A History of Vector Analysis

Michael J. Crowe
Distinguished Scholar in Residence
Liberal Studies Program and Department of Mathematics
University of Louisville
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Introduction

Permit me to begin by telling you a little about the history of the book[1] on which this talk[2] is based. It will help you understand why I am so delighted to be presenting this talk.

On the very day thirty-five years ago when my *History of Vector Analysis* was published, a good friend with the very best intentions helped me put the book in perspective by innocently asking: "Who was Vector?" That question might well have been translated into another: "Why would any sane person be interested in writing such a book?" Moreover, a few months later, one of my students recounted that while standing in the corridor of the Notre Dame Library, he overheard a person expressing utter astonishment and was staring at the title of a book on display in one of the cases. The person was pointing at my book, and asking with amazement: "Who would write a book about that?" It is interesting that the person who asked "Who was Vector?" was trained in the humanities, whereas the person in the library was a graduate student in physics. My student talked to the person in the library, informing him he knew the author and that I appeared to be reasonably sane. These two events may suggest why my next book was a book on the history of ideas of extraterrestrial intelligent life.

My *History of Vector Analysis* did not fare very well with the two people just mentioned, nor did it until now lead to any invitations to speak. The humanities departments at Notre Dame assumed that my subject was too technical, the science and math departments must have assumed that it was not technical enough. In any case, never in the thirty-five intervening years did I ever have occasion to talk on my topic. My response when recently asked to talk about the subject was partly delight—I had always wanted to do this—but also some hesitation—this was a topic I researched nearly forty years ago! But it has turned out to be fun.

Publishing the book has also proved interesting. Although it is not for everyone, the hardbound printing of about 1200 copies gradually nearly sold out, based partly on a number of very favorable reviews. It is rare that academic books sell that many copies. As it was about to go out of print, I hit on the idea of asking Dover whether they would want to take it over. This resulted in its re-publication in 1985 with a new preface updating the bibliography; by that time, there had appeared a few dozen papers and books shedding new light on various aspects of the subject. In the early 1990s, a curious development occurred. Nearly twenty-five years after the book had been published, a research center in Paris (La Maison des Sciences de l'Homme) announced a prize competition for a study on the history of complex and hypercomplex numbers). As you can imagine, I was quite pleased to submit my book. Some months later I was notified that I was being awarded a Jean Scott Prize, which included a check for $4000. At this point, Dover decided to do a new printing of the book, which includes an announcement of the prize. In any case, the book has now been continuously in print for 35 years and has led to all sorts of interesting letters and exchanges.

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[1]This talk is based on the following book: Michael J. Crowe, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System* (Notre Dame, Indiana: University of Notre Dame Press, 1967); paperback edition with a new preface (New York: Dover, 1985); another edition with new introductory material (New York: Dover, 1994). Quotations not fully referenced in this paper are fully referenced in that volume.

[2]Warm thanks to Professor Richard Davitt of the Department of Mathematics at the University of Louisville for his very helpful comments on drafts of this presentation.

Section I: Three Early Sources of the Concept of a Vector and of Vector Analysis

Comment: When and how did vector analysis arise and develop? Vector analysis arose only in the period after 1831, but three earlier developments deserve attention as leading up to it. These three developments are (1) the discovery and geometrical representation of complex numbers, (2) Leibniz's search for a geometry of position, and (3) the idea of a parallelogram of forces or velocities.

1545 Jerome Cardan publishes his *Ars Magna*, containing what is usually taken to be the first publication of the idea of a complex number. In that work, Cardan raises the question: "If someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 30 or 40, it is evident that this case or question is impossible." Cardan then makes the surprising comment: "Nevertheless, we shall solve it in this fashion," and proceeds to find the roots $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. When these are added together, the result is 10. Then he stated: "Putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$, making 25 – (–15) which is +15. Hence this product is 40."[3] As we shall see, it took more than two centuries for complex numbers to be accepted as legitimate mathematical entities. During those two centuries, many authors protested the use of these strange creations.

1679 In a letter to Christiaan Huygens, Gottfried Wilhelm Leibniz proposes the idea (but does not publish it) that it would be desirable to create an area of mathematics that "will express situation directly as algebra expresses magnitude directly." Leibniz works out an elementary system of this nature, which was similar in goal, although not in execution, to vector analysis.

1687 Isaac Newton publishes his *Principia Mathematica*, in which he lays out his version of an idea that was attaining currency at that period, the idea of a parallelogram of forces. His statement is: "*A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately.*" Newton did not have the idea of a vector. He was, however, getting close to the idea, which was becoming common in that period, that forces, because they have both magnitude and direction, can be combined, or added, so as to produce a new force.

1799 Caspar Wessel, a Norwegian surveyor, publishes a paper in the memoirs of the Royal Academy of Denmark in which he lays out for the first time the geometrical representation of complex numbers. His goal was not only to justify complex numbers, but also to investigate "how we may represent direction analytically." Not only does Wessel publish for the first time the now standard geometrical interpretation of complex numbers as entities that can be added, subtracted, multiplied, and divided, he also seeks to develop a comparable method of analysis for three-dimensional space. In this, he fails. Moreover, his 1799 paper fails to attract many readers. It becomes known only a century later, by which time various

[3]Girolamo Cardan, *The Great Art or The Rules of Algebra,* trans. and ed. by T. Richard Widmer (Cambridge: Massachusetts Institute of Technology Press, 1968), pp. 219–20.

other authors had also published the geometrical representation of complex quantities.

Comment: It seems somewhat remarkable that in three cases in the period from 1799 to 1828 two authors independently and essentially simultaneously work out the geometrical representation of complex numbers. This happened in 1799 (Wessel and Gauss), 1806 (Argand and Buée) and 1828 (Warren and Mourey). In fact, we shall see other cases of independent simultaneous discovery in this history.

