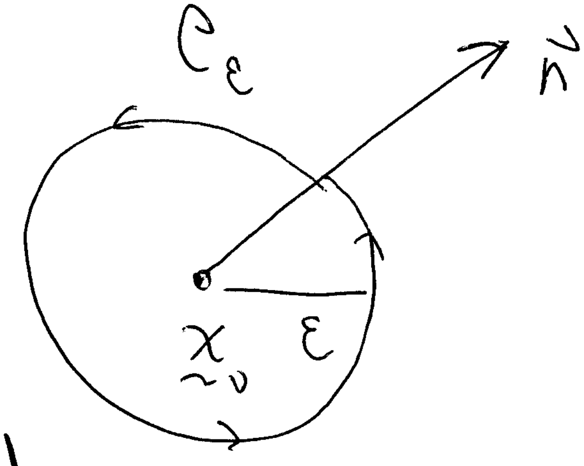
evaluated at $\underset{\sim}{x}_0 = (x_0, y_0, z_0)$

This interpretation follows from Stokes Thm

- Choose unit vector $\vec{n}$ positioned at $\underset{\sim}{x}_0$.
- Let C_ε denote the circle of center $\underset{\sim}{x}_0$, radius ε.
- Let D_ε be the disk with bounday C_ε



$\operatorname{Curl}\vec{F}\cdot\vec{n}$ = circulation per area around C



Stokes $\Rightarrow$

$$\oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds = \iint_{D_\varepsilon} \operatorname{Curl}\vec{F}\cdot\vec{n}\,d\sigma$$

$$\oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds = \iint_{D_\varepsilon} \text{Curl}\vec{F}\cdot\vec{n}\,d\sigma$$

Circulation of $\vec{F}$ around C_ε

Curl $\vec{F}\cdot\vec{n}$ almost constant on D_ε for $\varepsilon \ll 1$

$$\therefore \oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds \approx \text{Curl}\vec{F}\cdot\vec{n} \iint_{D_\varepsilon} d\sigma$$

$$= \text{Curl}\,\vec{F}\cdot\vec{n}\,(\text{Area of } D_\varepsilon)$$

$$\text{Curl}\vec{F}\cdot\vec{n} \approx \frac{1}{\text{Area } D_\varepsilon} \oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds$$

$$\text{Curl}\vec{F}\cdot\vec{n} = \lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \oint_{C_\varepsilon} \vec{F}\cdot\vec{T}\,ds$$

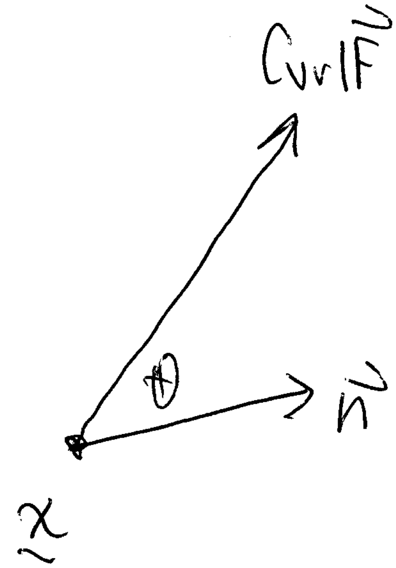
= "Circulation per area around a disc $\perp$ to $\vec{n}$"

Ex: Show that $\text{Curl}\vec{F}$ gives the axis around which $\frac{\text{Circulation}}{\text{Area}}$ is maximal:

Soln: $\text{Curl}\vec{F}\cdot\vec{n} = |\text{Curl}\vec{F}||\vec{n}|\cos\theta$

$= |\text{Curl}\vec{F}|\cos\theta$

$\cos\theta = 1$ when $\theta = 0 \Rightarrow \vec{n} = \dfrac{\text{Curl}\vec{F}}{|\text{Curl}\vec{F}|}$



$\therefore$ $\frac{\text{Circulation}}{\text{Area}}$ around axis $\vec{n}$

$= \text{Curl}\vec{F}\cdot\vec{n}$

is maximal when $\vec{n} = \dfrac{\text{Curl}\vec{F}}{|\text{Curl}\vec{F}|}$

$\Rightarrow$ "max'al circulation around axis in direction of Curl"

Ex: ~~Ex~~ Find the maximum circulation per area of $\vec{F}$ at $\underset{\sim}{x}_0$

