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Student ID#: Solutions

Section: ____________

Midterm Exam 2

Monday March 2

MAT 21D, Temple, Winter 2015

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): A particle moves along a trajectory given by

$$\mathbf{r}(t) = r_0 \cos \omega t \ \mathbf{i} + r_0 \sin \omega t \ \mathbf{j} + (t-1) \ \mathbf{k}.$$

(a) Find the velocity vector $\mathbf{v}(t)$ and the speed $v = \frac{ds}{dt}$.

$$\vec{r}'(t) = \vec{v}(t) = -\omega r_0 \sin \omega t \underset{\sim}{i} + \omega r_0 \cos \omega t \underset{\sim}{j} + 1 \underset{\sim}{k}$$

$$v = \frac{ds}{dt} = \|\vec{v}\| = \sqrt{\omega^2 r_0^2 + 1}$$

(b) Express ω in terms of r_0 and v.

$$v = \sqrt{\omega^2 r_0^2 + 1} \Leftrightarrow v^2 = \omega^2 r_0^2 + 1 \Leftrightarrow \frac{v^2 - 1}{r_0^2} = \omega^2$$

$$\omega \ \frac{\sqrt{v^2 - 1}}{r_0}$$

(c) Find the acceleration vector $\mathbf{a}(t)$

$$\vec{a}(t) = \vec{v}'(t) = -\omega^2 r_0 \cos \omega t \underset{\sim}{i} - \omega^2 r_0 \sin \omega t \underset{\sim}{j}$$

(d) Find the unit tangent vector $\mathbf{T}(t)$

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = -\frac{\omega r_0 \sin \omega t}{\sqrt{1 + \omega^2 r_0^2}} \underset{\sim}{i} + \frac{\omega r_0 \cos \omega t}{\sqrt{1 + \omega^2 r_0^2}} \underset{\sim}{j} + \frac{1}{\sqrt{1 + \omega^2 r_0^2}} \underset{\sim}{k}$$

(e) Find the unit normal $\mathbf{N}(t)$ $\quad \vec{N} = \dfrac{\frac{d\vec{T}}{dt}}{\left\| \frac{d\vec{T}}{dt} \right\|}$

$$\frac{d\vec{T}}{dt} = -\frac{\omega^2 r_0}{\sqrt{1+\omega^2 r_0^2}} \cos\omega t \, \underset{\sim}{i} - \frac{\omega^2 r_0 \sin\omega t}{\sqrt{1+\omega^2 r_0^2}} \, \underset{\sim}{j}$$

$$\vec{N} = -\cos\omega t \, \underset{\sim}{i} - \sin\omega t \, \underset{\sim}{j}$$

(f) Find the curvature $\kappa(t)$. $\quad \kappa v = \left\| \frac{d\vec{T}}{dt} \right\| = \dfrac{\omega^2 r_0}{\sqrt{1+\omega^2 r_0^2}}$

$$\kappa = \frac{1}{v} \left\| \frac{d\vec{T}}{dt} \right\| = \frac{1}{\left(\sqrt{\omega^2 r_0^2 + 1}\right)} \frac{\omega^2 r_0}{\sqrt{1+\omega^2 r_0^2}} = \frac{\omega^2 r_0}{1+\omega^2 r_0^2}$$

(g) Find the length of the component of $\mathbf{a}(t)$ in direction of $\mathbf{N}$.

$$\vec{a} \cdot \vec{N} = \omega^2 r_0 \qquad \underline{\underline{\text{or}}} \qquad \vec{a} \cdot \vec{N} = v^2 \kappa = \cancel{(1+\omega^2 r_0^2)} \cdot \frac{\omega^2 r_0}{\cancel{1+\omega^2 r_0^2}}$$

Problem #2 (20pts): ((a) Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field, where M, N, P are assumed to be given functions of (x, y, z). Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t)$ denotes any parameterization of curve C.)

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \mathbf{v}\, dt = \int_C M dx + N dy + P dz.$$

Given $\vec{r}(t)$ we have: $\frac{d\vec{r}}{dt} = \vec{v} = \frac{ds}{dt}\vec{T}$

Thus $\vec{T} ds = \vec{v}\, dt = d\vec{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$

$$\therefore \int_C \vec{F} \cdot \underbrace{\vec{T}\, ds}_{\vec{v}\, dt} = \int_C \vec{F} \cdot \underbrace{\vec{v}\, dt}_{d\vec{r}} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \overrightarrow{(M, N, P)} \circ \overrightarrow{(dx, dy, dz)}$$

$$= \int_C M dx + N dy + P dz$$

(b) Assume further that $F = -\nabla U$ for some scalar function $U(x, y, z)$, and that $\mathbf{F} = m\mathbf{a}$ is the total force creating the motion of a particle along a curve $\mathbf{r}(t)$ between two points of motion $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$. State and prove the principle of Conservation of Energy. Give complete arguments. (Hint: evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ two different ways.)

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= -(U(B) - U(A))$$

If $\vec{F} = m\vec{a} = m\vec{v}\,'(t)$ then:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b m\vec{v}\,'(t) \cdot \vec{v}(t)\, dt = \frac{1}{2}\int_a^b m \frac{d}{dt}(\vec{v}(t) \cdot \vec{v}(t))\, dt$$

$$= \frac{1}{2} m v(b)^2 - \frac{1}{2} m v(a)^2$$

$\therefore$ subtracting: $\frac{1}{2} m v(b)^2 + U(B) = \frac{1}{2} m v(a)^2 + U(A)$

$\Rightarrow$ Energy = KE + PE constant as B was arbitrary

Problem #3 (20pts): Let

$$\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 - 3)\mathbf{k}.$$

(a) Find Curl $\mathbf{F} = \nabla \times \mathbf{F}$.