1799 Around this time, Carl Friedrich Gauss works out the geometrical interpretation of complex quantities, but publishes his results only in 1831. Like Wessel, Gauss is seeking entities comparable to complex numbers that could be used for three-dimensional space.

1806 Jean Robert Argand publishes the geometrical interpretation of complex numbers, and in a follow-up publication of 1813 attempts to find comparable methods for the analysis of three-dimensional space. Also in 1806, the Abbé Buée publishes a somewhat comparable essay in which he comes close to the geometrical representation of complex numbers.

1828 England's John Warren and France's C. V. Mourey, both writing independently of the authors who had already published the geometrical representation of complex numbers, publish books setting forth the geometrical representation of complex numbers. Warren does not discuss extending his system to three dimensions, whereas Mourey states that such a system is possible, but does not publish such a system.

1831 Carl Friedrich Gauss publishes the geometrical justification of complex numbers, which he had worked out in 1799. Whereas the former five authors on this subject attracted almost no attention, the prestige and proven track record of Gauss ensures the widespread acceptance of this representation followed upon his publication. Ironically, Gauss himself did not accept the geometrical justification of imaginaries as fully satisfactory. It is also interesting to note that Felix Klein argued in 1898 that Gauss had anticipated Hamilton in the discovery of quaternions, which claim Peter Guthrie Tait and C. G. Knott vigorously disputed. Grassmann learns of Gauss's paper only in 1844 and Hamilton in 1852.

Section II: William Rowan Hamilton and His Quaternions

Comment: Hamilton searched for thirteen years for a system for the analysis of three-dimensional space, that search culminating in 1843 with his discovery of quaternions, one of the main systems of vector analysis. This section treats the creation and development of the quaternion system from 1843 to 1866, the year after Hamilton had died and the year in which his most extensive publication on quaternions appeared.

1805 Birth of William Rowan Hamilton in Dublin, Ireland.

1818 Hamilton at age thirteen attains fame for many intellectual achievements, including being "in different degrees acquainted with thirteen languages," including Greek, Latin, Hebrew, Syriac, Persian, Arabic, Sanskrit, Hindoostanee, Malay, French, Italian, Spanish, and German.

1823 Hamilton enters Trinity College, Dublin, placing first in the entrance exam.

1826 Even before the end of an undergraduate career, which had merited him many awards, Hamilton is named Andrews Professor of Astronomy in the University of Dublin and Royal Astronomer of Ireland. He holds these positions for the remainder of his life.

1832 Verification by Humphey Lloyd of Hamilton's mathematical prediction of internal and external conical refraction, one of the most famous scientific predictions of the century. This discovery, which comes out of Hamilton's very important papers on "Theory of Systems of Rays," further enhances his fame.

1835 Hamilton knighted.

1837 Hamilton publishes a long paper interpreting complex numbers as ordered couples of numbers, an alternate justification of such numbers, which now is seen as preferable. Hamilton also argues that algebra can be understood as the science of pure time as geometry is the science of pure space. In that paper, Hamilton mentions his hope to publish a "Theory of Triplets," i.e., a system that would do for the analysis of three-dimensional space what imaginary numbers do for two-dimensional space. Hamilton had been searching for such triplets from at least 1830. It is significant to note that in this paper Hamilton makes clear that he understands the nature and importance of the associative, commutative, and distributive laws, an understanding rare at a time when no exceptions to these laws were known.

1843 Having searched for his triplets for thirteen years, Hamilton discovers quaternions. In a letter he later wrote to one of his children about the discovery, he recounts that his children used to ask him each morning at breakfast: "Well, Papa, can you multiply triplets?" To this he would reply, "No, I can only *add* and subtract them." On 16 October 1843, his search ends with his discovery of mathematical entities he calls "quaternions." These are higher complex numbers of the form $a + xi + yj + zk$, where a, x, y, z are real numbers and i, j, and k are three distinct imaginary numbers obeying the following rules of multiplication: $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$, $ii = jj = kk = -1$. From this we see that for two quaternions in which the first part, the real number, is equal to zero
$Q = xi + yj + zk$ and
$Q' = + x''i + y''j + z''k$,,
their product
$QQ' = -(xx' + yy' + zz') + i(yz' - zy') + j(zx' - xz') + k(xy' - yx')$.
Hamilton immediately becomes convinced that he had made an important discovery, stating that "this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions [the calculus] was for the close of the seventeenth." He proceeds to devote the remaining twenty-two years of his life to writing one hundred and nine papers and two immense books on his quaternions.

Comment: One good way (especially in the present context) of describing Hamilton's search for quaternions is to state that his search was for numbers with the following six characteristics, all of which are found in ordinary complex numbers: (1) associativity for multiplication and division, (2) commutativity for addition and multiplication, (3) the distributive property, (4) the property that division is unambiguous, (5) the property that the numbers obey the law of the moduli,[4] (6) the property of being useful for the analysis of three-dimensional space. Quaternions possess all of six characteristics, with the exception that they are not commutative for multiplication. One can get a sense of why quaternionists objected to modern vector analysis when it is noted that modern vector analysis involves two forms of multiplication, the scalar (dot) and vector (cross) products. For the scalar product, associativity is irrelevant, and both the law of the moduli and unambiguity of division must be abandoned. For the vector product, the associative and commutative properties must be abandoned, division is not unambiguous, and the law of the moduli fails as well.

Comment: It was around this time that the ideas of the founders of non-Euclidean geometry, Nicholas Lobachevski and Janos Bolyai, were becoming known. It is important to realize that Hamilton, by creating the first extensive and consistent algebraic system that departed from at least one of the standard properties of traditional mathematics issued in a development that was probably as significant for algebra as the non-Euclidean systems were for geometry. Perhaps the most significant message carried by Hamilton's creation is that it is legitimate for mathematicians to create new algebraic systems that break traditional rules. Although some mathematicians resisted this claim, others soon took advantage of it by creating new algebraic systems.