Soln: $\dfrac{\text{Circulation}}{\text{Area}}$ around $\vec{n}$ = $\text{Curl}\,\vec{F}\cdot\vec{n}$

Max occurs when $\vec{n} = \dfrac{\text{Curl}\,\vec{F}}{|\text{Curl}\,\vec{F}|}$

$\therefore$ $\text{Max}\,\dfrac{\text{Circulation}}{\text{Area}} = \text{Curl}\,\vec{F}\cdot\dfrac{\text{Curl}\,\vec{F}}{|\text{Curl}\,\vec{F}|} = |\text{Curl}\,\vec{F}|$

Conclude: $\text{Curl}\,\vec{F}$ points in direction of the axis around which max $\dfrac{\text{circulation}}{\text{area}}$ occurs

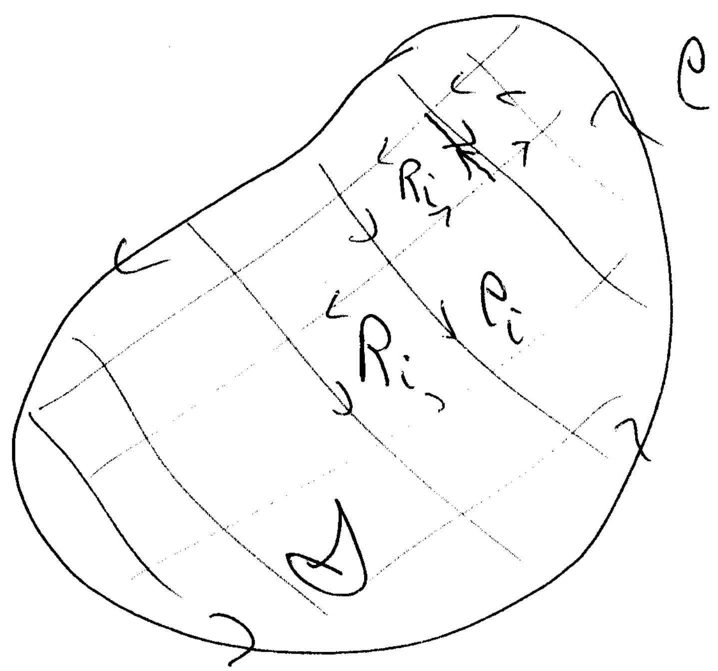
$|\text{Curl}\,\vec{F}|$ = max circulation per area.

Picture

$$\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \sum_i \underbrace{\int_{\mathcal{C}_i} \vec{F}\cdot\vec{T}\,ds}_{\text{Circulatn around } \mathcal{C}_i} \approx \sum_i \underbrace{\text{Curl}\vec{F}\cdot\vec{n}_i}_{\text{Circulation per area around } \vec{n}_i} \underbrace{\Delta\sigma_i}_{\text{area}}$$

$$\approx \iint_{\mathcal{S}} \text{Curl}\vec{F}\cdot\vec{n}\,d\sigma$$

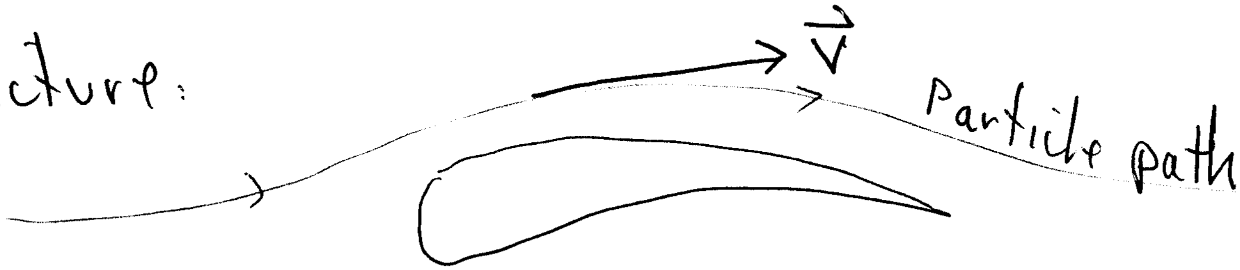
"Total circulatn around $\mathcal{C}$
= integral of circulation
per area over $\mathcal{S}$"