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2+2yz & y^2-3 \end{vmatrix} = \underset{\sim}{i}\left(\cancel{2y-2y}\right) - \underset{\sim}{j}\left(\cancel{0-0}\right)$$

$$+\underset{\sim}{k}\left(\cancel{2x-2x}\right) = 0 \checkmark$$

(b) Find f such that $\mathbf{F} = \nabla f$.

$$f(x,y,z) = \int_x 2xy\,dx = x^2 y + \underbrace{g(y,z)}$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} \stackrel{?}{=} x^2 + 2yz \Rightarrow \frac{\partial g}{\partial y} = 2yz$$

$$g = \int_y 2y\,dy = \overbrace{y^2 z + h(z)}$$

$$\frac{\partial f}{\partial z} = y^2 + h'(z) \stackrel{?}{=} y^2 - 3 \qquad h'(z) = -3 \quad h(z) = -3z$$

$$\boxed{f(x,y,z) = x^2 y + y^2 z - 3z}$$

(c) Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\; ds$ along any smooth curve C taking $A = (1, -1, 2)$ to $B = (-1, 1, -1)$.

$$\int_C \vec{F}\cdot\vec{T}\,ds = f(-1,1,-1) - f(1,-1,2)$$

$$= \{(-1)^2 1 + 1^2(-1) - 3(-1)\} - \{1^2(-1) + (-1)^2 2 - 3\cdot 2\}$$

$$= \cancel{1} - \cancel{1} + 3 + 1 - 2 + 6 = 10 - 2 = \boxed{8}$$

(d) Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\; ds$ by *parameterization* along the curve $\mathcal{C}$ given by $y = x^2$, $z = 0$ between $0 \le x \le 1$.

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \overrightarrow{(2t\cdot t^2, t^2, t^4-3)}\cdot\overrightarrow{(1, 2t, 0)}\,dt$$

$$\vec{r}(t) = \overrightarrow{(t, t^2, 0)} \qquad \vec{v} = \overrightarrow{(1, 2t, 0)}$$

$$0 \le t \le 1$$

$$= \int_0^1 2t^3 + 2t^3\,dt = \int_0^1 6t^3\,dt = \frac{6}{4}t^4\Big]_0^1$$

$$= \boxed{\frac{3}{2}}$$

Problem #4 (20pts): Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ denote the position vector of a point (x, y, z).

(a) Derive the formula $\frac{\partial}{\partial x}\|\mathbf{r}\| = \frac{x}{\|\mathbf{r}\|}$ where $\|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$, and write down the corresponding formulas for $\frac{\partial}{\partial y}\|\mathbf{r}\|$ and $\frac{\partial}{\partial z}\|\mathbf{r}\|$.

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\|\vec{r}\|}$$

$$\frac{\partial}{\partial y}\|\vec{r}\| = \frac{y}{\|\vec{r}\|} \qquad \frac{\partial}{\partial y}\|\vec{r}\| = \frac{z}{\|\vec{r}\|}$$

(b) Use **(a)** obtain a formula for $\nabla\frac{1}{\|\mathbf{r}\|}$.

$$\nabla\frac{1}{\|\vec{r}\|} = -\frac{\vec{r}}{\|\vec{r}\|^3}$$ by chain rule –

$$\nabla\left(\frac{1}{\|\vec{r}\|}\right) = -\frac{1}{\|\vec{r}\|^2}\nabla\|\vec{r}\| = -\frac{\vec{r}}{\|\vec{r}\|^3}$$

(c) What is the energy that is conserved all along the planets orbit if the total force on it is the Newtonian force $\mathbf{F} = M_P\mathbf{a} = -G\frac{M_P M_S}{\|\mathbf{r}\|^3}\mathbf{r}$?

$$\vec{F} = GM_pM_s\nabla\frac{1}{\|\vec{r}\|} \quad \text{so} \quad U = -\frac{GM_pM_s}{\|\vec{r}\|} = -f$$

$$\text{Energy} = \frac{1}{2}M_pv^2 + U(\vec{r}) \qquad \vec{F} = \nabla f = -\nabla U$$

$$\boxed{E = \frac{1}{2}M_pv^2 - \frac{GM_pM_s}{\|\vec{r}\|}} \leftarrow \text{constant}$$

Problem #5 (20pts): Let $\mathcal{C}$ denote a (simple) closed curve in the xy-plane surrounding a region $\mathcal{R}$ of area $A = 5$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds$ where $\mathbf{F} = (y - x)\mathbf{i} - (x + y)\mathbf{j}$. (Hint: Use Green's Theorem.)

$$\iint_R \operatorname{Curl} \vec{F} \cdot \underset{\sim}{k} \, dA = \int_C \vec{F} \cdot \vec{T} \, ds$$

$$\operatorname{Curl} \vec{F} \cdot \underset{\sim}{k} = N_x - M_y = -1 - 1 = -2$$

$$\iint_R \operatorname{Curl} \vec{F} \cdot \underset{\sim}{k} \, dA = -2 \iint_R dA = -2A = -10$$

$$\int_C \vec{F} \cdot \vec{T} \, ds = -10 \checkmark$$