1846 Hamilton publishes a paper in which he introduces the terms *scalar* and *vector,* referring respectively to the real and the imaginary parts of his quaternion. Thus he writes regarding a quaternion Q = a + bi + cj + dk, that "Q = Scal. Q + Vect. Q = S.Q + V.Q or simply Q = SQ + VQ." In other words, SQ = a, whereas VQ = bi + cj + dk. This led to quaternionists writing equations such as the following: if we have two quaternions both having their scalar parts equal to 0, Q = xi + yj + zk and Q´ = x´i + y´j + z´k, then the laws of quaternion multiplication dictate that SQQ´ = –(xx´ + yy´+ zz´) and VQQ´ = i(yz´ – zy´) + j(zx´ – xz´) + k(xy´ – yx´). What is important to note about this is that the scalar portion of this new quaternion can be seen as mathematically equal to the negative of the modern scalar or dot product, and the vector part as equal to the modern cross product. This will be very significant historically; in fact, it was precisely along this path that modern vector analysis originated.

1847 By this year, Hamilton receives prizes for his discovery from the Royal Irish Academy and the Royal Society of Edinburgh and publishes at least thirty-four papers on quaternions, which had been endorsed by some leading mathematical and scientific figures, including John Herschel.

1853 Hamilton publishes his *Lectures on Quaternions,* a 737–page volume, not counting its 64–page largely philosophical preface and 72–page table of contents.

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[4]This law as applied to imaginaries specifies that if three complex numbers combine so that $(a_1 + b_1 i)(a_2 + b_2 i) = (a_3 + b_3 i)$, then $(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_3^2 + b_2^3)$ ..

1865 Death of William Rowan Hamilton, who by this time had published 109 of the 150 papers that had been published on quaternions. During this period, quaternion analysis had been much praised but little practiced. Hamilton had, however, secured one energetic and talented disciple, the Scottish mathematician and scientist Peter Guthrie Tait, who took up Hamilton's mantle—or was it, as some thought, Hamilton's mania?

1866 Publication of Hamilton's *Elements of Quaternions*, which was one and one half times longer than Hamilton's immense *Lectures on Quaternions*.

Section III: Other Early Vectorial Systems, Especially Grassmann's Calculus of Extension

Comment: Hamilton was not alone in creating a vectorial system during the period around 1843. In fact, in that period six other authors from four countries were developing systems that were more or less vectorial in character. The six men were August Ferdinand Möbius, Giusto Bellavitis, Comte de Saint-Venant, Augustin Cauchy, Matthew O'Brien, and above all Hermann Günther Grassmann.

1809 Hermann Günther Grassmann was born in Stettin in Pomerania, where he spent the majority of his life. He was a son of a mathematics teacher at Stettin Gymnasium, Justus Günther Grassmann.

Grassmann

1827 Grassmann enters the University of Berlin, where he studies mainly theology and philology. After graduating, he returns to Stettin to study various subjects including mathematics in preparation for taking the state examination to qualify as a teacher.

1827 August Ferdinand Möbius of Leipzig University publishes *Der barycentrische Calcul* (*The Barycentric Calculus*), a system developed around the idea of taking the centroid of a system of weighted points.

1835 Giusto Bellavitis publishes his first exposition of his system of equipollences, which has some features in common with the now traditional vector analysis, as is suggested in his definition of equipollent: "Two straight lines are called *equipollent* if they are equal, parallel and directed in the same sense." His lines in fact behave in exactly the same manner as complex numbers behave, but it is important to note that he viewed his lines as essentially geometric entities, not as geometric representations of algebraic entities; in fact, he was opposed to complex numbers as "unworthy to belong to a science based on reason alone." Bellavitis devoted a long period to an unsuccessful attempt to extend his system to three dimensions.

1836 Grassmann takes a teaching position in Stettin, which he teaches for the remainder of his life.

1840 Grassmann completes the writing of his *Theorie der Ebbe und Flut* (*Theory of the Ebb and Flow*) and submits this 200+ page essay as evidence of his competence for teaching. This work on tidal theory contains the first system of spatial analysis based on vectors and is reasonably close to the modern system. Grassmann dated the origin of these ideas to 1832 and traced his fundamental idea to reflections on negative numbers and to the idea of adding and subtracting directed lines. He traced his idea of a geometrical product to textbooks written by his father and entitled *Raumlehre* (*Space Theory*) and *Trigonometrie*, the first having been published in 1824, the latter in 1835. In particular, his father had written in 1824: "The rectangle itself is the *true geometrical product,* and the construction of it ... is really *geometrical* multiplication." H. G. Grassmann's work did not come out of the geometrical representation of complex numbers tradition; in fact, he learned of that representation only in December, 1844.
Comment: Grassmann's 1840 *Theorie der Ebbe and Flut* presents among other matters the addition and subtraction of lines *(strecken)* and also what is numerically equivalent to the modern cross product, with this difference that whereas the product of two vectors in the modern system is another vector, in Grassmann's system it is a geometrical entity, the directed area of the parallelogram between the two strecken or vectors. Grassmann also presents in this treatise the "linear product" of two strecken, this being identical to the modern dot or scalar product. He also treated vector differentiation.

1844 Grassmann publishes the first full exposition of his system, his *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik dargestellt and durch Anwendungen auf die ubrigen Zweige der Mathematick, wie auch auf der Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert*. Whereas on the title page of Hamilton's *Lectures on Quaternions*, Hamilton was identified by a large array of titles and memberships, Grassmann's title page identified him only as "Lehrer an der Friedrich Wilhelms Schule zu Stettin." The book attracts almost no attention and about 600 copies of it were in 1864 used for waste paper.
Comment: Grassmann's *Die lineale Ausdehnungslehre* (*Linear Extension Theory*) demonstrated deep mathematical insights. It also in one sense contained much of the modern system of vector analysis. This, however, was embedded within a far broader system, which included n-dimensional spaces and as many as sixteen different products of his base entities (including his inner and outer products, which are respectively somewhat close to the our modern dot and cross products). Moreover, Grassmann justifies his system by philosophical discussions that may have put off many of his readers. The abstractness of his presentation and the originality of his insights also contributed to the difficulties readers had in comprehending Grassmann's message, as is evident from comments made by various mathematicians who had struggled with the book. Möbius, for example, labeled it unreadable, Baltzer reported that reading the book made him feel "dizzy and to see sky blue before my eyes," and Hamilton wrote De Morgan that to read the *Ausdehnungslehre* he would have to learn to smoke.