Ex: A fluid is moving with velocity vector

$$\vec{F} = \vec{V} = x\,\underset{\sim}{i} + xyz\,\underset{\sim}{k}$$

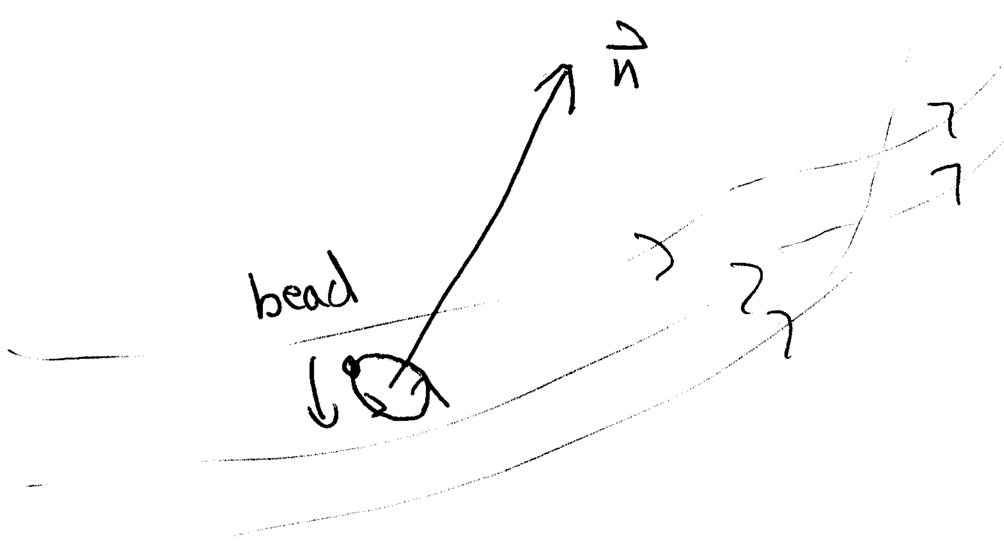
Picture:



(1) Find the axis around which the fluid is circulating most rapidly at $P = (1, -1, 2)$

(2) Find maximal circulation rate.

(3) Find the # of rev's/sec at which a bead would circulate around circle of radius ε, center (x_0, y_0, z_0) $\perp$ to axis $\vec{n}$



(1) $\text{Curl}\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ x & 0 & xyz \end{vmatrix} = \underset{\sim}{i}(xz) - \underset{\sim}{j}(yz) + \underset{\sim}{k}(0)$

$\text{Curl}\vec{F} = xz\,\underset{\sim}{i} - yz\,\underset{\sim}{j} = \overrightarrow{(xz, -yz, 0)}$

@ $P = (1, -1, 2)$, $\text{Curl}\vec{F} = (2)\,\underset{\sim}{i} + (2)\,\underset{\sim}{j} = 2\overrightarrow{(1,1,0)}$

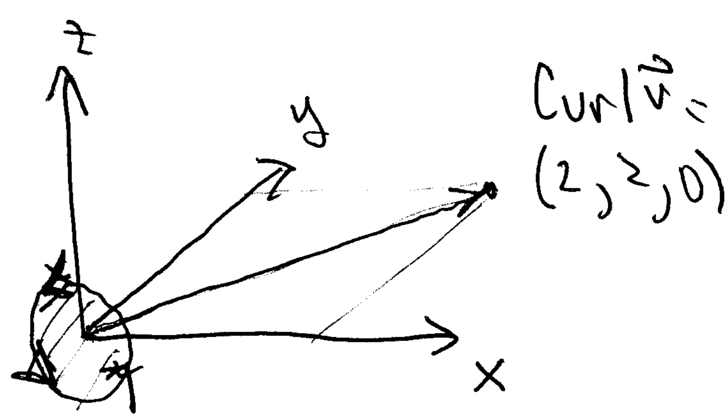
$\therefore$ Axis of maximal circulation is

$$\vec{n} = \frac{\text{Curl}\vec{F}}{\|\text{Curl}\vec{F}\|} = \frac{2\overrightarrow{(1,1,0)}}{2\sqrt{1^2+1^2+0^2}} = \frac{\overrightarrow{(1,1,0)}}{\sqrt{2}}$$

(2) Maximal circulation rate is $|\text{Curl}\vec{F}|$

$$\|\text{Curl}\vec{F}| = 2\sqrt{1^2+1^2+0} = 2\sqrt{2}$$

Which Way?
By RHR



③ |Curl| gives $\frac{\text{circulation}}{\text{Area}}$

$$|\text{Curl}\,\vec{v}| = \left| \frac{\int_{C_\varepsilon} \vec{v}\cdot\vec{T}\,ds}{\text{Area}\,C_\varepsilon} \right| = \text{max'al}\ \frac{\text{circulation}}{\text{area}}$$

Q: How does circulation translate into rev's per second around C_ε?