1845 After Grassmann's unsuccessful efforts to persuade Möbius to write a review of his book, Grassmann at Möbius's urging, writes a review of his own book—the only review his book received! He also publishes a paper containing a new discovery in electrical theory that he had derived using his new methods. The result: more neglect, until the 1870s when Clausius made the same electrical discovery and published it, only then realizing that Grassmann had preceded him.

1845 Adhémar Barré, Comte de Saint-Venant, publishes a short paper titled "Mémoire sur les sommes et les différences géométriques, et sur leur usage pour simplifier la mécanique," in which he lays out a number of the fundamental ideas of vector analysis, including a version of the cross product, the difference being that his product was viewed not as another vector but as a spatially oriented area. Grassmann and Saint-Venant correspond for a time, but Saint-Venant's ideas do not seem to have attracted significant attention. They do show, however, that the search for a vectorial system was "in the air."

1847 Möbius, hearing of a prize competition sponsored by the Jablonowski Gesellschaft for investigations fulfilling Leibniz's idea of a geometry of position urges Grassmann to enter the competition. Grassmann submits an essay, which wins the competition and is published in this year, along with some commentary by Möbius. Result: more neglect. In the same year, Grassmann applies to become a university professor. The mathematician E. E. Kummer is asked to evaluate Grassmann's writings, and finds them very unclear. Grassmann did not get a university position at that time; in fact, none was ever offered him.

1853 The prominent French mathematician Augustin Cauchy publishes his "Sur les clefs algébriques," in which he presents methods he had devised for the solution of various algebraic problems, for example, finding the roots of equations. Not only had Grassmann published what were essentially the same methods in his *Ausdehnungslehre*, but he had also sent two copies of that book to Cauchy in 1846 in an unsuccessful effort to get a copy to Saint-Venant. Friends alert Grassmann that there is a priority issue regarding Cauchy's methods, leading Grassmann to write the French Academy to arbitrate the priority issue. A three person committee is formed, which includes Cauchy! Nothing is ever settled, probably because Cauchy died in 1857.

1852 Matthew O'Brien of King's College, London publishes the most significant of a number of his papers setting out in a less than satisfactory manner a system of vector analysis, which was developed, it seems, partly in terms of Hamilton's quaternions. The most serious defect in O'Brien's system is his failure to investigate the associativity of his vectorial entities.

1853 Leaving aside Grassmann's own writings, only two published comments on Grassmann's work appeared before the 1860s. One of these consists in the commentary by Möbius included with the publication of Grassmann's prize essay; the other comment appears in the Preface to Hamilton's *Lectures on Quaternions*, published in this year. In 1852, Hamilton somehow learned of Grassmann's book and read through it, concluding with much relief that Grassmann had not discovered quaternions. In his private correspondence he waxes and wanes about the merits of Grassmann's insights, but in his Preface he devotes a paragraph to Grassmann, describing his book as "very original and remarkable" and its author as "profound and philosophical," but also stressing that their systems are "distinct and independent of each other," although sharing some features.

1860 Luigi Cremona praises Grassmann's ideas in a published article, thus bringing the number of authors who had publicly noted his writings to a total of three.

1862 Grassmann, convinced of the merits of his ideas but frustrated by the almost total neglect of his publications, publishes his system in a new form under the title *Die Ausdehnungslehre: Vollständing und in strenger Form bearbeitet*. Three hundred copies are printed in the shop of Grassmann's brother and all at Grassmann's expense. For this volume, Grassmann wisely decides to remove the philosophical discussions included in his earlier *Ausdehnungslehre* and to present his system in Euclidean dress, a decision that Friedrich Engel, the editor of Grassmann's works and one of his two biographers, labeled a "disastrous mistake." Grassmann himself wrote in 1877: "this new work met with even less attention than the first."

1867 In his *Theorie der complexen Zahlensysteme*, Hermann Hankel, a young and promising mathematician who had studied with Riemann, praises Grassmann's ideas, but soon thereafter dies (1873).

1872 Rudolf Clebsch praises Grassmann's work in his "Zum Gedachness an Julius Plücker" published in this year, which is also the year of Clebsch's death.

1872 Victor Schlegel, who had come to know Grassmann while teaching in Stettin from 1866 to 1868, publishes in this year his *System der Raumlehre nach dem Prinzipien der Grassmann'schen Ausdehnungslehre und als Einleitung in Dieselbe*, with a second part appearing in 1875. This re-presentation of Grassmann's ideas in a more elementary form was not, however, very successful. Engel argues that the biography of Grassmann that Schlegel publishes in 1878 was more influential in drawing attention to his work. Schlegel continued to champion Grassmann's idea for the rest of life (d. 1905), publishing over 25 papers in the Grassmannian tradition.

1877 Death of Hermann Günther Grassmann, a brilliant if isolated schoolmaster whose achievements extended far beyond mathematics. For example, in the 1870s, he published his *Wörterbuch zum Rig-Veda* (1784 pages) and his translation of the *Rig-Veda* (1123 pages), which led to his receiving an honorary doctorate from the University of Tübingen. He also published textbooks on the Latin and German languages, various religious and musical writings, and a book on German botanical terminology.

1878 Because of the increased interest in Grassmann during the 1870s and the short supply of the earlier *Ausdehnungslehre*, a second edition of the earlier *Ausdehnungslehre* is published in this year.

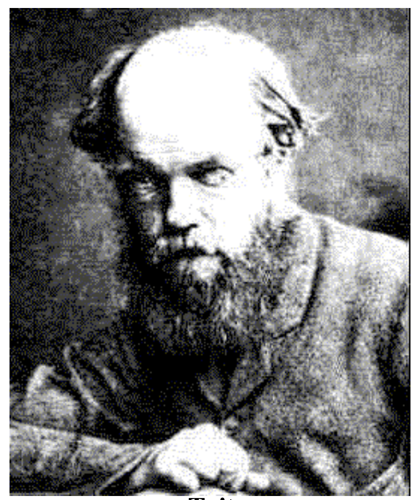
Section IV: The Middle Period in the Development of the Modern System of Vector Analysis.

Comment: It is useful to analyze the development of modern vector analysis in terms of three periods, the first extending up to 1865, by which time the two main traditions, the Hamiltonian quaternionic and the Grassmannian tradition had arisen. The second or middle period runs from about 1865 to about 1880. By the beginning of this period, Hamilton (because of his death) and Grassmann (who concentrated on other areas) had ceased to be major contributors. Other mathematicians had gradually assumed positions of leadership. In the third period, which began around 1880, the modern system of vector analysis came into existence through the work of Josiah Willard Gibbs and Oliver Heaviside and by 1910 had established itself as the dominant system, although not without a struggle against the Hamiltonian and Grassmannian systems. The leading figures in this middle period were Peter Guthrie Tait, Benjamin Peirce, James Clerke Maxwell, and William Kingdon Clifford.

If one asks whether the Hamiltonian or Grassmannian systems was more vigorous in the period from the early 1840s up to 1900, the answer is certainly the Hamiltonian. Quaternionic publications in that nearly sixty year period numbered 594, whereas Grassmannian publications came to 217. Looked at geographically, quaternionic interest was strongest in Britain and Ireland, with the United States ranking next; Grassmannian publications were primarily written in German, but both systems had followers far beyond the countries of their origin.

Two key questions should be kept in mind in the remainder of this discussion. It is true that the Grassmannian system contained within it most of modern vector analysis. Consequently, it is possible that the modern system of vector analysis could have originated from it. It is also true that the quaternionic system was significantly different from modern vector analysis, but possessed some similarities to it. From which system did the modern system originate? And related to this question, how did this take place? The answers to these questions may prove quite surprising.

1858 To the aging William Rowan Hamilton, it may have seemed in 1858 that he would die without a successor in the quaternionic cause. In that year, however, he began to correspond with Peter Guthrie Tait (1831–1901), a Scot who had graduated from Cambridge University in 1853 as Senior Wrangler and First Smith's Prizeman, and who in 1858 was teaching mathematics at Queen's College, Belfast, but who in 1860 became the Professor of Natural Philosophy at Edinburgh University. Tait had become very interested in quaternions, especially in their usefulness for physical science. One measure of the intensity of their correspondence is that one letter alone runs 96 pages. Hamilton could hardly have have hoped for a more energetic successor. By 1859, Tait had published the first of his about seventy papers on quaternions and soon began writing an elementary presentation of that system, which he graciously held back from publication until after the 1866 publication of Hamilton's *Elements of Quaternions*.



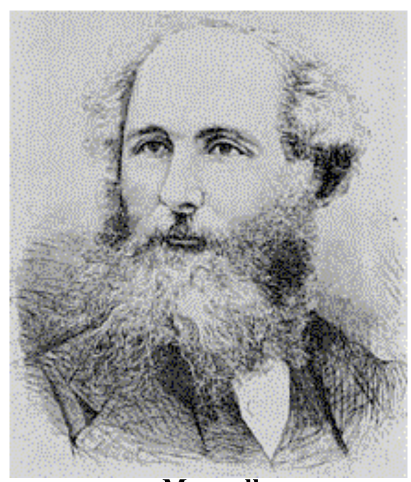
Tait

1867 Tait publishes his *Elementary Treatise of Quaternions,* which went through later editions in 1873 and 1890 as well as translations into German and French. He also co-authored with Philip Kelland *An Introduction to Quaternions* (1873; later editions in 1882, and 1904). A noteworthy feature of Tait's *Treatise* was the extensive attention that he gave (as Hamilton had not) to physical applications. Partly for this reason, his books tended to be filled with cases in which the scalar portion or the vector portion of the full quaternion product was separated out, to the point that those books look much like modern day vector analysis books, with of course the major difference that the scalar part of the product of two quaternionic vectors was the negative of the scalar product in modern vector analysis. Tait included extensive treatment of the operator $\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$. In fact, Maxwell correctly if somewhat enigmatically described Tait as the "Chief Musician upon Nabla." All in all, Tait's book looks very similar to modern vector analysis books, even in much of its terminology, e.g., the scalar and vector products.

1867 Tait's best known publication of 1867 was co-authored with William Thomson (known from 1892 as Lord Kelvin). Sometimes referred to as the *Principia* of the nineteenth century, it was their *Treatise on Natural Philosophy*. Correspondence between these two leading British physicists shows that Tait had unsuccessfully urged the inclusion of quaternionic methods in this major publication, Thomson resisting. In fact, as Thomson wrote in 1901, he and Tait "had a thirty-eight years' war over quaternions." Tait also corresponded extensively about quaternions with another great Scottish physicist and close personal friend, James Clerke Maxwell, in this case with better but far from complete success (see 1873).

1870 Benjamin Peirce of Harvard publishes in lithograph his *Linear Associative Algebra,* described by Dirk Struik as "the first major original contribution to mathematics produced in the United States." An early enthusiast for quaternions and the chief source of the interest in quaternions in the U.S., Peirce in this publication, working from Hamilton's discovery of the possibility of new algebras, lays out and classifies 162 different algebras. He describes his goal as developing "so much of hyper-complex numbers as would enable him to enumerate all inequivalent, pure, nonreciprocal number systems in less than seven units."