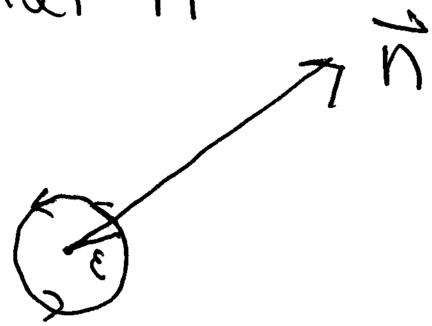
check units: $[\text{Curl}\,\vec{v}] = [\partial_x v] = \frac{1}{L}\frac{L}{T} = \frac{1}{T}$

$\Rightarrow$ has dimensions of freq $= \frac{\text{cycles}}{\text{sec}}$

Another Way - Stokes Thm $\Rightarrow$

$$[\text{Curl}\,\vec{v}] = \left[\frac{\int_{C_\varepsilon} \vec{v}\cdot\vec{T}\,ds}{\text{Area}\,C_\varepsilon} \right] = \frac{\frac{L}{T}\cdot L}{L^2} = \frac{1}{T} \checkmark$$

So let C_ε be small circle center $(x_0, y_0, z_0) = P_0$ with normal $\vec{n}$



- $\text{Curl}\,\vec{V}\cdot\vec{n} = \dfrac{\text{Circulation}}{\text{Area}}$ around $\vec{n}$

$$\cong \underbrace{\frac{1}{\pi\varepsilon^2}}_{\text{Area}} \int_{C_\varepsilon} \vec{V}\cdot\vec{T}\,ds$$

$$\int_{C_\varepsilon} \vec{V}\cdot\vec{T}\,ds \underset{ds=\varepsilon\,d\theta}{\uparrow=} \int_0^{2\pi} (\vec{V}\cdot\vec{T})\,\varepsilon\,d\theta$$

$$\approx 2\pi\varepsilon\,\frac{1}{2\pi}\int_0^{2\pi} \underbrace{\vec{V}\cdot\vec{T}}(\theta)\,d\theta$$

$\vec{V}\cdot\vec{T} = \dfrac{\text{dist.}}{\text{time}}$ around C_ε

$$\approx 2\pi\varepsilon\,\bar{V}$$, $\bar{V}$ = average velocity around C_ε

$\therefore \ \text{Curl}\,\vec{v}\cdot\vec{n} = \dfrac{2\pi\varepsilon}{\pi\varepsilon^2}\,\bar{v} = \dfrac{2}{\varepsilon}\,\bar{v}$

<u>Now</u>: at average velocity $\bar{v}$, the bead make one revolution around $2\pi\varepsilon$ in time

$$\underbrace{2\pi\varepsilon = \bar{v}\,T}_{\text{dist} = \text{vel} \times \text{time}} \iff T = \frac{2\pi\varepsilon}{\bar{v}}$$

time of one revolution

$$\therefore \quad \frac{\#\,\text{rev}}{\text{sec}} = \overbrace{\frac{1}{\text{time of one rev}} = \frac{1}{T}}^{\#\text{ of times } T \text{ in 1sec}} = \frac{\bar{v}}{2\pi\varepsilon} = \omega$$

$$\omega = \frac{\bar{v}}{2\pi\varepsilon} = \frac{1}{2\pi\varepsilon}\cdot\frac{\varepsilon}{2}\,\text{Curl}\,\vec{v}\cdot\vec{n} = \frac{1}{4\pi}\,\text{Curl}\,\vec{v}\cdot\vec{n}$$

rev/sec

Our example: $P = (1, -1, 2)$ ...

• For $\vec{n} = \dfrac{\text{Curl}\,\vec{v}}{|\text{Curl}\,\vec{v}|}$, $\text{Curl}\,\vec{v}\cdot\vec{n} = |\text{Curl}\,\vec{v}|$

$= 2\sqrt{2} \Rightarrow \text{max freq} = \dfrac{1}{4\pi}\cdot 2\sqrt{2} = \dfrac{\sqrt{2}}{2\pi}\ \dfrac{\text{rev}}{\text{sec}}$