1873 James Clerke Maxwell (1831–1879) publishes his *Treatise on Electricity and Magnetism*, arguably the most important treatment of the dominant physical science of the nineteenth century. Although Maxwell had not used quaternionic methods at all in working out this four famous papers on electricity and magnetism, Maxwell, by 1870, partly under the influence of Tait, who had been his friend since childhood, had begun to read quaternionic works. Moreover, Maxwell expresses many of the results presented in his *Treatise* not only in Cartesian form, but also in their quaternionic equivalent. This happened frequently enough that readers could easily assume that Maxwell himself preferred these methods, but had decided not to force them upon readers of his book. In fact, Maxwell's position, stated explicitly in various publications, was somewhat different. For example, in his famous paper of 1871 titled "On the Mathematical Classification of Physical Quantities," Maxwell states: "The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all sorts of science." Some read this as an endorsement of quaternions as a mathematical method; in fact, what Maxwell was saying was that vectorial methods provide an invaluable way of thinking, but in actual practice, quaternionic methods


Maxwell

are not satisfactory. What we shall see is that later authors, proceeding precisely from such statements as this claim by Maxwell and from their own experience in physical science, proceed to rearrange and shape the quaternionic system into the modern system of vector analysis.

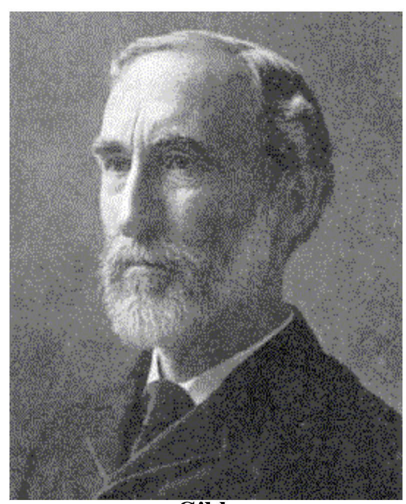
1877 William Kingdon Clifford (1845–1879) publishes his *Elements of Dynamic*, an elementary treatise on mechanics. Clifford, a brilliant if short lived English mathematician teaching at University College, London, was one of the few mathematicians in this period who knew both the quaternionic and the Grassmannian systems. He had lectured on quaternions at University College in 1877 and in 1878 had published a paper employing Grassmannian methods. What is significant for our purposes in his *Elements of Dynamic* is that near the middle of his book, in a section titled "Product of Two Vectors," he actually introduces two products, one of which calls the "vector product," which is the standard vector product as employed by both Grassmannians and Hamiltonians as well as in modern vector analysis. His other product, which he calls the scalar product, he describes qualitatively, but leaves the question of its sign undefined. As we have seen, this product for Grassmannians was positive, whereas for quaternionists it was negative. Not only that, it really was not a product at all, but rather nothing more than a part of the full quaternion product. Nearly a hundred pages later he returned to his scalar product and defined it as "as the negative sum of the products of their components along the axes," i.e., he specified it in a quaternionic manner. What we see in Clifford is what was already beginning to appear in somewhat different forms in Tait and Maxwell: a separation of the full quaternionic product into two parts, along with the treatment of these parts as separate products.

Section V: The Creation of the Modern System of Vector Analysis.

Comment: Two individuals played a key role in the creation of modern vector analysis. They were Josiah Willard Gibbs and Oliver Heaviside, who independently developed the system that is almost universally taught at the present time.

V, Part One: Gibbs

1881 Josiah Willard Gibbs (1839–1903) prints the first half of his *Elements of Vector Analysis*, which presents what is essentially the modern system of vector analysis. This had its basis in the course he had begun to give at Yale in 1879 on vector analysis. The draft of a long letter that Gibbs wrote in 1888 to Victor Schlegel makes very clear how he came to create his system. From reading Maxwell's *Treatise on Electricity and Magnetism*, "where Quaternion notations are considerably used, I became convinced that to master those subjects, it was necessary for me to commence by mastering those methods. At the same time I saw, that although the methods were called quaternionic, the idea of the quaternion was quite foreign to the subject. I saw that there were two important functions (or products) called the vector part & the scalar part of the product, but that the union of the two to form what was called the (whole) product did not advance the theory as an



Gibbs

instrument of geom. investigation." Gibbs then "began to work out ab initio" a new form of vector analysis that involved two distinct products as well as various other features of modern vector analysis. Gibbs then explains to Schlegel that after this, he came to learn something of Grassmann's work, first through a paper by Grassmann on electricity. After learning a little of Grassmann, "I saw that the methods wh. I was using, while nearly those of Hamilton, were almost exactly those of Grassmann." Gibbs recounts that he then procured copies of Grassmann's *Ausdehnungslehre* volumes, struggled with them, but adds: "I am not however conscious that Grassmann's writings exerted any particular influence on my VA, although I was glad enough in the introductory paragraph to shelter myself behind one or two distinguished names...." In summary, Gibbs adds that that he hopes Schlegel will be interested to know "how commencing with some knowledge of Ham[ilton]'s methods & influenced simply by a desire to obtain the simplest algebra ... I was led essentially to Grassmann's algebra of vectors, independently of any influence from him...." Although the point is not mentioned by Gibbs, a side-by-side comparison of Gibbs's *Vector Analysis* and the second edition of Tait's *Treatise on Quaternions*, makes it very clear that Gibbs had learned many of his methods and much of his notation from that book, and then translated it into the form of modern vector analysis. Keeping in mind that Gibbs called our scalar or dot product the "skew product" and wrote it as "$\alpha.\beta$," whereas he called our cross product the "direct product" and wrote it as "$\alpha \times \beta$," we can see the closeness of the treatments of Gibbs and Tait by examining a few of their equations:

Gibbs (2.21) $\alpha.\beta = \beta.\alpha$
Tait (3;43) $S\alpha\beta = S\beta\alpha$
Gibbs (2.21) $\alpha \times \beta = -\beta \times \alpha$
Tait (3;43) $V\alpha\beta = -V\beta\alpha$

1884 Gibbs publishes the second half of his *Elements of Vector Analysis,* which concentrates on the more advanced parts of vector analysis, especially linear vector functions, that is, vector functions of such a nature that a function of the sum of any two vectors is equal to the sum of the functions of the vectors. In doing this, Gibbs introduces the terms and concepts of "dyad" and "dyadic." Moreover, during the 1880s Gibbs frequently teaches a course on vector analysis, and does so every year during the 1890s. In later years, the course consists of as many as 90 lectures.

1886 Gibbs publishes one of his most important and creative papers in mathematics. Entitled "On Multiple Algebra," it makes a case for increased attention to multiple algebra, praises Grassmannian methods, and concludes with the famous line "We begin by studying *multiple algebras*; we end, I think, by studying MULTIPLE ALGEBRA."

1901 Edwin Bidwell Wilson, a former student of Gibbs, publishes the first book-length formally published presentation of modern vector analysis in English: *Vector Analysis: A Text Book for the Use of Students of Mathematics and Physics and Founded upon the Lectures of J. Willard Gibbs,* which becomes a classic.

1903 Death of Josiah Willard Gibbs.

V, Part Two: Heaviside

1883 Oliver Heaviside (1850–1925), an Englishman whose schooling had ended at age 16, begins to introduce vectorial methods into his writing on

electrical theory, which were in the Maxwellian tradition; indeed, Heaviside is frequently seen as the chief successor to Maxwell in electromagnetic theory.

1885 Heaviside in one of his electrical papers gives his first unified presentation of his system of vector analysis, which is essentially identical to that of Gibbs and to the modern system. Heaviside, known for his wit, later explained how he came to develop his system, beginning by describing the experiences of a boy who, enchanted by the word quaternion, tried to learn its meaning by reading Hamilton's books.

> He took these books home and tried to find out. He succeeded after some trouble, but found some of the properties of vectors professedly proved were wholly incomprehensible. How could the square of a vector be negative? And Hamilton was so positive about it. After the deepest research, the youth gave it up [and] died.
>
> My own introduction to quaternions took place in quite a different manner. Maxwell exhibited his main results in quaternionic form in his treatise. I went to Prof. Tait's treatise to get information, and to learn how to work them. I had the same difficulties as the deceased youth, but by skipping them, was able to see that quaternions could be explored consistently in vectorial form. But on proceeding to apply quaternionics to the development of electrical theory, I found it very inconvenient. ... So I dropped out the quaternions altogether, and kept to pure scalars and vectors....

Heaviside then recounts that in 1888, he received a copy of Gibbs's privately printed text, a sort of "condensed synopsis of a treatise," finding that it was essentially the same system to which he had been independently led. He then adds: "I appeased Tait considerably ... by disclaiming any idea of discovering a new system. I professedly derived my system from Hamilton and Tait by elimination and simplification...."

In short, by an almost identical path to that followed by Gibbs and in entire independence of him, Heaviside had arrived at essentially the same system. It appears that Heaviside first learned of Grassmann's system only in 1888 when he found Grassmann's name in Gibbs's text.. There is no reason to think Heaviside ever read any of Grassmann's writings.

1893 Heaviside publishes the first volume of his *Electromagnetic Theory*, which contains as Chapter 3, "The Elements of Vectorial Algebra and Analysis," a 173-page presentation of the modern system of vector analysis. This is the first extensive published treatment of that system and contains an endorsement of Gibbs's presentation although not of his notation. Of course, Heaviside's presentation appeared in a specialized book on electrical theory; on the other hand, Heaviside's association of vector analysis with the ever expanding area of electrical science was very helpful in ensuring its spread, as will be evident in what follows.

1925 Heaviside, by then suffering from poverty, deafness, and isolation, dies.

Section VI: A "Struggle for Existence" in the 1890s among the Systems of Vector Analysis.

Comment: In an 1888 letter, Gibbs predicted that "a Kampf ums Dasein [struggle for existence] is just commencing between the different methods and notations of multiple algebra, especially between the ideas of Grassmann & of Hamilton." Gibbs's prediction was fulfilled: In the years 1890 to 1894, a widespread and vigorous debate on vectorial methods took place. No less than eight journals, twelve scientists, and thirty-eight publications came forth. Lord Rayleigh aptly characterized the spirit of the debate by a paraphrase of Tertullian: "Behold how these vectorists love one another." Sample quotations follow in chronological order.

Peter Guthrie Tait, who wrote with vigor but also a certain impatience:

> Even Professor Gibbs must be ranked as one of the retarders of Quaternion progress, in virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann.

Josiah Willard Gibbs, who wrote with almost unwavering tact and good sense:

> The merits or demerits of a pamphlet printed for private distribution a good many years ago do not constitute a subject of any great importance, but the assumptions implied in the sentence quoted are suggestive of certain reflections and inquiries which are of broader interest, and seem not untimely at a period when the methods and results of the various forms of multiple algebra are attracting so much attention. It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain unchallenged, especially when supported by so high an authority. The criticism relates particularly to the notations, but I believe that there is a deeper question of notions underlying that of notations. Indeed, if my offence had been solely in the matter of notations, it would have been less accurate to describe my productions as a monstrosity, than to characterize its dress as uncouth.

Oliver Heaviside, who wrote with much insight and at time with scarcely less wit:

> ... the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics; but to find out quaternions required a genius.

Peter Guthrie Tait, who showed the penchant of vectorists for metaphors when late in the debate responding to Arthur Cayley's claim on behalf of the importance of using Cartesian coordinates by comparing quaternions to a pocket map that repeatedly needs to be unfolded:

> A much more natural and adequate comparison would ... liken Co-ordinate Geometry ... to a steam-hammer, which an expert may employ on any destructive or constructive work *of one general kind,* say the cracking of an egg-shell, or the welding of an anchor. But you must have your expert to manage it, for without him it is useless. He has to toil amid the heat, smoke, grime, grease, and perpetual din of the suffocating engine-room. The work has to be brought to the hammer, for it cannot usually be taken to its work.... Quaternions, on the other hand, are like the elephant's trunk, ready at *any* moment for *anything*, be it to pick up a crumb or a field gun, to strangle a tiger, or to uproot a tree. Portable in the extreme, applicable anywhere ... directed by a little native who requires no special skill or training, and who can be transferred from one elephant to another without much hesitation. Surely this, which adapts itself to its work, is the grander instrument! But then, *it* is the natural, the other the artificial, one.

Comment: One effect of this widespread and colorful debate was to alert the scientific public that there were a number of vectorial systems, that they were somewhat different, and that major mathematicians and physical scientists were concerned about the issues. Moreover, as Gibbs and Heaviside became ever more widely known for their contributions to science, their mutually supportive advocacy of what became the modern system must have been taken more seriously. Another offshoot of the debate, or at least of the issues underlying it, was the successful issuing in 1895 of a call for the formation of what became the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics, which from 1900 to 1913 published a journal.[5]

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[5]*Bulletin of the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics*.

Section VII: Emergence of the Modern System of Vector Analysis: 1894–1910

Comment: We have seen that by 1893 three main systems of vector analysis had been created and received substantial attention: the systems of Hamilton, Grassmann, and Gibbs-Heaviside, the third being the most recent. Which would come to dominance and how would this happen? The evidence indicates that by 1910, the Gibbs-Heaviside, i.e., the modern system had won out in the struggle for existence. This section provides some of the evidence for that claim. Moreover, it suggests that although Gibbs had presented his system with more sophistication and originality, it was Heaviside's writings, particularly his association of vector analysis with Maxwellian electrical theory, that was more influential in establishing the modern system. The evidence will take the form of a brief examination of eleven major publications presenting the modern system of vector analysis in the period from 1894 to 1910, and also enough information on the publishing history, if any, for these books to allow an assessment of their success.

1894 August Föppl publishes his *Einfuhrung in die Maxwell'sche Theorie de Elektricität*. Its first three chapters (84 pages) consist of an exposition of Heaviside's presentation of vector analysis. Very influential, both in electrical theory and applied mathematics. Föppl also used vector analysis in other publications.
Subsequent history: 2nd. ed, 1904; 3rd., 1907, 4th, 1912, 1st English ed., 1932, 16th German ed., 1957.

1899 Galileo Ferraris publishes his *Lezioni di Elettrotechnica,* presenting both electricity and vector analysis in the Heaviside tradition.

1901 Edwin Bidwell Wilson publishes *Vector Analysis: A Text Book for the Use of Students of Mathematics and Physics and Founded upon the Lectures of J. Willard Gibbs*. This is the first formally published book devoted entirely to presenting the modern system of vector analysis. Trained as a Harvard undergraduate in quaternions by J. M. Peirce, Wilson, upon graduation in 1899, proceeded to Yale where he reluctantly took Gibbs's vector analysis course, and agreed to write this book, which became quite successful.
Subsequent history: 2nd. ed, 1909, 8th printing, 1943, paperback reprinting, 1960.

1903 Alfred Heinrich Bucherer publishes his *Elemente der Vektor-Analysis mit Beispielen aus der theoretischen Physik*. Author had background in electricity. This is the first German book devoted solely to presenting the modern system of vector analysis.
Subsequent history: 2nd. ed, 1905.

1905 Eugen Jahnke publishes his *Vorlesungen über die Vektorenrechnung mit Anwendungen auf Geometrie, Mechanic und mathematische Physik*. Jahnke, a mathematician, draws on both the Grassmannian and Gibbs-Heaviside formulations.
Subsequent history: no later editions known.

1906 Gibbs's *Elements of Vector Analysis* published as part of his collected works, which also reprinted some of his creative and his polemical articles on vector analysis.
Subsequent history: Paperback reprinting, 1961.

1907 Pavel Osipovich Somoff publishes in Russian the first book in that language on vector analysis. Explicitly states that he is following in the tradition of "Maxwell, Heaveside [sic], Gibbs, and Föppl."

1907 Siegfried Valentiner publishes his *Vektoranalysis*. A physicist, he includes many applications. Eclectic in approach, drawing on both the Gibbs-Wilson and the Heaviside-Föppl formulations.
Subsequent history: 2nd. ed, 1912; 7th ed., 1950; reprint of the 7th ed., 1954.

1909 Cesare Burali-Forti (a mathematician) and Roberto Marcolongo (a physicist) publish their *Elementi di calcolo vettoriale con numerose applicazioni alla geometria, alla meccanica e alla Fisica-Matematica*. Written to a substantial extent from a Grassmannian perspective, but with some attention to the Gibbs-Heaviside formulation.
. Subsequent history: No later editions, but a French translation, 1910.

1909 Joseph George Coffin publishes his *An Introduction to Vector Methods and Their Various Applications to Physics and Mathematics*. The American physicist Coffin supplied with this book the need for a shorter and more elementary presentation in the Gibbs and also Heaviside tradition.
Subsequent history: 2nd. ed, 1911; 1st French ed., 1914; 2nd ed., 6th impression, 1923; 9th reprinting, 1959.

1909–10 W. V. Ignatowsky publishes in two parts his *Die Vektoranalysis und ihre Anwendung in der theoretischen Physik*. This book is chiefly in the Heaviside tradition.
Subsequent history: 3rd ed., 1926.

Comment: It is striking that whereas most of the books using the Gibbs-Heaviside approach went into a number of later editions, those using the Grassmannian approach attained no later editions. This suggests that not only in the *creation*, but also in the *acceptance* of modern vector analysis, the Grassmannian tradition played no major role. Moreover, the evidence provided above shows that although both the Gibbs and the Heaviside traditions were quite influential, the Heaviside tradition, with its association with electromagnetic theory, was more important.